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On Generalized and Extended Generalized ϕ-recurrent Sasakian Finsler Structures

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Abstract: In this research, generalized and extended generalized ϕ-recurrent Sasakian Finsler structures on horizontal and vertical tangent bundles and their various geometric properties are studied.

Genelleştirilmiş ve Genişletilmiş Genelleştirilmiş ϕ-tekrarlı Sasakian Finsler Yapılar Üzerine

Anahtar Kelimeler
Genelleştirilmiş ϕ-tekrarlı,
Genişletilmiş genelleştirilmiş ϕ-tekrarlı,
Sasakian Finsler yapı,
Einstein manifoldu

Özet: Bu araştırmada, yatay ve dikey tanjant demetleri üzerinde genelleştirilmiş ve genişletilmiş genelleştirilmiş ϕ-tekrarlı Sasakian Finsler yapılar ve bunların çeşitli geometrik özellikleri çalışıldı.

1. Introduction

Ruse defined a Riemannian space of the recurrent curvature for which the covariant derivation of the Riemannian curvature tensor R satisfies the relation:

$$(\nabla_s R)(p,q)r = A(s)R(p,q)r \tag{1}$$

at all points for the non-zero 1-form A, in 1949 [13]. In this relation, if A vanishes so the space is reduced to a locally symmetric manifold. Besides, generalized recurrent manifolds take part in the literature with Dubey's study in 1979 [8]. Dubey weakened the recurrence condition that defined in (1) in the following way:

$$(\nabla_s R)(p,q)r = A(s)R(p,q)r + B(s)g(p,q)r \tag{2}$$

for all vector fields p,q,r,s and non-zero 1-forms A and B satisfying:

$$A(s) = g(s,p_1), B(s) = g(s,p_2) \tag{3}$$

where p_1, p_2 are vector fields associated with A and B, respectively and the Riemannian metric tensor g is defined as follows:

$$g(p,q)r = g(q,r)p - g(p,r)q, \tag{4}$$

The Riemannian space satisfying (2) (so, (3) and (4)) is called generalized (Riemann) recurrent manifold. Additionally, generalized Ricci recurrent and generalized concircular recurrent manifolds are defined with the following relations, respectively:

$$(\nabla_s S)(p,q)r = A(s)S(p,q)r + B(s)g(p,q)r, \tag{5}$$

$$(\nabla_s C)(p,q)r = A(s)C(p,q)r + B(s)g(p,q)r \tag{6}$$

for all vector fields p,q,r,s where S is Ricci curvature tensor and C is concircular curvature tensor.
The Sasakian manifold satisfying

$$\phi^2((\nabla_s R)(p,q)r) = 0 \tag{7}$$

is introduced as locally ϕ-symmetric manifold by Takahashi, in 1977 [17]. In addition, generalized ϕ-recurrency is one type of the weakened extensions of locally ϕ-symmetry. ϕ-recurrency of the spaces introduced by De, Shaikh and Biswas for Sasakian manifolds in 2003 [6] in which ϕ-recurrent Sasakian manifold satisfies the following relation:

$$\phi^2((\nabla_s R)(p,q)r) = A(s)(R(p,q)r) \tag{8}$$

for all vector fields p,q,r,s and if $A = 0$ it turns to a locally ϕ-symmetric manifold. Then generalized ϕ-recurrency of Kenmotsu manifolds are studied by Basari and Murathan [2]. Furthermore, generalized ϕ-recurrent spaces, like Sasakian [1], P-Sasakian [15], LP-Sasakian [14], Kenmotsu [4] and trans-Sasakian [7], are discussed in many studies. In [1], the generalized ϕ-recurrent Sasakian manifold is defined with the following relation:

$$\phi^2((\nabla_s R)(p,q)r) = A(s)R(p,q)r + B(s)g(p,q)r \tag{9}$$

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for all vector fields p,q,r,s. By taking $R=C$ and $R=P$ in (9) the Sasakian manifold said to be generalized $C-\phi$-recurrent and generalized $P-\phi$-recurrent where C,P are concircular and projective curvature tensors, respectively. Moreover, extended generalized ϕ-recurrency is one type of the extensions of ϕ-recurrency and discussed by Prakasha [11] and Jaiswal and Yadav [9] for Sasakian and trans-Sasakian manifolds. In [11], extended generalized ϕ-recurrent Sasakian manifold is defined in the following way:

