Süleyman Demirel Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 29, Sayı 3, 715-725, 2025

Süleyman Demirel University
Journal of Natural and Applied Sciences
Volume 29, Issue 3, 715-725, 2025

DOI: 10.19113/sdufenbed.1818549

Quantitative Relations between Commutativity, Surjectivity, and Homomorphism Degrees in Finite Groups

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(Alınış / Received: 06.11.2025, Kabul / Accepted: 26.11.2025, Online Yayınlanma / Published Online: 25.12.2025)

Keywords
Commutativity degree,
Group homomorphism,
Surjectivity.

Abstract: This article introduces two new probabilistic measures, the surjectivity degree and the homomorphism degree, for the purpose of structural analysis in finite groups. An analytical framework is developed that establishes a relationship between these measures and the previously introduced commutativity degree. New lower and upper bounds for the commutativity degree, depending on the surjectivity degree, are obtained; the homomorphism properties of functions between groups are quantitatively investigated. The relationships between the concepts are supported by theorems and examples, and SageMath code is provided for some examples. These findings contribute to a deeper probabilistic understanding of structural homomorphisms and provide new analytical tools for quantifying algebraic relationships within finite groups.

Sonlu Gruplarda Komütatiflik, Sürjektiflik ve Homomorfizma Dereceleri Arasındaki Nicel İlişkiler

Anahtar Kelimeler
Komütatiflik derecesi,
Grup homomorfizması,
Sürjektiflik.

Öz: Bu makale, sonlu gruplarda yapısal analiz amacıyla iki yeni olasılıksal ölçüyü, sürjektiflik derecesi ve homomorfizm derecesini tanıtmaktadır. Bu ölçütler ile daha önce tanıtılan komütatiflik derecesi arasında ilişki kuran analitik bir çerçeve geliştirilmiştir. Sürjektiflik derecesine bağlı olarak, komütatiflik derecesi için yeni alt ve üst sınırlar elde edilmiş ve gruplar arasındaki fonksiyonların homomorfizma özellikleri nicel olarak incelenmiştir. Kavramlar arasındaki ilişkiler teoremler ve örneklerle desteklenmiş olup, bazı örnekler için SageMath kodu verilmiştir. Bu bulgular yapısal homomorfizmlerin daha derinlemesine olasılıksal olarak anlaşılmasına katkıda bulunmakta ve sonlu gruplar içindeki cebirsel ilişkilerin niceliksel olarak belirlenmesi için yeni analitik araçlar sağlamaktadır.

1. Introduction

The probability that two randomly selected group elements commute is known as the *commutativity degree* $d(G)$ of a finite group G. This concept was first introduced by P. X. Gallagher in 1970 [1]. Gallagher aimed to measure the degree of commutativity between group elements by expressing it as a numerical ratio. In this way, he intended to quantitatively determine how close a group is to an abelian structure. This probabilistic notion is also related to the class equation such that for a finite group G, the commutativity degree is also characterized via the number of the conjugacy classes of G, which is denoted by $k(G)$, as $d(G) = \frac{k(G)}{|G|}$. In abelian groups, all elements commute, hence $d(G) = 1$. In contrast, $d(G) < 1$ in non-abelian groups. W. H. Gustafson (1973) presented one of the first fundamental results in this field: For every non-abelian finite group, the inequality $d(G) \leq 5/8$ holds [2]. Equality is satisfied if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This upper bound sets the maximum level at which a non-commutative group can exhibit commutativity. Also, D. J. Rusin calculated this probability for various classes of groups and used it especially in the study of the lower orders of nilpotent groups in 1979 [3].

This paper aims to reveal the structural relations between two probabilistic measures, the commutativity and surjectivity degrees, in finite groups. First, the surjectivity degree of a group homomorphism with respect to the target group is defined formally. Based on this concept, for the commutativity degree various lower and upper bounds are obtained. New inequalities are presented that explain how the degree of co-

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mutativity behaves under group homomorphisms. Especially, known results indicating that the commutativity degree can increase under epimorphisms are generalized. For instance, an adapted version of the classical Center-Image Theorem is presented, taking into account the degree of surjection. Also, the concept of homomorphism degree (a quantitative measure of how closely a map behaves like a group homomorphism) has been developed for functions that are only partially homomorphic. It studies the structural effects and measurements of functions that behave similarly to group homomorphisms. Its effect on commutativity degree has been analyzed. It is shown that the homomorphism degree imposes a lower bound that forces the identity element of group in the domain to be mapped to the identity element of group in the range. This lower bound is given explicitly. In this context, a relationship has been established between the commutativity degree and the homomorphism degree. This relationship has led to new structural characterizations in the context of finite groups. An example is also presented, illustrating that the multiplication function behaves homomorphically only for a certain number of commutative pairs, and that the homomorphism degree is equal to the commutativity degree within the group. These results in the study offer a new viewpoint in group theory for studying the structure and impact of group homomorphisms using quantitative methods. This perspective helps deepen the analysis of how homomorphisms determine finite group structures. The surjectivity and homomorphism degrees provide quantitative tools for measuring how strongly a mapping preserves group structure, complementing classical algebraic invariants.

These probabilistic measures also allow a finer analysis of homomorphic images, partial homomorphisms, and structural stability under group operations. These probabilistic measures also have potential applications beyond group theory. The surjectivity and homomorphism degrees provide quantitative tools for analysing how strongly a map preserves algebraic structure, which is useful in computational group theory, coding-related structures, and the study of approximate homomorphisms. These notions also offer research potential for extending such probabilistic invariants to group rings, semigroups, and operator-theoretic settings.

Recent works in operator theory and Banach algebras also suggest that probabilistic invariants can play an important structural role in analytic settings. Studies on extended eigenvalues in the Wiener algebra [4], reproducing kernels and Engliš algebras [5], Banach-algebraic techniques in analysing operator behaviour [6], structural properties of concrete operators [7], and extended eigenvalue theory for shift operators [8] indicate that quantitative structural tools—such as those introduced in this paper—may eventually be applicable in these analytic frameworks as well. These connections underline the broader mathematical relevance and potential impact of the probabilistic measures studied here.

2. Material and Method

By building on the fundamental results of Gallegher, Gustafson, and Rusin, calculating or estimating the commutativity degree of a finite group is based on counting commuting pairs in two different ways. Let G be a finite group and $d(G)$ is the commutativity degree of G. First way: Count directly the number of commuting pairs $(x, y) \in G \times G$ such that $xy = yx$. It is mathematically defined by

$$d(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}. \qquad (1)$$

Second way: Fix an element $x \in G$. For each such x, the number of elements $y \in G$ that commute with x is exactly $|C_G(x)|$, where $C_G(x)$ is the centralizer of x in G. Therefore, the total number of commuting pairs in $G \times G$ is $\sum_{x\in G} |C_G(x)|$. Hence, the commutativity degree can also be calculated by $d(G) = \frac{1}{|G|^2} \sum_{x\in G} |C_G(x)|$.

