Sakarya University Journal of Science, 22 (6), 1868-1873, 2018.

SAKARYA UNIVERSITY
JOURNAL OF SCIENCE

e-ISSN: 2147-835X
http://www.saujs.sakarya.edu.tr

Received
21-05-2018
Accepted
24-09-2018

Doi
10.16984/saufenbilder.425519

Curves According to the Successor Frame in Euclidean 3-Space

Melek Masal *

Abstract

In the present study, the successor formulae of the successor curves defined by Menninger [1] are given. Then, by defining the successor planes, the geometric meanings of the successor curvatures are investigated and the relations across the components of the position vectors of successor curves are found. Furthermore, in this study, it is proven that lies in the 3rd.type successor plane, lies in the 1st type successor plane and by defining the involute-evolute S-pair, the distance between the corresponding points of these curves is found.

Keywords: Successor frame; Successor curves; Slant helix, Involute-evolute curves.

1. INTRODUCTION

The geometry of the curves may be surrounded by the topics on general helices, involute-evolute curves, Mannheim curves and Bertrand curves (see [2-10]). Such special curves are investigated and used to solve some real-world problems; such as problems of mechanical design or robotics by the help of well-known Frenet-Serret equations since the curves can be thought as the path of a moving particle in the Euclidean Space.After that, some researchers in the field aimed to determine another moving frame for a regular curve [11,12,13]. Menninger, for example, pioneered “Successor frame” using parallel vector fields [1].

In the original part of this study, the successor formulae of the successor curves in 3-dimensional Euclidean space E^3 are provided, and the successor curvatures of the successor curves in a geometrical treatment are described by specifying the i^{th} successor plane. Afterwards, by referring to the position vector of a successor curve as $\alpha = v_1T_1 + v_2N_1 + v_3B_1$, the relations between the components v_i. are obtained. In the fourth section, we define helix concerning the successor system and prove that T_1-helix and B_1-helix, respectively, lie in the 3rd type successor plane and the 1st type successor plane. We also see that there is no successor curve as N_1-helix in E^3. In the fifth section, we define the involute-evolute S-pair, and then, we find the distance between the corresponding points of these curves.

2. SUCCESSOR TRANSFORMATION OF FRENET APPARATUS

The Euclidean 3-space provided with the standard flat metric is given by

$$\langle , \rangle = dx_1^{\ 2} + dx_2^{\ 2} + dx_3^{\ 2}$$

where $\{x_1, x_2, x_3\}$ is a rectangular coordinate system of E^3. Recall that the norm of an arbitrary vector X is given by $\|X\| = \sqrt{\langle X, X \rangle}$. Let $\beta : I \subset \mathbb{R} \to E^3$ be an arbitrary curve in the Euclidean space E^3. The curve β is stated to be a unit speed if the inner product

* Corresponding Author: Sakarya University, Faculty of Education, Mathematics and Science Education Deparment, 54300 Hendek, Sakarya, Turkiye, mmasal@sakarya.edu.tr

$\left\langle \frac{d\beta}{ds}, \frac{d\beta}{ds} \right\rangle = 1$ is satisfied. Throughout this paper, we will assume that all curves are unit speed curves. For any arbitrary unit speed curve, the Frenet-Serret Formulae are given by

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N. \end{aligned} \tag{1}$$

Here T, N, B are completely determined by the curvature κ and torsion τ, as a function of parameter s, [4].

Definition 2.1

Let T be the unit tangent vector of the curve $\beta : \beta(s)$. A curve $\alpha : \alpha(s)$ that has T as the principal normal is called the successor curve of the curve β, and the frame $\{T_1, N_1, B_1\}$ is called the successor frame of the Frenet frame $\{T, N, B\}$ if $N_1 \equiv T$, [1].