$$\begin{aligned}((\nabla_s R)(p,q)r) &= A(s)\phi^2(R(p,q)r)\\ &+B(s)\phi^2(g(p,q)r)\end{aligned} \tag{10}$$

for all vector fields p,q,r,s. Particularly, substituting R by C and P respectively, the Sasakian manifold is called extended generalized $C-\phi$- recurrent and extended generalized $P-\phi$-recurrent respectively.

These studies motivated us to discuss generalized ϕ-recurrent and extended generalized ϕ-recurrent Sasakian Finsler structures.

2. Preliminaries

Assume that $M^{m=(2n+1)}$, $F^m=(M,F)$ and g be be a smooth manifold, a Finsler manifold and a Finsler metric tensor with $g_{ij}(x,y)=\frac{1}{2}[\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ coefficients respectively. Besides, $x=(x^1,\ldots,x^m)$ are the local coordinates of M, T_xM is an m-dimensional tangent space at $x\in M$ and $y=y^i\frac{\partial}{\partial x^i}\in T_xM$. So, TM denotes $2m$-dimensional slit tangent bundle of M and $u=(x,y)\in TM$ [10].

Furthermore, T_uTM is the tangent space to TM at u and $\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\}$ are the canonical frames for T_uTM. The differential map $\pi_*: T_uTM \to T_{\pi(u)}M$ satisfy $X_u\in\pi(X_u)=X_u$. Hence, the vertical subbundle TMV is derived from $ker(\pi)$. The horizontal subbundle $TMH=(N_i^j(x,y))$ is a non-linear connection on TM where $N_i^j=\frac{\partial N^j}{\partial y^i}$ are obtained via spray coefficients $N^j=\frac{1}{4}g^{jk}(\frac{\partial^2F^2}{\partial y^k\partial x^h}y^h-\frac{\partial F^2}{\partial x^k})$.

It enables to define $p\in T_uTM$ with these coefficients in the following way: $p=p^i(\frac{\partial}{\partial x^i}-N_i^j(x,y)\frac{\partial}{\partial y^j})+(N_i^j(x,y)p^i+p^j)\frac{\partial}{\partial y^j}=p^i\frac{\delta}{\delta x^i}+p^j\frac{\partial}{\partial y^j}$. Here, $(dx^i,\delta y^j)$ are the dual frames of $(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})$ where $\delta y^j=dy^j+N_i^j dx^i$. In this manner, $T_uTM=T_u^HTM\oplus T_u^VTM$ at $u\in TM$ gives rise to complementary distributions $TMH=\bigcup_{u\in TM_H}T_u^HTM$ and $TMV=\bigcup_{u\in TM_V}T_u^VTM$ [5].

Furthermore, distributing $\eta=\eta_i dx^i+\eta_j\delta y^j\in(T_uTM)^*$ to horizontal and vertical parts, we have $\eta^H\in(T_u^HTM)^*$ and $\eta^V\in(T_u^VTM)^*$.

The Sasaki-Finsler metric G on TM is defined as follows:

$$G(p,q)=G^H(p^H,q^H)+G^V(p^V,q^V)$$

So, some warped contact structures with Finsler coefficients can be constructed like in [12] and [16].