In the 2000s, Patrick Lescot's contributions revitalized work on this concept. After his work in 1995 [9], Lescot examined the relationship between the commutativity degree and the structure of the group in the context of central extensions and isoclinism. Lescot showed that the value of $d(G)$ remains unchanged under isoclinism, implying that the commutativity structure is preserved under a broader equivalence than isomorphism [10]. He also presented classifications of groups whose degrees of commutativity fall within certain ranges. His results also revealed a strong connection between $d(G)$ and both the center and the commutator subgroup of the group. Moreover, $d(G)$ exhibits a multiplicative property under direct products such that if $G = G_1 \times G_2$, then $d(G) = d(G_1) \cdot d(G_2)$.

Following Lescot's foundational results, important generalizations regarding the degree of commutativity were put forward in the 2000s. In particular, Moghaddam and collaborators (2005) defined the *n-th nilpotency degree* $d^{(n)}(G)$ by extending the concept of commutativity to $(n + 1)$-tuples of elements [11]. This generalization satisfies $d^{(1)}(G) = d(G)$, and it was shown that $d^{(n)}(G)$ monotonically approaches 1 as n increases. In later studies, Erfanian, Rezaei, and Lescot (2007) introduced the concept of the *relative commutativity degree* $d(H, G)$ for a subgroup $H \leq G$, generalizing the classical $d(G)$ concept to the subgroup level [12]. It is mathematically defined by

$$d(H, G) = \frac{|\{(h, g) \in H \times G \mid hg = gh\}|}{|H| \cdot |G|}. \qquad (2)$$

In this context, they proved the inequalities $d(H,G) \leq 3/4$ if H is not contained in the center $Z(G)$, and $d(H,G) \leq 5/8$ if H is also non-abelian. They also analyzed in detail the n-nilpotent generalizations of $d(H,G)$. This extension allowed for the measurement of commutativity at the level of interactions between subgroups. Following this development, Marius Tărnăuceanu introduced the *subgroup commutativity degree*, which focuses on the probability that a pair of subgroups of a group chosen at random will commute, in 2009 [13]. This criterion facilitates structural analysis at the subgroup lattice level. Erfanian et al. conducted more research on finite groups with precisely three different degrees of relative commutativity, offering a structural categorization based on the center and quotient [14]. Furthermore, Rezaei and Erfanian studied isoclinism invariance in relative commutativity contexts and established stricter boundaries [15]. Pournaki and Sobhani (2008) provided a lower bound for $d(G)$ under the assumption that $|\mathrm{cd}(G)| = 2$, where $\mathrm{cd}(G)$ is the set of degrees of the irreducible complex characters of G [16]. Thereafter, Nath and Das (2010) removed this restriction and established a general lower bound valid for all finite groups: $d(G) \geq \frac{1}{|G'|}\left(1 + \frac{|G'|-1}{|G:Z(G)|}\right)$ [17]. This result shows that the commutativity degree is not only a probabilistic measure but also reflects the internal structural features of the group. New interpretations of generalized commutativity degrees were made possible by Hashemi and Pirzadeh's investigation of the number of solutions to commutator equations in two-generator nilpotent groups [18]. Additionally, Ghaneei and Azadi established new lower bounds in quasi-commutative algebraic contexts and introduced the n-th commutativity degree for semigroups [19].

R. K. Nath and A. K. Das (2011) studied the generalized forms of the degree of commutativity and obtained new boundary inequalities using character theory [20]. In particular, they account for the impact of a group's representation-theoretic properties on $d(G)$ by introducing bounds based on character degrees. The bounds provided by Nath and Das can be expressed in terms of the order of the commutator subgroup $|G'|$ and character degrees. These relations highlight the connection between $d(G)$ and structural properties such as nilpotency and solvability.

The degree of commutativity has become a significant focus of attention in recent years. Many remarkable studies have emerged in the literature on this subject. Chashiani and Rezaei defined the commutativity degree for the group ring $F[G]$ in 2021 [21]. They showed that the commutativity degree of the group rings over groups that are isoclinic and whose centers are of equal order, are the same. This result established a meaningful connection between group theory and ring theory. Subsequently, Arvasi and collaborators (2022) introduced the notion of commutativity degree for crossed module structures (S, R, ∂), presenting a two-dimensional generalization of the classical concept in terms of isoclinism and structural classification [22]. The invariant they define can be computable through the GAP software and contributes to understanding structures beyond classical groups. A similar approach is also suitable for adaptation to more advanced algebraic structures based on crossed modules of finite groups such as crossed modules with action, since the additional action intrinsic to the structure is related to the commutativity of the underlying crossed module [23].

As a result, commutativity degree serves as an important tool in group theory, both theoretically and structurally. It finds various applications through its connections with character theory and subgroup structure. It also plays a role in classifying groups in terms of properties such as nilpotency and solvability.

3. Results

3.1. The relationships among commutativity and surjectivity degrees

The next theorem reveals the limiting effect of a homomorphism defined between finite groups on the commutativity degree of the group in its domain.

Theorem 3.1.1. *If $\varphi: G \to H$ is a group epimorphism, then $d(G) \leq d(H)$.*

Proof. Suppose that $|\ker\varphi| = n$. Say $\ker\varphi = g_0 = e, g_1, \dots, g_{n-1}$. Let $h, h' \in H$. Then there are $k, k' \in G$ with $h = \varphi(k)$, $h' = \varphi(k')$, so that $h = \varphi(kg_i)$ and $h' = \varphi(k'g_j)$ for all $0 \leq i, j \leq n-1$.

Assume that $hh' \neq h'h$. In this case $\varphi(kg_i) \neq \varphi(k'g_j)$ for all $0 \leq i, j \leq n-1$. Hence, corresponding to each element of the set $\{(h,h') \in H \times H : hh' \neq h'h\}$, the set $\{(g,g') \in G \times G : gg' \neq g'g\}$ has $n \times n = n^2$ distinct elements. This means that $n^2|\{(h,h') \in H^2 : hh' \neq h'h\}| \leq |\{(g,g') \in G^2 : gg' \neq g'g\}|$.

By $|\ker\varphi| = n$, $|G| = n \cdot |H|$ and

$$
\begin{aligned}
d(H) &= \frac{|\{(h,h') \in H^2 : hh' = h'h\}|}{|H|^2} \\
&= 1 - \frac{|\{(h,h') \in H^2 : hh' \neq h'h\}|}{\left(\frac{|G|}{n}\right)^2} \\
&\geq 1 - \frac{\frac{1}{n^2}|\{(g,g') \in G^2 : gg' \neq g'g\}|}{\frac{1}{n^2}|G|^2} \\
&= \frac{|\{(g,g') \in G^2 : gg' = g'g\}|}{|G|^2} = d(G).
\end{aligned}
$$

□

The cases where a homomorphism can have an effect not only on the group in its domain but also on the commutativity behavior of its subgroups are discussed in the following theorem.