Theorem 2.1

Every Frenet curve has a family of successor curves. Given a Frenet system $F = \{T, N, B, \kappa, \tau\}$, the totality of successor systems $F_1 = \{T_1, N_1, B_1, \kappa_1, \tau_1\}$ is as follows:

$$\begin{aligned} T_1 &= -\cos\varphi N + \sin\varphi B, \\ N_1 &= T, \\ B_1 &= \sin\varphi N + \cos\varphi B, \end{aligned} \tag{2}$$

$$\kappa_1 = \kappa\cos\varphi, \; \tau_1 = \kappa\sin\varphi, \; \varphi(s) = \varphi_0 + \int \tau(s) ds \tag{3}$$

Depending on a parameter, φ_0 is a constant real number. The Darboux vector of the successor frame is $D_1 = \kappa B$, [1].

Remark 2.1

The inverse of the successor transformation may be denoted as predecessor transformation. Bilinski described it for the case, but it is not well-defined in general, [1].

3. SUCCESSOR CURVES

In this section, initially the successor formulae of the successor curves in 3-dimensional Euclidean space are given, and the successor curvatures of the successor curves are interpreted geometrically by describing the successor plane. Afterwards, by referring to the position vector of the successor curve as, the relations among the components are obtained.

Theorem 3.1

If α is the successor curve of the Frenet curve β is given with the Frenet system $\{T, N, B, \kappa, \tau\}$, and if $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ is the successor system of α in E^3, then there exists the following formulae:

$$\begin{aligned} T_1' &= \kappa_1 N_1, \\ N_1' &= -\kappa_1 T_1 + \tau_1 B_1, \\ B_1' &= -\tau_1 N_1. \end{aligned}$$

Proof

Let α be the successor curve of the curve β given with the Frenet system $\{T, N, B, \kappa, \tau\}$ and let $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ be the successor system of α in E^3. Then, if we differentiate each side of the Equation (2) with respect to s, the following is found:

$$\begin{aligned} T_1' &= \varphi' \sin\varphi N - \cos\varphi N' + \varphi' \cos\varphi B + \sin\varphi B', \\ N_1' &= T', \\ B_1' &= \varphi' \cos\varphi N + \sin\varphi N' - \varphi' \sin\varphi B + \cos\varphi B'. \end{aligned}$$

If the Frenet-Serret formulae, Equations (2), (3) and the Remark 2.1 are substituted in these last equations, then following equations are obtained:

$$\begin{aligned} T_1' &= \kappa_1 N_1, \\ N_1' &= -\kappa_1 T_1 + \tau_1 B_1, \\ B_1' &= -\tau_1 N_1. \end{aligned}$$

After that, the formulae, defined in Theorem 3.1 , will be called as Successor Formulae.

Definition 3.1

Let α be the successor curve of the curve β given with the Frenet system $\{T, N, B, \kappa, \tau\}$ and let $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ be the successor system of α in E^3. The subspace $Sp\{T_1, N_1\}$ is called the 1st type successor plane of α, the subspace $Sp\{T_1, B_1\}$ is called the 2nd type successor plane of α , and the subspace $Sp\{N_1, B_1\}$ is called the 3rd type successor plane of α.

Theorem 3.2

The Let α be the successor curve of the Frenet curve β in E^3.

i) If α is a successor curve, then the successor approximation of the successor curve α can be obtained as;

$$\hat{\alpha}(s)=\alpha(0)+s(-\lambda_0\kappa_{10})T_{10}+s(1+\lambda_0')N_{10}+s(\lambda_0\tau_{10})B_{10}.$$

ii) If $\kappa_1=0$, then the successor curve α lies in the 3rd type successor plane.
iii) If $\tau_1=0$, then the successor curve α lies in the 1st type successor plane.

Proof

Let α be the successor curve of the curve β given with the Frenet system $\{T,N,B,\kappa,\tau\}$ and let $\{T_1,N_1,B_1,\kappa_1,\tau_1\}$ be the successor system of α in E^3. From the Definition 2.1, we can write, (Figure 3.1)