Definition 2.1. Suppose that $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ be Sasakian Finsler structures on TM_H and TM_V, respectively. Then we have the below relations:

$$\phi.\phi=-I_n+\eta^H\otimes\xi^H+\eta^V\otimes\xi^V, \tag{11}$$

$$\phi\xi^H=\phi\xi^V=0, \tag{12}$$

$$\eta^H(\xi^H)=\eta^V(\xi^V)=1, \tag{13}$$

$$\eta^H(\phi p^H)=0,\eta^V(\phi p^V)=0,\eta^H(\phi p^V)=0, \tag{14}$$

$$\begin{aligned}G^H(\phi p^H,\phi q^H)&=G^H(p^H,q^H)-\eta^H(p^H)\eta^H(q^H),\\ G^V(\phi p^V,\phi q^V)&=G^V(p^V,q^V)-\eta^V(p^V)\eta^V(q^V),\end{aligned} \tag{15}$$

$$G^H(p^H,\xi^H)=\eta^H(p^H),G^V(p^V,\xi^V)=\eta^V(p^V) \tag{16}$$

where $p^H,q^H,r^H\in T_u^HTM$ and $p^V,q^V,r^V\in T_u^VTM$, ξ is the Reeb vector field, η is the 1-form, G is the Finsler metric structure and the $(1,1)$ tensor field ϕ denotes the endomorphism [18].

In the Sasakian Finsler manifolds TM_H and TM_V following relations hold:

$$\begin{aligned}G^H(\phi p^H,q^H)&=-G^H(p^H,\phi q^H)\\ G^V(\phi p^V,q^V)&=-G^V(p^V,\phi q^V)\end{aligned} \tag{17}$$

$$\nabla_p^H\xi^H=-\frac{1}{2}\phi p^H,\nabla_p^V\xi^V=-\frac{1}{2}\phi p^V \tag{18}$$

$$\begin{aligned}R(p^H,q^H)\xi^H&=\frac{1}{4}[\eta^H(q^H)p^H-\eta^H(p^H)q^H],\\ R(p^V,q^V)\xi^V&=\frac{1}{4}[\eta^V(q^V)p^V-\eta^V(p^V)q^V]\end{aligned} \tag{19}$$

$$\begin{aligned}R(p^H,\xi^H)q^H&=\frac{1}{4}[\eta^H(Y^H)p^H-G^H(p,q)\xi^H],\\ R(p^V,\xi^V)q^V&=\frac{1}{4}[\eta^V(Y^V)p^V-G^V(p,q)\xi^V]\end{aligned} \tag{20}$$

$$\begin{aligned}R(p^H,q^H)r^H&=\frac{1}{4}[G(r^H,q^H)p^H-G(r^H,p^H)q^H],\\ R(p^V,q^V)r^V&=\frac{1}{4}[G(r^V,q^V)p^V-G(r^V,p^V)q^V]\end{aligned} \tag{21}$$

$$S(p^H,\xi^H)+S(p^H,\xi^H)=\frac{n}{2}(\eta^H(p^H)+\eta^V(p^V)) \tag{22}$$

$$\begin{aligned}S(\phi p,\phi q)&=S(p^H,q^H)-\frac{n}{2}\eta^H(p^H)\eta^H(q^H),\\ S(\phi p,\phi q)&=S(p^V,q^V)-\frac{n}{2}\eta^V(p^V)\eta^V(q^V)\end{aligned} \tag{23}$$

for all vector fields and where R is the Riemann curvature tensor field, S is the Ricci tensor field and ∇ is the Finsler connection on TM [18].

However Sasakian Finsler structures can be constructed both on horizontal and vertical tangent subbundles, in this paper; it is studied for TM_H, for briefness.

3. Generalized ϕ-recurrent Sasakian Finsler structures on TM_H

Definition 3.1. The Sasakian Finsler structure $(\phi^H, \xi^H, \eta^H, G^H)$ on TM_H is called generalized ϕ-recurrent if the following relation holds:

$$\phi^2((\nabla^H_s R)(p^H, q^H)r^H) = A^H(s^H)(R(p^H, q^H)r^H) + B^H(s^H)(G(p^H, q^H)r^H) \quad (24)$$

for $p^H, q^H, r^H, s^H \in T^H_u TM$ where A^H and B^H are the non-zero 1-forms defined by

$$A^H(s^H) = G^H(s^H, p^H_1), B^H(s^H) = G^H(s^H, p^H_2) \quad (25)$$

and p^H_1, p^H_2 are vector fields associated with 1-forms A^H and B^H, respectively and G is defined as follows:

$$G(p^H, q^H)r^H = \frac{1}{4}[G^H(q^H, r^H)p^H - G^H(p^H, q^H)r^H]. \quad (26)$$