Theorem 3.1.2. *Let G and H be finite groups, $M \leq G$. Let $\varphi: G \to H$ be a group epimorphism. Then, $d(M,G) \leq d(\varphi(M),H)$.*

Proof. Let $\varphi: G \to H$ be a group epimorphism and $M \leq G$ be a subgroup. By the definition of relative commutativity degree in [12],

$$d(M,G) := \frac{|\{(m,g) \in M \times G \mid mg = gm\}|}{|M| \cdot |G|}$$

and

$$d(\varphi(M),H) := \frac{|\{(a,b) \in \varphi(M) \times H \mid ab = ba\}|}{|\varphi(M)| \cdot |H|}.$$

Since φ is surjective, for every $h \in H$, there are exactly $|\ker\varphi|$ elements in G mapping to h, and similarly for each $a \in \varphi(M)$, there are $|M \cap \ker\varphi|$ elements in M mapping to a. Hence, for each pair $(a,b) \in \varphi(M) \times H$, the number of lifts in $M \times G$ is $|\ker\varphi| \cdot |M \cap \ker\varphi|$.

Now suppose that $a = \varphi(m)$ and $b = \varphi(g)$. If a and b do not commute, then neither do m and g, because

$$\varphi(mg) = \varphi(m)\varphi(g) = ab, \quad \varphi(gm) = ba,$$

and $ab \neq ba \Rightarrow mg \neq gm$. Thus,

$$\big(\varphi(m),\varphi(g)\big) \in \varphi(M) \times H \text{ is non-commuting}$$
$$\Longrightarrow (m,g) \in M \times G \text{ is non-commuting.}$$

This defines an injective relation from the set of non-commuting pairs in $\varphi(M) \times H$ to those in $M \times G$, with each pair in the domain having $|\ker\varphi| \cdot |M \cap \ker\varphi|$ images.
Therefore,

$$|\{(a,b) \in \varphi(M) \times H \mid ab \neq ba\}| \leq \frac{|\{(m,g) \in M \times G \mid mg \neq gm\}|}{|ker\ \varphi| \cdot |M \cap ker\ \varphi|}$$

Hence,

$$d(\varphi(M),H) = 1 - \frac{|\{(a,b) \in \varphi(M) \times H \mid ab \neq ba\}|}{|\varphi(M)| \cdot |H|}$$
$$\geq 1 - \frac{|\{(m,g) \in M \times G \mid mg \neq gm\}|}{|\ker\varphi| \cdot |M \cap \ker\varphi| \cdot |\varphi(M)| \cdot |H|}.$$

Using the fact that, $|M| = |M \cap \ker\varphi| \cdot |\varphi(M)|$, $|G| = |\ker\varphi| \cdot |H|$, we get

$$|M| \cdot |G| = |\ker\varphi| \cdot |M \cap \ker\varphi| \cdot |\varphi(M)| \cdot |H|.$$

Substituting

$$\frac{1}{|\ker\varphi| \cdot |M \cap \ker\varphi| \cdot |\varphi(M)| \cdot |H|} = \frac{1}{|M| \cdot |G|}$$

we get

$$d(\varphi(M),H) \geq 1 - \frac{|\{(m,g) \mid mg \neq gm\}|}{|M| \cdot |G|} = d(M,G).$$

□

Definition 3.1.3. For a group homomorphism $\varphi: G \to H$ with $|G|, |H| < \infty$, the **surjectivity degree** *is*

$$\sigma(\varphi) = \frac{|\{h \in H: \exists g \in G,\ \varphi(g) = h\}|}{|H|} = \frac{|\mathrm{Im}\,\varphi|}{|H|} \quad (3)$$

and $0 < \sigma(\varphi) \leq 1$.

If φ is not surjective, $\sigma(\varphi) < 1$.

As shown in the upcoming theorem, the structural constraints that a group homomorphism imposes on the target group, both in terms of the commutativity degree and the surjectivity degree.

Theorem 3.1.4. *If $\varphi: G \to H$ is a group homomorphism, then $d(H) \geq d(G) \cdot [\sigma(\varphi)]^2$.*

Proof. Since φ defines an epimorphism from G onto $\mathrm{Im}\,\varphi$, we have $d(G) \leq d(\mathrm{Im}\,\varphi)$ by Theorem 3.1.1.

$$d(\mathrm{Im}\,\varphi) = \frac{|\{(h,h') \in (\mathrm{Im}\,\varphi) \times (\mathrm{Im}\,\varphi): hh' = h'h\}|}{|\mathrm{Im}\,\varphi|^2}$$

Hence, there are at least $d(\mathrm{Im}\,\varphi) \cdot |\mathrm{Im}\,\varphi|^2$ commuting pairs in $\mathrm{Im}\,\varphi$, and therefore in H. Thus, we have

$$d(H) \geq \frac{d(\mathrm{Im}\,\varphi) \cdot |\mathrm{Im}\,\varphi|^2}{|H|^2} = d(\mathrm{Im}\,\varphi) \cdot [\sigma(\varphi)]^2$$
$$\geq d(G) \cdot [\sigma(\varphi)]^2.$$

□

A concrete application of Theorem 3.1.4 is examined in the following example, and the effect of the homomorphism on the commutativity degree is shown numerically.

Example 3.1.5. Let $D_4 = e, a, a^2, a^3, b, ba, ba^2, ba^3$ be the dihedral group of order 8. Table 1 is the commutativity table where each cell indicates whether the pair (g,g') satisfies $gg' = g'g$. (✓ means they commute; × means they do not.)

Table 1. Commutativity table for D_4.

$g \setminus g'$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	✓	✓	✓	✓	✓	✓	✓	✓
a	✓	✓	✓	✓	×	×	×	×
a^2	✓	✓	✓	✓	✓	✓	✓	✓
a^3	✓	✓	✓	✓	×	×	×	×
b	✓	×	✓	×	✓	×	✓	×
ba	✓	×	✓	×	×	✓	×	✓
ba^2	✓	×	✓	×	✓	×	✓	×
ba^3	✓	×	✓	×	×	✓	×	✓

Number of total commutative pairs is 40. Number of total possible pairs is $8 \times 8 = 64$. Hence, $d(D_4) = \frac{40}{64} = \frac{5}{8}$. Let $V_4 = \varepsilon, x, y, xy$ be the Klein four group. The group V_4 is known to be abelian and $d(V_4) = 1$.