$$\alpha(s)=\beta(s)+\lambda(s)T(s)$$

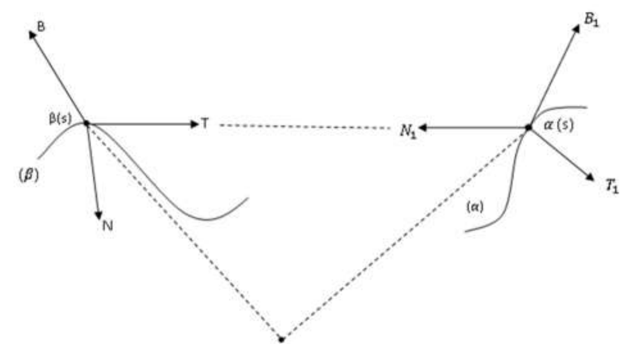

Figure 3.1

For Taylor expansion of the successor curve α in a neighborhood of s_0, we can write

$$\alpha(s)\approx\alpha(0)+s\alpha'(0)+\frac{s^2}{2}\alpha''(0)+\frac{s^3}{6}\alpha'''(0)+\ldots \quad (5)$$

where $s_0=0$. If successor system is called by $\{T_{10},N_{10},B_{10},\kappa_{10},\tau_{10}\}$ at the point $\alpha(0)$ and the Frenet system is called by $\{T_0,N_0,B_0,\kappa_0,\tau_0\}$ at the point $\beta(0)$, and if we differentiate each side of the Equation 4 with respect to s, following equations are obtained.

$$\alpha'(0)=(-\lambda_0\kappa_{10})T_{10}+(1+\lambda_0')N_{10}+(\lambda_0\tau_{10})B_{10},$$

$$\alpha''(0)=\left(\left(-2\lambda_0'-1\right)\kappa_{10}-\lambda_0\kappa_{10}'\right)T_{10}+$$
$$\left(\lambda_0''-\lambda_0\left(\kappa_{10}^2+\tau_{10}^2\right)\right)N_{10}+\left(\left(2\lambda_0'+1\right)\tau_{10}+\lambda_0\tau_{10}'\right)B_{10},$$

$$\alpha'''(0)=\left(\kappa_{10}\left(-3\lambda''+\lambda\left(\kappa_{10}^2+\tau_{10}^2\right)\right)-\left(3\lambda_0'+1\right)\kappa_{10}'-\lambda_0\kappa_{10}''\right)T_{10}$$
$$+\left(\left(-3\lambda_0'-1\right)\left(\kappa_{10}^2+\tau_{10}^2\right)+\lambda_0'''-3\lambda_0\left(\kappa_{10}\kappa_{10}'+\tau_{10}\tau_{10}'\right)\right)N_{10}$$
$$+\left(3\lambda_0'''\tau_{10}+\left(3\lambda_0'+1\right)\tau_{10}'+\lambda_0\left(\tau_{10}\left(\kappa_{10}^2+\tau_{10}^2\right)+\tau_{10}''\right)\right)B_{10}.$$

If the above equations are put in the Equation (5), the following equation is found:

$$\alpha(s)\approx\alpha(0)$$
$$+(s\left(-\lambda_0\kappa_{10}\right)+\frac{s^2}{2}\left(\left(-2\lambda_0'-1\right)\kappa_{10}-\lambda_0\kappa_{10}'\right)+$$
$$\frac{s^3}{6}\left(\kappa_{10}\left(-3\lambda_0''+\lambda_0\left(\kappa_{10}^2+\tau_{10}^2\right)\right)-\left(3\lambda_0'+1\right)\kappa_{10}'-\lambda_0\kappa_{10}''\right)+\ldots)T_{10}$$
$$+(s\left(1+\lambda_0'\right)+\frac{s^2}{2}\left(\lambda_0''-\lambda_0\left(\kappa_{10}^2+\tau_{10}^2\right)\right)+$$
$$\frac{s^3}{6}\left(\left(-3\lambda_0'-1\right)\left(\kappa_{10}^2+\tau_{10}^2\right)+\lambda_0'''-3\lambda_0\left(\kappa_{10}\kappa_{10}'+\tau_{10}\tau_{10}'\right)\right)+\ldots)N_{10}$$
$$+(s\left(\lambda_0\tau_{10}\right)+\frac{s^2}{2}\left(\left(2\lambda_0'+1\right)\tau_{10}+\lambda_0\tau_{10}'\right)+$$
$$\frac{s^3}{6}\left(3\lambda_0'''\tau_{10}+\left(3\lambda_0'+1\right)\tau_{10}'+\lambda_0\left(-\tau_{10}\left(\kappa_{10}^2+\tau_{10}^2\right)+\tau_{10}''\right)\right)+\ldots)B_{10}$$