Lemma 3.2. *In a generalized ϕ-recurrent Sasakian Finsler manifold TM_H, the relation $A^H + B^H = 0$ is satisfied.*

Theorem 3.3. *A generalized ϕ-recurrent Sasakian Finsler manifold TM_H with the quadruple $(\phi^H, \xi^H, \eta^H, G^H)$ is of constant curvature $\frac{1}{4}$.*

Proof. Due to the manifold is generalized ϕ-recurrent then (24) is satisfied. Applying ϕ both sides of (24), and replacing r^H with ξ^H, by using (11) we have

$$-((\nabla^H_s R)(p^H, q^H)\xi^H) + \eta^H((\nabla^H_s R)(p^H, q^H)\xi^H)\xi^H = A^H(s^H)[-R(p^H, q^H)\xi^H + \eta^H(R(p^H, q^H)\xi^H)\xi^H] + B^H(s^H)[G(p^H, q^H)r^H - \eta^H(G(p^H, q^H)\xi^H)\xi^H]. \quad (27)$$

By the use of (19) and (26) the following relation holds:

$$(\nabla^H_s R)(p^H, q^H)\xi^H = \eta^H((\nabla^H_s R)(p^H, q^H)\xi^H)\xi^H + \frac{1}{4}[\eta^H(q^H)p^H - \eta^H(p^H)q^H][A^H(s^H) + B^H(s^H)]. \quad (28)$$

With the help of (17) and (18), we get

$$(\nabla^H_s R)(p^H, q^H)\xi^H = \frac{1}{8}[G^H(s^H, \phi q^H)\eta^H(p^H) - G^H(s^H, \phi p^H)\eta^H(q^H)] + \frac{1}{2}(R(p^H, q^H)\phi s^H). \quad (29)$$

On the other hand, by applying η^H to the (29) and using (20), it is found that $\eta^H((\nabla^H_s R)(p^H, q^H)\xi^H) = 0$ and so (29) takes the following form:

$$(\nabla^H_s R)(p^H, q^H)\xi^H = \frac{1}{4}[\eta^H(q^H)p^H - \eta^H(p^H)q^H][A^H(s^H) + B^H(s^H)] \quad (30)$$

By virtue of the right parts of (28) and (29),

$$(R(p^H, q^H)\phi s^H) = \frac{1}{4}[G^H(s^H, \phi q^H)\eta^H(p^H) - G^H(s^H, \phi X^H)\eta^H(Y^H)] + \frac{1}{2}[\eta^H(q^H)p^H - \eta^H(p^H)q^H][A^H(s^H) + B^H(s^H)] \quad (31)$$

Using Lemma 3.2, (31) takes the following form:

$$R(p^H, q^H)\phi s^H = \frac{1}{4}[G^H(s^H, \phi q^H)\eta^H(p^H) - G^H(s^H, \phi p^H)\eta^H(q^H)] \quad (32)$$

By applying ϕ both sides of (32) and via (11) and (19), the below equation is satisfied:

$$R(p^H, q^H)s^H = \frac{1}{4}[G^H(q^H, s^H)p^H - G^H(p^H, s^H)q^H]. \quad (33)$$

So, the generalized ϕ-recurrent Sasakian Finsler structure on TM_H is of the constant curvature $\frac{1}{4}$. □

Theorem 3.4. *Assume that TM_H be a (2n+1)-dimensional generalized ϕ-recurrent Sasakian Finsler manifold with the structure $(\phi^H, \xi^H, \eta^H, G^H)$ and p^H_1 be the associated vector field of 1-form A^H given in (25). Then the Eigen value of the Ricci tensor corresponding to the Eigen vector p^H_1 is $\frac{n}{2}$.*

Proof. By virtue of (24) and (11), the following relation holds:

$$((\nabla^H_s R)(p^H, q^H)r^H) = \eta^H((\nabla^H_s R)(p^H, q^H)r^H)\xi^H - A^H(s^H)R(p^H, q^H)r^H - \frac{1}{4}B^H(s^H)[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)] \quad (34)$$

