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An Inequality on M-Matrices

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Abstract: Let A_0 be a nonsingular symmetric M-matrix. For a sufficiently large t, $A_t = tI + A_0$ is a new nonsingular symmetric M-matrix and the following inequalities hold for the sum of the principal minors of new matrix A_t:

$$\sum_{C_n^1} |A(1)| < \sum_{C_n^2} |A(1,2)| < \cdots < \sum_{C_n^n} |A(1,2, \dots, n)|.$$

Keywords: Non-negative matrix, Nonsingular symmetric matrix

M-Matrisleri Üzerine Bir Eşitsizlik

Öz: A_0 tekil olmayan simetrik bir M- matrisi olsun. Yeteri kadar büyük bir t değeri için $A_t = tI + A_0$ şeklinde oluşturulan M- matrisinin esas minörlerinin toplamları arasında

$$\sum_{C_n^1} |A(1)| < \sum_{C_n^2} |A(1,2)| < \cdots < \sum_{C_n^n} |A(1,2, \dots, n)|.$$

eşitsizliği vardır.

Anahtar Kelimeler: Negatif olmayan matris, Tekil olmayan simetrik matris

1. Introduction

Definition 1: Let $A = (a_{ij})$ be a real valued matrix for $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$. If $a_{ij} \geq 0$ then matrix A is said to be a non-negative matrix (Gantmacher, 1956).

Definition 2: Let $B = (b_{ij})$ be a non-negative an n dimensional square matrix and I be a n dimensional unit matrix. Further let $\rho(B)$ be the spectral radius of B. Then let A be defined by

$$A = sI - B \qquad (1)$$

A is called a M-matrix for $s \geq \rho(B)$. If $s = \rho(B)$ then A is a singular M-matrix, and conversely, if $s > \rho(B)$ then A is nonsingular M-matrix (Berman and Plemmons, 1979).

Theorem 1: let A_0 be a nonsingular symmetric M-matrix

$$A_t = tI + A_0 \qquad (2)$$

is also a nonsingular symmetric M-matrix for sufficiently large positive integer t.

Proof: A_0 is a nonsingular symmetric M-matrix. Thus, $A_0 = sI - B$ when $s > \rho(B)$ where s is defined as in equation (1). We can write $A_0 = sI - B$, and putting this value into equation (2) we have $A_t = tI + (sI - B)$. Then it follows that $A_t = (s+t)I - B$. This shows that A_t is a nonsingular M-matrix (Berman and Plemmons, 1979).

Theorem 2: Let $A_0 = \left(A_{0_{ij}}\right)$ be an n dimensional nonsingular symmetric M-matrix and A_t be a matrix defined by $A_t = tI + A_0$ for $t \geq n$. Then, for new matrix A_t,

$$\sum_{C_n^1} |A_t(1)| < \sum_{C_n^2} |A_t(1,2)| < \cdots < \sum_{C_n^n} |A_t(1,2,\ldots,n)|, \quad (3)$$

where C_n^r $(r = 1,2,\ldots,n)$ is a binomial coefficient and

$|A_t(i_1, i_2, \ldots, i_n)|$ with

$1 \leq \mathrm{i}_1 < \mathrm{i}_2 < \cdots < \mathrm{i}_\mathrm{r} \leq \mathrm{n}$ is the corresponding principal minor of A_t.

Proof: We will prove this theorem by induction using the properties of similar matrices, Since the characteristic polynomials of similar matrices are the same and the eigenvalues of a real symmetric matrix are real numbers (Mirsky, 1955). It is easily seen that A_t can be written explicitly in the following form.

$$A_t = \begin{bmatrix} t + a_{0_{11}} & a_{0_{12}} & \ldots & a_{0_{1n}} \\ a_{0_{21}} & t + a_{0_{22}} & \ldots & a_{0_{2n}} \\ \ldots & \ldots & \ldots & \ldots \\ a_{0_{n1}} & a_{0_{n2}} & \ldots & t + a_{0_{nn}} \end{bmatrix}$$

A_t is similar to such a triangular matrix whose elements on the main diagonal equal to the eigenvalues of A_t. Thus if the eigenvalues of A_0 are λ_i for i= 1,2,…,n, then the eigenvalues of A_t will be $t + \lambda_i$ for i= 1,2,…,n. In this case, we have a nonsingular n dimensional triangular matrix

$$PA_tP^{-1} = R_t = \begin{bmatrix} t+\lambda_1 & \gamma_{12} & \gamma_{13} & \ldots & \gamma_{1n} \\ 0 & t+\lambda_2 & \gamma_{23} & \ldots & \gamma_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & t+\lambda_n \end{bmatrix}, \quad (4)$$

where $P = (p_{ij})$ is a nonsingular matrix. Now we will prove that equation (3) holds for R_t matrix. For $n = 2$, equation (4) can be written as

$$R_t = \begin{bmatrix} t+\lambda_1 & \gamma_{12} \\ 0 & t+\lambda_2 \end{bmatrix}. \quad (5)$$