We define a group homomorphism $\varphi: V_4 \to D_4$ by the rule

$$\varphi(\varepsilon) = e, \varphi(x) = a^2, \varphi(y) = e,$$
$$\varphi(xy) = \varphi(x)\varphi(y) = a^2 \cdot e = a^2.$$

Indeed, since V_4 is abelian and all elements square to the identity:

$$\varphi(x)^2 = a^2a^2 = a^4 = e; \ \ \varphi(y)^2 = e;$$
$$\varphi(xy) = \varphi(x)\varphi(y) = a^2e = a^2.$$

Thus, φ is a group homomorphism. $\operatorname{Im}\varphi = \{e, a^2\} \cong C_2$. Hence, $\sigma(\varphi) = \frac{|\operatorname{Im}\varphi|}{|D_4|} = \frac{2}{8} = \frac{1}{4}$. We should have $d(D_4) \geq d(V_4) \cdot [\sigma(\varphi)]^2$. Indeed, since $d(D_4) = \frac{5}{8}$, $d(V_4) = 1$ and $[\sigma(\varphi)]^2 = \frac{1}{16}$, we get $\frac{5}{8} \geq 1 \cdot \frac{1}{16}$.

If $\sigma(\varphi) = 1$ (i.e., φ is onto), the inequality $d(H) \geq d(G) \cdot [\sigma(\varphi)]^2$ reduces to inequality $d(H) \geq d(G)$ in Theorem 3.1.1. If $\sigma(\varphi) \approx 0$, the bound becomes the trivial $d(H) \geq 0$. The inequality provides a quantitative bridge between how far φ is from being surjective and how large the commutativity degree of H can be, relative to G. If $\varphi: G \to H$ is an isomorphism, $|H| = |G|$; so $\sigma(\varphi) = 1$. Hence, we get $0 < \sigma(\varphi) \leq 1$.

If $|H| \geq |G|$, $\varphi(e_G) = e_H \in \operatorname{Im}(\varphi)$ where e_G and e_H are identity elements of G and H, respectively. Hence, $\sigma(\varphi) \geq \frac{1}{|H|}$. Since $\frac{G}{\ker\varphi} \cong \operatorname{Im}\varphi$, then $\sigma(\varphi) = \frac{|\operatorname{Im}\varphi|}{|H|} = \frac{|G|}{|\ker\varphi|\cdot|H|}$. Hence, $\sigma(\varphi) \geq \frac{1}{|\ker\varphi|\cdot|H|}$; and this means that the larger the kernel, the smaller the surjectivity degree will be. If $\varphi: G \to H$ is non-trivial and not surjective homomorphism, $\sigma(\varphi) \leq \frac{1}{2}$, clearly. Hence,

$$\frac{1}{|\ker\varphi| \cdot |H|} \leq \sigma(\varphi) \leq \frac{1}{2}.$$

Example 3.1.6. $H = S_4$, $|H| = 24$. The alternating subgroup $A_4 \leq S_4$ has index 2 and order 12. The inclusion $\varphi: A_4 \hookrightarrow S_4$ satisfies

$$d(\varphi) = \frac{12}{24} = \frac{1}{2}.$$

The next theorem establishes a maximum for the surjectivity degree of homomorphisms, whose domain exhibits a specific prime factor structure to a finite abelian target group.

Theorem 3.1.7. *Let $G = \Pi_{p||H|} C_p$ (p is prime divisor), H be a finite abelian group; and let $p_{\min} := \min\{p \mid p$ be a prime and $p \mid |H|\}$. Then, for every non-surjective homomorphism $\varphi: G \to H$, $\sigma(\varphi) \leq \frac{1}{p_{\min}}$.*

Proof. Since for each prime divisor p of $|H|$, the group G has a quotient isomorphic to the cyclic group C_p, there exists a group homomorphism $\varphi: G \to H$ such that $|\operatorname{Im}\varphi| = p$. Moreover, $\varphi(G) \leq H$, and $|\varphi(G)|$ divides both $|G|$ and $|H|$. The largest proper subgroup of H has order $|H|/p_{\min}$; hence its relative size is $\frac{|H|/p_{\min}}{|H|} = \frac{1}{p_{\min}}$. Every finite group H possesses a maximal subgroup $M \leq H$ of prime index $p_{\min}$. Taking the inclusion homomorphism $\varphi : M \hookrightarrow H$ yields $\operatorname{Im}\varphi = M$ and therefore,

$$\sigma(\varphi) = \frac{|M|}{|H|} = \frac{\frac{|H|}{p_{min}}}{|H|} = \frac{1}{p_{min}}.$$

□

Below, Theorem 3.1.8 explains how the behavior of surjectivity is transferred in homomorphism chains by showing that the degree of surjectivity in compound homomorphisms is subject to a multiplicative lower bound.

Theorem 3.1.8. *Let there be a chain of finite groups $K \leq H \leq G$ with two successive group homomorphisms p: $K \to H$ and r: $H \to G$ and let their composition be $s := r \circ p: K \to G$. Then, $\sigma(s) \geq \sigma(p) \cdot \sigma(r)$.*

Proof. We denote $p(K)$ as the $\operatorname{Im} p$ for ease of writing if necessary. Let $\operatorname{Im} p = H_0 \leq H$, and $\operatorname{Im} r = G_0 \leq G$. Then, since $|H_0 \cap \ker r| \leq |\ker r|$ we have

$$\frac{1}{|H_0 \cap \ker r|} \geq \frac{1}{|\ker r|} \Rightarrow \frac{|H_0|}{|H_0 \cap \ker r|} \geq \frac{|H_0|}{|H|} \cdot \frac{|H_0|}{|\ker r|}$$

$$\Rightarrow |r(H_0)| \geq \frac{|H_0|}{|H|} \cdot |r(H)|$$

$$\Rightarrow |\operatorname{Im} s| \geq \frac{|\operatorname{Im} p|}{|H|} \cdot |\operatorname{Im} r|.$$

So, we get

$$\sigma(s) = \frac{|\operatorname{Im} s|}{|G|} \geq \frac{|\operatorname{Im} p|}{|H|} \cdot \frac{|\operatorname{Im} r|}{|G|} = \sigma(p) \cdot \sigma(r).$$

Therefore, $\sigma(s) \geq \sigma(p) \cdot \sigma(r)$. □

The following theorem establishes a meaningful relation between the surjectivity degree of a homomorphism and the relative commutativity of its image within the target group.

Theorem 3.1.9. *Let $\varphi: G \to H$ be a homomorphism of finite groups and suppose that $\operatorname{Im}\varphi$ is abelian. Then, $\sigma(\varphi) \leq d(\operatorname{Im}\varphi, H)$.*

Proof. Set $I := \operatorname{Im}\varphi \leq H$. Since I is abelian, $I \subseteq C_H(a)$ for all $a \in I$, hence $|C_H(a)| \geq |I|$. So, we get $\sum_{a\in I} |C_H(a)| \geq \sum_{a\in I} |I| = |I|^2$.