Substituting s^H, p^H, r^H cyclically in (34), three equations are found from which it follows that

$$((\nabla^H_s R)(p^H, q^H)r^H) + ((\nabla^H_s R)(q^H, s^H)p^H) + ((\nabla^H_s R)(s^H, p^H)r^H) = \eta^H((\nabla^H_s R)(p^H, q^H)r^H)\xi^H + \eta^H((\nabla^H_s R)(q^H, s^H)r^H)\xi^H + \eta^H((\nabla^H_s R)(s^H, p^H)r^H)\xi^H - A^H(s^H)R(p^H, q^H)r^H - A^H(p^H)R(q^H, s^H)r^H - A^H(q^H)R(s^H, p^H)r^H - \frac{1}{4}\{B^H(s^H)[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)] - B^H(p^H)[G(s^H, r^H)q^H - \eta^H(G(q^H, r^H)s^H)] - B^H(q^H)[G(p^H, r^H)s^H - \eta^H(G(s^H, r^H)p^H)]\} \quad (35)$$

Using Second Bianchi identity and applying Lemma 3.2,

$$A^H(s^H)[R(p^H, q^H)r^H + A^H(p^H)[R(q^H, s^H)r^H + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)]] + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)]] + A^H(q^H)[R(s^H, p^H)r^H + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)]] = 0. \quad (36)$$

Setting $q^H = r^H = e^H_i$ in (36),

$$A^H(s^H)[S(p^H, t^H)r^H - \frac{2n-1}{4}G(p^H, t^H)] + A^H(p^H)[-S(s^H, t^H) + \frac{2n-1}{4}G(p^H, t^H)] - A^H(R(s^H, p^H)t^H) - G(R(s^H, p^H)t^H, p^H_1) = 0. \quad (37)$$

By the use of (23) and contracting (37), below relation is satisfied:

$$S(s^H, p^H_1) = \frac{n}{2}A^H(s^H). \quad (38)$$

□

4. Extended generalized ϕ-recurrent Sasakian Finsler structures on TM_H

Definition 4.1. The Sasakian Finsler manifold TM_H admitting the quadruple $(\phi^H, \xi^H, \eta^H, G^H)$ is called extended generalized ϕ-recurrent if the following relation holds:

$$\phi^2((\nabla_s^H R)(p^H, q^H)r^H) = A^H(s^H)\phi^2(R(p^H, q^H)r^H) + B^H(s^H)\phi^2(G(p^H, q^H)r^H) \quad (39)$$

for $p^H, q^H, r^H, s^H \in T_u^H TM$.

Theorem 4.2. *An extended generalized ϕ-recurrent Sasakian Finsler manifold TM_H is Einstein. Also the 1-forms A^H and B^H are related as*

$$A^H = -B^H. \quad (40)$$

Proof. Accept that TM_H be a ϕ-recurrent Sasakian Finsler manifold. Then using (11) and (24), we obtain

$$\phi^2(\nabla_s^H R)(p^H, q^H)r^H = \eta^H((\nabla_s^H R)(p^H, q^H)r^H)\xi^H + A^H(s^H)[R(p^H, q^H)r^H - \eta^H(R(p^H, q^H)r^H)\xi^H] + B^H(s^H)[G(p^H, q^H)r^H - \eta^H(G(p^H, q^H)r^H)\xi^H] \quad (41)$$

From the above relation it can be seen that;

$$G((\nabla_s^H R)(p^H, q^H)r^H, t^H) = \eta^H((\nabla_s^H R)(p^H, q^H)r^H)\eta^H(t^H) + A^H(s^H)[G(R(p^H, q^H)r^H, t^H) - \eta^H(R(p^H, q^H)r^H)\eta^H(t^H) + B^H(s^H)[G(G(p^H, q^H)r^H, t^H) - \eta^H(R(p^H, q^H)r^H)\eta^H(t^H)] \quad (42)$$