If s^2, s^3, s^4,.... are omitted here, and the obtained piece is denoted by $\hat{\alpha}$, this is found

$$\hat{\alpha}(s)=\alpha(0)+s(-\lambda_0\kappa_{10})T_{10}+s(1+\lambda_0')N_{10}+s(\lambda_0\tau_{10})B_{10} \quad (6)$$

This equation will be called as the successor approximation of the successor curve α.

As a result of Equation (6), if $\kappa_{10}=0$, the curve lies in a plane spanning by $\{N_{10},B_{10}\}$ and also if $\tau_{10}=0$, the curve lies in a plane spanning by $\{T_{10},N_{10}\}$. The geometric mean of κ_1 measures measures to an extent which the successor curve departs from a 3rd type successor plane whereas τ_1 measures to an extent which the successor curve departs from a 1st type successor plane.

Theorem 3.3

If $\alpha=v_1T_1+v_2N_1+v_3B_1$ is the position vector of the successor curve α, then the coefficients $v_i=v_i(s)$ *and* $i=1,2,3$ satisfy the following relations:

$$v_1'=\kappa_1\left(v_2-\lambda\right),$$
$$v_2'=1+\lambda'-v_1\kappa_1+v_3\tau_1,$$
$$v_3'=-\tau_1\left(v_2-\lambda\right).$$

Where, the distance of the successor curve α to the Frenet curve β is λ.

Proof

Let $\alpha=v_1T_1+v_2N_1+v_3B_1$ *and* $v_i=v_i(s)$ be the position vectors of the successor curve α. If we take the derivative of the position vector of the successor curve in view of the Theorem 3.1, the following can be obtained:

$$\alpha'=\left(v_1'-v_2\kappa_1\right)T_1+\left(v_1\kappa_1+v_2'-v_3\tau_1\right)N_1+\left(v_2\tau_1+v_3'\right)B_1 \quad (7)$$

Furthermore, if equation 4 is differentiated, then the

$$\alpha' = \left(-\lambda\kappa_1\right)T_1 + \left(1+\lambda'\right)N_1 + \left(\lambda\tau_1\right)B_1 \qquad (8)$$

equation where the distance of the successor curve α to the Frenet curve β is λ is obtained.

Thus, the Equations (7) and (8) give us

$$v_1' = \kappa_1\left(v_2 - \lambda\right),$$

$$v_2' = 1 + \lambda' - v_1\kappa_1 + v_3\tau_1,$$

$$v_3' = -\tau_1\left(v_2 - \lambda\right).$$

So, from the Definition 3.1, Theorem 3.2 and Theorem 3.3, we can reach the following result:

Corollary 3.1

Let $\alpha = v_1T_1 + v_2N_1 + v_3B_1$ be the position vector of the successor curve α , and the distance of the successor curve α to the Frenet curve β is λ.

i) If α is in the 1st type successor plane, then we get the following equation:

$$v_1' + v_2' = \kappa_1\left(v_2 - v_1\right) + \lambda' - \lambda\kappa_1 + 1$$

ii) If α is in the 2nd type successor plane, then we get the following equation:

$$\lambda'' = \lambda\left(\tau_1^2 - \kappa_1^2\right) + v_1\kappa_1' + v_3\tau_1'$$

iii) If α is in the 3rd type successor plane, then we get the following equation:

$$v_2' + v_3' = \tau_1\left(v_3 - v_2\right) + \lambda' + \lambda\tau_1 + 1$$

4. HELIX ACCORDING TO SUCCESSOR SYSTEM

In this section, the helix concerning the successor system is defined, and furthermore, T_1-helix and B_1-helix, respectively, lie in the 3rd type successor plane, and the 1st type successor plane is proven. It can also be observed that there is no successor curve as N_1-helix in E^3.