By the definition $d(I, H) = \frac{1}{|I||H|}\sum_{a\in I} |C_H(a)|$, and thus,

$$d(I, H) \geq \frac{|I|^2}{|I||H|} = \frac{|I|}{|H|} = \sigma(\varphi)$$

□

Theorem 3.1.10. *Let G and H be finite groups and let $\varphi: G \to H$ be a group homomorphism. Then the following two statements are equivalent:*

1. $\sigma(\varphi) \leq d(\operatorname{Im}\varphi, H)$;
2. $\sum_{a\in\operatorname{Im}\varphi} |C_H(a)| \geq |\operatorname{Im}\varphi|^2$.

Proof. Write $I := \operatorname{Im}\varphi \leq H$.

(1) ⇒ (2). Assume $\sigma(\varphi) \leq d(I, H)$. Multiply both sides by $|I||H|$ such that

$$|I|^2 \leq \sum_{a\in I} |C_H(a)|.$$

(2) ⇒ (1). Conversely, suppose (2) holds. Dividing the inequality $\sum_{a\in I} |C_H(a)| \geq |I|^2$ by $|I||H|$ gives

$$\frac{|I|}{|H|} \leq \frac{1}{|I||H|}\sum_{a\in I} |C_H(a)| = \mathrm{d}(\mathrm{I},\mathrm{H}).$$

□

The next theorem shows how homomorphisms with high surjectivity degree integrate into the central structure of the target group by examining the effect of the central image of a homomorphism on the centralizer in the target group.

Theorem 3.1.11. *Let $Z(G), Z(H)$ be the centers of the finite groups G and H, respectively. Let $\varphi: G \to H$ be a group homomorphism. If $\mathrm{Im}\varphi \subseteq C_H\left(\varphi\big(Z(G)\big)\right)$, we have $\left|C_H\left(\varphi\big(Z(G)\big)\right)\right| \geq \sigma(\varphi)|H|$, and the index is bound by $\left[H: C_H\left(\varphi\big(Z(G)\big)\right)\right] \leq \frac{1}{\sigma(\varphi)}$. Moreover, if $\sigma(\varphi) > \frac{1}{2}$, then $C_H\left(\varphi\big(Z(G)\big)\right) = H$ and $\varphi\big(Z(G)\big) \subseteq Z(H)$.*

Proof. For each $h \in \mathrm{Im}\,\varphi$, there is $g \in G$ with $h = \varphi(g)$. Then for all $z \in Z(G)$, we have

$$\begin{aligned} h\varphi(z) &= \varphi(g)\varphi(z) = \varphi(gz) \\ &= \varphi(zg) = \varphi(z)\varphi(g) = \varphi(z)h. \end{aligned}$$

So, $\mathrm{Im}\,\varphi \subseteq C_H\left(\varphi\big(Z(G)\big)\right) := \{h \in H \mid h\varphi(z) = \varphi(z)h, \forall z \in Z(G)\}$. Hence, $\left|C_H\left(\varphi\big(Z(G)\big)\right)\right| \geq |\mathrm{Im}\,\varphi| = \sigma(\varphi)\cdot|H|$, and so the index inequality follows. If $\sigma(\varphi) > 1/2$, then the index is ≤ 1, that is the centralizer is the whole group H, and therefore $\varphi\big(Z(G)\big) \subseteq Z(H)$. □

In the case of epimorphism ($\sigma(\varphi) = 1$), we recover the classical result $\varphi\big(Z(G)\big) \subseteq Z(H)$.

3.2. Homomorphism degree of functions between finite groups

Definition 3.2.1. Let $f: G \to H$ be a function. We define the **homomorphism degree** of f as

$$\chi(f) = \frac{|\{(g,g') \in G^2: f(gg') = f(g)f(g')\}|}{|G|^2}. \quad (4)$$

Example 3.2.2. Let $f: \mathbb{Z}_5 \to \mathbb{Z}_6$ be a function defined by

$f(\bar{0}) = \bar{0},\ f(\bar{1}) = \bar{3},\ f(\bar{2}) = \bar{5},\ f(\bar{3}) = \bar{1},\ f(\bar{4}) = \bar{2}.$

We construct Table 2 to test the homomorphism condition $f(g+g') = f(g) + f(g')$.

Table 2. Test for the homomorphism condition for $f: \mathbb{Z}_5 \to \mathbb{Z}_6$.

$g \setminus g'$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	✓	✓	✓	✓	✓
$\bar{1}$	✓	×	×	×	×
$\bar{2}$	✓	×	×	✓	×
$\bar{3}$	✓	×	✓	×	×
$\bar{4}$	✓	×	×	×	×

In this case, there are 11 checkmarks on the table, so $\chi(f) = \frac{11}{25}$.

For groups, the commutativity degree never takes the value 0, because at least the identity element commutes with all elements and every element commutes with itself. However, the homomorphism degree can take the value 0. An illustrative example is the following.

Example 3.2.3. Consider the constant function $f: \mathbb{Z}_3 \to \mathbb{Z}_5$, given by $f(x) = \bar{3}$ for all $x \in \mathbb{Z}_3$. Then, for a pair $(x,y) \in \mathbb{Z}_3 \times \mathbb{Z}_3$

$$\begin{aligned} f(x+y) = f(x) + f(y) \quad &\Leftrightarrow \bar{3} = \bar{3} + \bar{3} \\ &\Leftrightarrow \bar{3} = \bar{1}, \end{aligned}$$

which is a contradiction. Hence, there is no pair (x,y) in $\mathbb{Z}_3 \times \mathbb{Z}_3$ satisfying the homomorphism conditions, and $\chi(f) = 0$.

Definition 3.2.4. Let define a function $f: G \to H$ (not necessarily a homomorphism) for finite groups G and H. For each element $x \in G$, the **left-homomorphismizer** of f at x, denoted by $LH_f(x)$, is defined as the subset of G consisting of all elements $y \in G$ for which the homomorphism identity holds when multiplied on the left by x. Formally,

$$LH_f(x) := \{y \in G \mid f(xy) = f(x)f(y)\}. \quad (5)$$

This set captures the collection of elements y with respect to which f behaves like a group homomorphism under left multiplication by a fixed element x.

The definition of the **right-homomorphismizer** parallels that of the left-homomorphismizer, with right multiplication replacing left multiplication. So,

$$RH_f(x) := \{y \in G: f(yx) = f(y)f(x)\}. \quad (6)$$

The homomorphism degree of f is defined by

$$\chi(f) := \frac{|\{(g,g') \in G \times G: f(gg') = f(g)f(g')\}|}{|G|^2}.$$

This means for $(x,y) \in G \times G$, $f(xy) = f(x)f(y) \Leftrightarrow y \in LH_f(x)$. Among the pairs starting with fixed x, the number of elements $y \in G$ satisfying $f(xy) = f(x)f(y)$ is exactly the size of $LH_f(x)$. So, the number of pairs $(x,y) \in G \times G$ satisfying $f(xy) = f(x)f(y)$ is $\sum_{x\in G}|LH_f(x)|$. Hence, by definition of $\chi(f)$, we get

$$\chi(f) = \frac{1}{|G|^2}\sum_{x\in G}|LH_f(x)|. \quad (7)$$

The subsequent theorem establishes a meaningful relation that imposes a lower bound on the commutativity degree of the target group when the group in the domain is commutative, and the given function satisfies a certain homomorphism degree.