When we evaluate the first and second order sums of principal minors, we get

$$\sum_{c_2^1} |R_t(1)| = (t+\lambda_1) + (t+\lambda_2) = 2t + \lambda_1 + \lambda_2 \quad (5.1)$$

$$\sum_{c_2^2} |R_t(1,2)| = (t+\lambda_1)(t+\lambda_2) = t^2 + (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2 \quad (5.2)$$

Supposing $t \geq 2$ and using equations (5.1) and (5.2) we have

$$\sum_{c_2^1} |R_t(1)| < \sum_{c_2^2} |R_t(1,2)|.$$

It follows by hypothesis that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ because the eigenvalues of A_0 are real numbers. Therefore, we have the same values for λ_i; $(i = 1,2)$. When the result is correct for the smallest λ in equation (5) then it is trivial to proof that the result is true for larger λ without losing any generality on chosing λ = min (λ_i); $(i = 1,2,\dots,n)$. This will help in improving Theorem 2. Now, let *n*=3, then the matrix in equation (4) can be written in the following form

$$R_t = \begin{bmatrix} t+\lambda & \gamma_{12} & \gamma_{13} \\ 0 & t+\lambda & \gamma_{23} \\ 0 & 0 & t+\lambda \end{bmatrix} \qquad (6)$$

Evaluating first

second and third order sums of the principal minors we will have

$$\sum_{c_3^1} |R_t(1)| = \mathrm{C}_3^1 \sum (t+\lambda) = 3t + 3\lambda = 3(t+\lambda) \qquad (6.1)$$

$$\sum_{c_3^2} |R_t(1,2)| = \mathrm{C}_3^2 \sum (t+\lambda)^2 = 3(t^2 + 2\lambda t + \lambda^2) \qquad (6.2)$$

and

$$\sum_{c_3^3} |R_t(1,2,3)| = \mathrm{C}_3^3 \sum (t+\lambda)^3 = t^3 + 3t^2\lambda + 3t\lambda^2 + t^3. \qquad (6.3)$$

Then for $t \geq 3$, using equations (6.1), (6.2) and (6.3) we obtain

$$\sum_{c_3^1} |R_t(1)| < \sum_{c_3^2} |R_t(1,2| < \sum_{c_3^3} |R_t(1,2,3|.$$

Now, assuming that the inequality in equation (3) holds for *n*-1, i.e,

$$\sum_{c_{n-1}^0} |R_t(0)| < \sum_{c_{n-1}^1} |R_t(1)| < \sum_{c_{n-1}^2} |R_t(1,2)| < \cdots < \sum_{c_{n-1}^{n-1}} |R_t(1,2,\dots,n-1)|$$

It follows that

$$C_{n-1}^0 \sum (t+\lambda)^0 < C_{n-1}^1 \sum (t+\lambda)^1 < C_{n-1}^2 \sum (t+\lambda)^2 < \cdots < C_{n-1}^{n-1} \sum (t+\lambda)^{n-1}.$$

replacing *n*-1 by *n* in the last expression we get

$$C_n^1 \sum (t+\lambda)^1 < C_n^2 \sum (t+\lambda)^2 < C_n^3 \sum (t+\lambda)^3 < \dots < C_n^n \sum (t+\lambda)^n.$$

it follows easily that

$$\sum_{c_n^1} |R_t(1)| < \sum_{c_n^2} |R_t(1,2)| < \cdots < \sum_{c_n^n} |R_t(1,2,\dots,n)|.$$

Since A_t is similar to R_t, then

$$\sum_{c_n^1} |A_t(1)| < \sum_{c_n^2} |A_t(1,2)| < \cdots < \sum_{c_n^n} |A_t(1,2,\dots,n)|.$$

This completes the proof.

2. Result and Discussion

In this study, some inequalities on M-matrices are examined by using the properties of M- matrices and benefiting from principle minors by taking advantage of studies on inequality of M- matrices by Ando (1980), Chun-Wei (1988), Furuichi and Lin (2010). As a result an inequality on the sum of the principal minors of M-matrices was proved.

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