Definition 4.1

Let $\{T_1(s), N_1(s), B_1(s)\}$ be the successor system of a successor curve α. If T_1 at any point of the successor curve α makes a constant angle with a fixed line, then α is called T_1-helix, and if N_1 at any point of the α makes a constant angle with a fixed line, then the α is called N_1-helix, and if B_1 at any point of α makes a constant angle with a fixed line, then α is called B_1-helix.

Theorem 4.1

If the successor curve α is a T_1-helix, then $\frac{\kappa_1}{\tau_1}$=constant

Proof

Assume that the successor curve α is a T_1-helix. In this case, the following can be written by taking the definition into consideration:

$$\langle T_1, U\rangle = \cos\theta = \text{constant} \neq 0 \qquad (9)$$

where U is aconstant vector. If we differentiate the Equation (9) and consider Theorem 3.1, we get;

$$\langle \kappa_1 N_1, U\rangle = 0.$$

This shows $U = S_P\{T_1, B_1\}$. Therefore, the following can be written:

$$U = u_1T_1 + u_2B_1, \quad u_i = \text{constant}. \qquad (10)$$

Taking derivative of the Equation (10), the Successor formulae give

$$u_1\kappa_1 - u_2\tau_1 = 0$$

Then, it is seen that

$$\frac{\kappa_1}{\tau_1} = \text{constant}$$

Theorem 4.2

There is no successor curve as N_1-helix in E^3.

Proof

Let the successor curve α be the N_1-helix, assuming that there is a constant vector V which satisfies

$$\langle N_1, V\rangle = \cos\gamma = \text{constant} \qquad (11)$$

Differentiating the Equation (11) and considering Theorem 3.1, the following is seen:

$$\left\langle -\kappa_1T_1 + \tau_1B_1, V\right\rangle = 0,$$

which means that $V = S_P\{N_1\}$. Then, it is seen that;

$$V = vN_1 \ , \ v = \text{constant} \qquad (12)$$

If we differentiate the Equation (12), we get $\kappa_1 = 0$ and $\tau_1 = 0$. The curve α cannot be a N_1-helix, due to the fact that κ_1 and τ_1 cannot vanish at the same time for any successor curve α .

Theorem 4.3

If the successor curve α is B_1-helix, the $\frac{\kappa_1}{\tau_1}$ = constant

Proof

Suppose that the successor curve α is -helix. Then, there is a constant vector W such as;

$$\langle B_1, W\rangle = \cos\psi = \text{constant} \quad (13)$$

From the equation presented above,

$$\langle \tau_1 N_1, W\rangle = 0 \quad (14)$$

is found. So, we can call

$$W = \omega_1 T_1 + \omega_2 B_1,\ \omega_i = \text{constant}.$$

If we differentiate the Equation (14), we get

$$\omega_1\kappa_1 - \omega_2\tau_1 = 0$$

and

$$\frac{\kappa_1}{\tau_1} = \text{constant}$$

5. INVOLUTE-EVOLUTE CURVES ACCORDING TO THE SUCCESSOR SYSTEM

In this section the involute-evolute S-pair is defined, and the distance between the corresponding points of these curves are found.

Definition 5.1

Let $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ be the successor system of the successor curve α and let $\{T_1^*, N_1^*, B_1^*, \kappa_1^*, \tau_1^*\}$ be the successor system of the successor curve α^*. If $\langle N_1, N_1^*\rangle = 0$, then the curve pair (α, α^*) is called the involute-evolute successor pair or shortly involute-evolute S-pair according to the successor system.

Theorem 5.1

Let α be the successor curve of the Frenet curve β and α^* be the successor curve of the Frenet curve β^*. If (α, α^*) is the involute-evolute S-pair, then

$$\alpha^*(s) = \alpha(s) + (\lambda + s + c)N_1(s)$$

and

$$\lambda^* + s^* = c_1$$

Where c and c_1 are constants of integration.