Theorem 3.2.5. *Suppose that $f: G \to H$ is a bijection and G is a finite abelian group, then we have $d(H) \geq \chi(f) - \frac{1}{2} + \frac{1}{2|G|}$.*

Proof. If $\chi(f) \leq \frac{1}{2} - \frac{1}{2|G|}$, then $\chi(f) - \frac{1}{2} + \frac{1}{2|G|} \leq 0$ and the inequality $d(H) \geq \chi(f) - \frac{1}{2} + \frac{1}{2|G|}$ holds trivially, as $d(H) \geq 0$ by the definition. Hence, we can assume that $\chi(f) \geq \frac{1}{2} - \frac{1}{2|G|}$. In this case, the number of ordered pairs $(g, g') \in G \times G$ satisfying $f(gg') = f(g)f(g')$ is at least $\chi(f) \cdot |G|^2 \geq \frac{|G|^2 - |G|}{2}$.
Every subset of $G \times G$ corresponds to a relation on G, and an asymmetric relation (that is, $a \sim b \Rightarrow b \nsim a$ for a relation $\sim$ on G) can contain at most $\frac{|G|^2 - |G|}{2}$ elements, since it cannot include any of the $|G|$ diagonal elements (g, g), and when half of the remaining $|G|^2 - |G|$ elements have been written, any new element to be added from that point on for every new pair (a, b) included, the corresponding (b, a) must be excluded. In other words, once half of the non-diagonal elements are used, any further addition of a new pair (a, b) would force (b, a) into the relation, violating asymmetry.
Therefore, there are at least $\chi(f) \cdot |G|^2 - \frac{|G|^2 - |G|}{2}$ many ordered pairs (a, b) and (b, a) in $G \times G$ for which both (a, b) and (b, a) is in the set $\{(g, g') \in G \times G : f(gg') = f(g)f(g')\}$. Hence, for $f(g), f(g') \in H$, we get $f(g)f(g') = f(gg') = f(g'g) = f(g')f(g)$, because G is abelian. Since f is bijective, it follows that in H there are at least $\chi(f) \cdot |G|^2 - \frac{|G|^2 - |G|}{2}$ commuting pairs of elements. Hence, we obtain

$$d(H) \geq \frac{\chi(f) \cdot |G|^2 - \frac{|G|^2 - |G|}{2}}{|H|^2} = \frac{\chi(f) \cdot |G|^2 - \frac{|G|^2 - |G|}{2}}{|G|^2} = \chi(f) - \frac{1}{2} + \frac{1}{2|G|}.$$

□

Example 3.2.6. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, and let $H = D_4 = \langle r, s \mid r^4 = e,\ s^2 = e,\ srs = r^{-1} \rangle$.
Define a bijection $f: G \to H$ by

$$f(\overline{m}, \overline{n}) := \begin{cases} r^m, & \overline{n} = \overline{0}, \\ sr^m, & \overline{n} = \overline{1}. \end{cases}$$

There are $64 = |G|^2$ ordered pairs in total. Moreover,

$$|\{(g, g') \in G^2 \mid f(g + g') = f(g)f(g')\}| = 48,$$

as indicated in Listing 3.3.7 in Subsection 3.3. Hence, $\chi(f) = \frac{48}{64} = \frac{3}{4}$. For the dihedral group D_{2n} with even n, $d(D_{2n}) = \frac{n+6}{4n}$. Taking $n = 4$ gives $d(H) = d(D_4) = \frac{4+6}{16} = \frac{5}{8}$. Substituting the values computed above,

$$\chi(f) - \frac{1}{2} + \frac{1}{2|G|} = \frac{3}{4} - \frac{1}{2} + \frac{1}{16} = \frac{5}{16}.$$

Since $d(H) = \frac{5}{8}$ and $\chi(f) - \frac{1}{2} + \frac{1}{2|G|} = \frac{5}{16}$, we indeed have

$$d(H) \geq \chi(f) - \frac{1}{2} + \frac{1}{2|G|},$$

which is asserted in Theorem 3.2.5.

Theorem 3.2.7 below shows that failure to preserve the identity element places a strict upper bound on the extent to which a function can satisfy the homomorphism condition.

Theorem 3.2.7. *Let define a function $f: G \to H$ (not necessarily a homomorphism) for finite groups G and H. Assume that $e_G \in G$ and $e_H \in H$ denote the identity elements of the respective groups. If $\chi(f) > 1 - \frac{2}{|G|} + \frac{1}{|G|^2}$, then $f(e_G) = e_H$.*

Proof. Suppose that $f(e_G) \neq e_H$. We will estimate how many ordered pairs $(x, y) \in G \times G$ fail to satisfy $f(xy) = f(x)f(y)$. First, observe that $\forall x \in G$, we get $f(xe_G) = f(x)$, but $f(x)f(e_G) \neq f(x)$, since $f(e_G) \neq e_H$. Therefore, for all values of $x \in G$, the pair (x, e_G) violates the homomorphism identity.
Similarly, $\forall y \in G$, we get $f(e_G y) = f(y)$, but $f(e_G)f(y) \neq f(y)$ in general. So, for all values of $y \in G$, the pair (e_G, y) also fails to satisfy the identity. Thus, we expect all $|G|$ pairs of the form (x, e_G), and all $|G|$ pairs of the form (e_G, y), to fail. However, the pair (e_G, e_G) is counted in both sets, so the total number of distinct failing pairs is at least $2|G| - 1$. Therefore, there are at most $|G|^2 - (2|G| - 1) = |G|^2 - 2|G| + 1$. successful pairs. Hence, the homomorphism degree satisfies

$$\chi(f) \leq \frac{|G|^2 - 2|G| + 1}{|G|^2} = 1 - \frac{2}{|G|} + \frac{1}{|G|^2}.$$

So, if $f(e_G) \neq e_H$, then $\chi(f) \leq 1 - \frac{2}{|G|} + \frac{1}{|G|^2}$. As a result, if $\chi(f) > 1 - \frac{2}{|G|} + \frac{1}{|G|^2}$, so $f(e_G) = e_H$. □

If $|G| = 4$, the condition in Theorem 3.2.7 becomes $\chi(f) > 1 - \frac{2}{4} + \frac{1}{16} = \frac{9}{16}$. If $|G| = 5$, then $\chi(f) > \frac{16}{25}$. If $|G| = 10$, then $\chi(f) > \frac{81}{100}$. If $|G| = 100$, then $\chi(f) > \frac{9801}{10000}$. It follows that as the order of the group increases, ensuring that the identity element is preserved under the mapping requires the function to behave homomorphically on almost all pairs.