Let $\{e_i^H\}$ $i = 1, 2, \ldots, 2n+1$, be the orthonormal basis of $T_u^H TM$. Then putting $p^H = t^H = e_i^H$ and taking summation over i, $1 \le i \le 2n+1$, we get

$$\sum G((\nabla_s^H R)(e_i^H, q^H)r^H, e_i^H) = \sum\{\eta^H((\nabla_s^H R)(e_i^H, q^H)r^H)\eta^H(e_i^H) + G(R(e_i^H, q^H)r^H, e_i^H) + A^H(s^H)[-\eta^H(R(e_i^H, q^H)r^H)\eta^H(e_i^H)] + B^H(s^H)[(G(e_i^H, q^H)r^H, e_i^H) - \eta^H(G(e_i^H, q^H)r^H)\eta^H(e_i^H)]\} \quad (43)$$

If the relations $\eta^H((\nabla_W^H R)(e_i^H, Y^H)Z^H)\eta^H(e_i^H) = 0$ and $\sum G(R(e_i^H, q^H)r^H, e_i^H) = S(q^H, r^H)$ are considered we have the following

$$(\nabla_s^H S)(q^H, r^H) = A^H(s^H)S(q^H, r^H) - A^H(s^H)\eta^H(R(\xi^H, q^H)r^H) + \frac{1}{4}B^H(s^H)[2nG(q^H, r^H)e_i^H - G(q^H, r^H) + \eta^H(q^H)\eta^H(r^H)]. \quad (44)$$

By using (21) in (44), we have

$$(\nabla_s^H S)(q^H, r^H) = A^H(s^H)S(q^H, r^H) - \frac{1}{4}A^H(s^H)[G(r^H, q^H) - \eta^H(q^H)\eta^H(r^H)] + \frac{1}{4}B^H(s^H)[(2n-1)G(q^H, r^H) + \eta^H(q^H)\eta^H(r^H)] \quad (45)$$

By putting $r^H = \xi^H$ in (45), we have the following;

$$(\nabla_s^H S)(q^H, \xi^H) = A^H(s^H)S(q^H, \xi^H) + \frac{n}{2}B^H(s^H)\eta^H(q^H) \quad (46)$$

By virtue of (22)

$$(\nabla_s^H S)(q^H, \xi^H) = \frac{n}{2}[A^H(s^H) + B^H(s^H)]\eta^H(q^H) \quad (47)$$

On the other hand, due to the relation between $(\nabla_s^H S)(q^H, \xi^H)$ and $(\nabla_s^H)(S(q^H, \xi^H))$ and by the use of (18) and (22) the below equation is satisfied:

$$(\nabla_s^H S)(q^H, \xi^H) = \frac{n}{4}G(s^H, \phi q^H) + \frac{1}{2}S(q^H, \phi s^H). \quad (48)$$

Substituting ϕq^H with q^H in (48) with the help of (23) we have

$$(\nabla_s^H S)(\phi q^H, \xi^H) = \frac{n}{4}G(s^H, q^H) + \frac{1}{2}S(q^H, s^H). \quad (49)$$

Plugging $q^H = \phi q^H$ in (42) with the help of (14) we get

$$(\nabla_s^H S)(\phi q^H, \xi^H) = 0. \quad (50)$$

It follows from (49) and (50),

$$S(q^H, s^H) = \frac{n}{2}G(s^H, q^H). \quad (51)$$

So, manifold is Einstein [3].
Further, contracting (48) by the use of (12) and (14) we get the relation $(\nabla_s^H S)(q^H, \xi^H) = 0$. In the same sence, contracting (47), we have (40).

□

Theorem 4.3. *Assume that TM_H be a (2n+1)-dimensional extended generalized ϕ-recurrent Sasakian Finsler manifold, then the following relation holds:*

$$(\nabla_s^H R)(p^H, q^H)r^H = \frac{1}{2}\{\frac{1}{4}[-G(Y^H, s^H)G(\phi p^H, r^H) + G(X^H, s^H)G(\phi q^H, r^H) + G(\phi R(p^H, q^H)s^H, r^H)]\xi^H - A^H(s^H)(R(p^H, q^H)r^H)\}. \quad (52)$$

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