Proof

Let α be the successor curve of the Frenet curve β and α^* be the successor curve of the Frenet curve β^*. If $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ are Frenet systems of the Frenet curves β and β^* respectively, and $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ and $\{T_1^*, N_1^*, B_1^*, \kappa_1^*, \tau_1^*\}$ are successor systems of the successor curves α and α^* respectively, then from the Definition 2.1, we can write;

$$\alpha(s) = \beta(s) + \lambda(s)T(s)$$
$$\alpha^*(s^*) = \beta^*(s^*) + \lambda^*(s^*)T^*(s^*) \quad (15)$$

If (α, α^*) is the involute-evolute S-pair, then from the Definition 5.1, the following can be written:

$$\alpha^*(s) = \alpha(s) + \lambda(s)N_1(s) \quad (16)$$

From Equations (15) and (16), we have;

$$\beta^*(s) + \lambda^*(s)T^*(s) = \beta(s) + \lambda(s)T(s) + a(s)N_1(s) \quad (17)$$

If we differentiate each side of the Equation (17) with respect to s and consider Theorem 3.1, the following is obtained:

$$\left\{\left(-\lambda^*\kappa_1^*\right)T_1^* + \left(1+\lambda^{*\prime}\right)N_1^* + \left(\lambda^*\tau_1^*\right)B_1^*\right\}\frac{ds^*}{ds} =$$
$$\left(-\kappa_1(\lambda + a)\right)T_1 + \left(1 + \lambda' + a'\right)N_1 + \left(\tau_1(\lambda + a)\right)B_1$$

From the Definition 5.1, we get

$$1 + \lambda' + a' = 0,\ 1 + \lambda^{*\prime} = 0$$

Thus,

$$\lambda^* = c_1 - s^*,\ a = \lambda + s + c \quad (18)$$

Where c and c_1 are constants of integration.

If the Equation (18) is replaced by the Equation (16), then

$$\alpha^*(s) = \alpha(s) + (\lambda + s + c)N_1(s)$$

is obtained.

6. CONCLUSION

In this paper, thegeometric meanings of the successor curvatures, the involute-evolute successor curves, the successor helices and some properties of these special curves have been introduced. This frame is new for the differential geometricians; thus, we expect that it will broaden the horizon of the geometricians in the field. We also hope that this new frame will attract geometricians as the other special frames do (e.g. Sabban frame, Bishop frame, Darboux frame and so forth).

REFERENCES

[1] Menninger, A. (2014) Characterization of the slant helix as successor curves of the general helix. *International Electronic Journal of Geometry*, 7(2):84-91.

[2] Ali, A.T. (2011) Position vectors of general helices in Euclidean 3-space. Bull. Math. Analy. Appl. 3 (2): 198-205.

[3] Bertrand, J. (1850) La theories de courbes a double courbure. J. Math. Pures et Appl. 15: 332-350.

[4] Do Carmo, M.P. (1976) Differential geometry of curves and surfaces. Prentice Hall, Englewood Cliffs, New Jersey.

[5] Fuchs, D. (2013) Evolutes and involutes of spatial curves. Amer. Math. Monthly, 120(3):217-231.

[6] Izumiya, S. & Takeuchi, N. (2002) Generic properties of helices and Bertand curves. J. Geom., 71(1): 97-109.

[7] Liu, H. & Wang F., (2008) Mannheim partner curves in 3-space. J. Geom., 88: 120-126.

[8] Lucas, P. & Ortega-Yagues, J.A.,(2012) Bertrand curves in the three-dimensional sphere. J. Geom. Phys., 62(9): 1903-1914.

[9] Orbay, K. & Kasap, E. (2009) On Mannheim partner curves in . Int. J. Phys. Sci., 4(5): 261-264.

[10] Struik, D.J. (1988) Lectures on classical differential geometry. Dover, New-York.

[11] Bektaş, Ö. & Yüce, S. (2013) Special involute-evolute partner D-curves in. European Journal of Pure and Applied Mathematics, 6(1):20-29.

[12] Bükcü, B. & Karacan, M.K. (2009) The slant helices according to Bishop frame,.World Academy of Science, Engineering and Technology, 59:1039-1042.

[13] Yılmaz, S. & Turgut, M. (2010) A new version of Bishop frame and application to spherical images. J. Math. Anal. Appl., 371: 764-776.