The following example shows that, for finite groups, the commutativity degree coincides with the homomorphism degree of a specific function. In this regard, the homomorphism degree can be viewed as a generalization of the commutativity degree in finite groups.

Example 3.2.8. Let G be a finite group and $f: G \times G \to G, f(a, b) \mapsto ab$.

$$f\big((a_1, b_1)(a_2, b_2)\big) = f(a_1 a_2, b_1 b_2) = a_1 a_2 b_1 b_2$$
$$f(a_1, b_1) f(a_2, b_2) = a_1 b_1 a_2 b_2.$$

Then, $f\big((a_1, b_1)(a_2, b_2)\big) = f(a_1, b_1) f(a_2, b_2) \Leftrightarrow a_2 b_1 = b_1 a_2$.

Let there be n commuting pairs in $G \times G$. Then, $n = d(G) \cdot |G|^2$. If (a, b) is one such pair, then for all $a', b' \in G$, we require

$$f\big((a', b)(a, b')\big) = f(a', b)f(a, b').$$

In this case, consider the function $f: G \times G \to G$, $f(a, b) = ab$ which satisfies the homomorphism condition for exactly $n \cdot |G|^2$ such pairs. Hence, the homomorphism degree is given by

$$\chi(f) = \frac{n \cdot |G|^2}{|G \times G|^2} = \frac{d(G) \cdot |G|^2 \cdot |G|^2}{|G|^4} = d(G).$$

Similarly, under the notation of Equation (2), if the multiplication function $f: G \times H \to G$, $f(g, h) = gh$ is considered, then $\chi(f) = d(G, H)$.

Example 3.2.9. The commutativity degree of a group G is also equal to the homomorphism degree of the inversion map $i: G \to G$, $i(g) = g^{-1}$; since $i(gh) = i(g)i(h)$ if and only if $(gh)^{-1} = g^{-1}h^{-1}$, that is $gh = hg$.

3.3. Listings

Implementations of SageMath code for some of the examples covered in the main text are shown in this chapter. For commutativity, surjectivity, and homomorphism degree, several examples are provided. SageMath version 10.6 on the CoCalc cloud platform was used for all computations.*

Listing 3.3.1. Commutativity Degree of the dihedral group D_4 (Example 3.1.5)

```
# Define the dihedral group D_4
G = DihedralGroup(4)  # Order 8 group: {e, a, a^2,
a^3, b, ba, ba^2, ba^3}

# Count commuting pairs
count = 0
total = 0

for x in G:
  for y in G:
    total += 1
    if x * y == y * x:
      count += 1

# Compute commutativity degree
commutativity_degree = count / total

# Output
print(f"Number of commuting pairs: {count} out of
{total}")
print("Commutativity degree d(G) = {:.4f}".for-
mat(float(commutativity_degree)))
```

Listing 3.3.2. Output of the commutativity degree computation for D_4

```
Number of commuting pairs: 40 out of 64
Commutativity degree d(G) = 0.6250
```

Listing 3.3.3. Surjectivity degree of the homomorphism $\varphi: V_4 \to D_4$ (Example 3.1.5)

```
# Define domain group V4 ~= Z_2 x Z_2
V4 = AbelianGroup([2, 2])
a, b = V4.gens()

# Define codomain group D4 = Dihedral group of or-
der 8
D4 = DihedralGroup(4)
r, s = D4.gens()  # r: rotation, s: reflection

# Define phi: V4 -> D4 by phi(e)=e, phi(a)=r^2,
phi(b)=e, phi(ab)=r^2
# Use tuples to represent V4 elements correctly

mapping = {
  V4((0, 0)): D4.identity(),  # phi(e) = e
  V4((1, 0)): r^2,            # phi(a) = r^2
  V4((0, 1)): D4.identity(),  # phi(b) = e
  V4((1, 1)): r^2             # phi(ab) = r^2
}

# Define phi as a function
def phi(g):
  return mapping.get(g, None)

# Compute the image set
image = set()
for g in V4:
  val = phi(g)
  if val is not None:
    image.add(val)

# Surjectivity degree: |Im phi| / |D4|
surjectivity_degree = len(image) / D4.order()

# Output
print(f"Image of phi: {image}")
print(f"Surjectivity degree sigma(phi) = {len(image)}
/ {D4.order()} = {float(surjectivity_degree):.4f}")
```

Listing 3.3.4. Output of the surjectivity degree calculation for φ

```
Image of phi: {(1,3)(2,4), ()}
Surjectivity degree sigma(phi) = 2 / 8 = 0.2500
```

Listing 3.3.5. Homomorphism degree of the function $f: \mathbb{Z}_5 \to \mathbb{Z}_6$ (Example 3.2.2)

* https://cocalc.com

```
# Define domain and codomain groups
Z5 = IntegerModRing(5)   # Z_5 under addition
Z6 = IntegerModRing(6)   # Z_6 under addition

# Define the function f: Z5 -> Z6
f_values = {
   Z5(0): Z6(0),
   Z5(1): Z6(3),
   Z5(2): Z6(5),
   Z5(3): Z6(1),
   Z5(4): Z6(2)
}

def f(x):
   return f_values.get(x, None)

# Count satisfying pairs
count = 0
total = 0

for a in Z5:
   for b in Z5:
      total += 1
      lhs = f(a + b)
      rhs = f(a) + f(b)
      if lhs == rhs:
         count += 1

# Compute homomorphism degree
hd = count / total

# Output
print("Number of pairs (a, b) satisfying f(a + b) = f(a)
+ f(b): {} out of {}".format(count, total))
print("Homomorphism degree = {:.4f}".for-
mat(float(hd)))
```

Listing 3.3.6. Output of the homomorphism degree calculation for f

```
Number of pairs (a, b) satisfying f(a + b) = f(a) + f(b):
11 out of 25
Homomorphism degree = 0.4400
```

Listing 3.3.7. Homomorphism degree of the function $f: \mathbb{Z}_4 \times \mathbb{Z}_2 \to D_4$ (Example 3.2.6)

```
# Define domain and codomain groups
G = AbelianGroup([4, 2])   # Z_4 x Z_2
H = DihedralGroup(4)       # D_4, order 8

# Get generators
r, s = H.gens()          # r: rotation of order 4, s: reflec-
tion

# Define the bijection f: G -> H by f((a,b)) = r^a if b=0,
else s*r^a
mapping = {}
for a in range(4):
   for b in range(2):
      g = G((a, b))
      if b == 0:
         mapping[g] = r**a
      else:
         mapping[g] = s * (r**a)

def f(x):
   return mapping.get(x, None)

# Count pairs satisfying the homomorphism condi-
tion f(x*y) = f(x)*f(y)
count = 0
total = 0
for x in G:
   for y in G:
      total += 1
      lhs = f(x * y)
      rhs = f(x) * f(y)
      if lhs == rhs:
         count += 1

# Compute homomorphism degree
homomorphism_degree = count / total

# Output
print("Number of pairs (x, y) satisfying f(x*y) =
f(x)*f(y): {} out of {}".format(count, total))
print("Homomorphism degree = {:.4f}".for-
mat(float(homomorphism_degree)))
```

Listing 3.3.8. Output of the homomorphism degree calculation for f

```
Number of pairs (x, y) satisfying f(x*y) = f(x)*f(y): 48
out of 64
Homomorphism degree = 0.7500
```

4. Discussion and Conclusion

This study presents a structural framework that connects the commutativity degree, the surjectivity degree and the homomorphism degree in finite groups and investigates the interaction of these probabilistic measures in the context of group homomorphisms. In the study, various new lower and upper bounds for the commutativity degree are derived depending on the concept of commutativity degree defined for group homomorphisms. These bounds allow to quantitatively evaluate the behavior of group structures under homomorphism. Thus, the extent to which homomorphic images preserve structure can be numerically analyzed. This approach allows to study the structural properties of groups with statistical measures. As a result, a deep connection has been established between probabilistic methods and structural aspects of group theory.

The study further explores how commutativity and homomorphic behavior are quantitatively related. To do this, it introduces the concept of homomorphism degree, which applies to functions that are only partly homomorphic. This notion serves as a means of assessing how well a function preserves the group structure. Moreover, it contributes to the analysis of probabilistic relations in group structures by determining the extent to which functions approach homomorphic structure. This approach allows for quantitative analysis of the structural effects of out-group functions. The homomorphism degree has been analyzed especially in terms of its structural implications regarding the preservation of the unit element. It is shown that when the homomorphism degree exceeds a certain threshold, the identity element in the domain of the function necessarily maps to the identity element in the range. This situation reveals that homomorphic behavior can lead to structural imperatives.

The concepts introduced here may also be applied to the study of group extensions, approximate homomorphisms, and structural properties of homomorphic images. Future work may extend these ideas to group rings, crossed modules, and operator-theoretic contexts, motivated by recent developments in Banach-algebra methods and Wiener-algebra techniques. A further possible direction is the study of a new quantitative invariant related to injective maps between groups, namely the embeddability degree, which may provide an additional framework for extending the ideas introduced here.

To conclude, the definitions, theorems and examples discussed in this study offer a new perspective to examine the structural effects of homomorphisms in finite groups through several probabilistic concepts. The results give a clear and combined way to understand the topic that analyzes commutativity and homomorphic behavior together in theory of finite groups. This approach allows for better quantitative understanding within the scope of group theory. Future studies could look into applying these ideas to larger classes of groups or new algebraic structures.

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

References

[1] Gallagher, P. X. 1970. The number of conjugacy classes in a finite group. Mathematische Zeitschrift, 118(3), 175–179.

[2] Gustafson, W. H. 1973. What is the probability that two group elements commute? The American Mathematical Monthly, 80(9), 1031–1034.

[3] Rusin, D. 1979. What is the probability that two elements of a finite group commute? Pacific Journal of Mathematics, 82(1), 237–247.

[4] Gürdal, M. 2009. Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra. Expositiones Mathematicae, 27(2), 153–160.

[5] Karaev, M., Gürdal, M., Huban, M. B. 2016. Reproducing kernels, Engliš algebras and some applications. Studia Mathematica, 232(2), 113–141.

[6] Karaev, M., Gürdal, M., Saltan, S. 2011. Some applications of Banach algebra techniques. Mathematische Nachrichten, 284(13), 1678–1689.

[7] Gürdal, M., Garayev, M. T., Saltan, S. 2015. Some concrete operators and their properties. Turkish Journal of Mathematics, 39(6), 970–989.

[8] Gürdal, M. 2009. On the extended eigenvalues and extended eigenvectors of shift operator on the Wiener algebra. Applied Mathematics Letters, 22(11), 1727–1729.

[9] Lescot, P. 1995. Isoclinism classes and commutativity degrees of finite groups. Journal of Algebra, 177(3), 847–869.

[10] Lescot, P. 2001. Central extensions and commutativity degree. Communications in Algebra, 29(10), 4451–4460.

[11] Moghaddam, M. R. R., Salemkar, A. R., Chiti, K. 2005. n-Isoclinism classes and n-nilpotency degree of finite groups. Algebra Colloquium, 12(2), 255–261.

[12] Erfanian, A., Rezaei, R., Lescot, P. 2007. On the relative commutativity degree of a subgroup of a finite group. Communications in Algebra, 35(12), 4183–4197.

[13] Tărnăuceanu, M. 2009. Subgroup commutativity degrees of finite groups. Journal of Algebra, 321(9), 2508–2520.

[14] Barzegar, R., Erfanian, A., Farrokhi, M. D. G. 2013. Finite groups with three relative commutativity degrees. Bulletin of the Iranian Mathematical Society, 39(2), 271–280.

[15] Rezaei, R., Erfanian, A. 2014. A note on the relative commutativity degree of finite groups. Asian-European Journal of Mathematics, 7(1), 1450017.

[16] Pournaki, M. R., Sobhani, R. 2008. Probability that the commutator of two group elements is equal to a given element. Journal of Pure and Applied Algebra, 212(4), 727–734.

[17] Nath, R. K., Das, A. K. 2010. On a lower bound of commutativity degree. Rendiconti del Circolo Matematico di Palermo, 59(1), 137–142.

[18] Pirzadeh, M., Hashemi, M. 2022. A generalization of the n^th-commutativity degree in finite groups. Computational Sciences and Engineering, 2(1), 33–40.

[19] Ghaneei, M., Azadi, M. 2021. The nth commutativity degree of semigroups. Journal of Linear and Topological Algebra, 10(3), 225–233.

[20] Nath, R. K., Das, A. K. 2011. On generalized commutativity degree of a finite group. The Rocky Mountain Journal of Mathematics, 41(6), 1987–2000.

[21] Chashiani, A., Rezaei, R. 2021. On the commutativity degree of a group algebra. Afrika Matematika, 32(5), 1137–1145.

[22] Arvasi, Z., Çağlayan, E. I., Odabaş, A. 2022. Commutativity degree of crossed modules. Turkish Journal of Mathematics, 46(1), 242–256.

[23] Çetin, S., Gürdal, U. 2024. Crossed modules with action. Ukrainian Mathematical Journal, 76(4), 649–668.