Characterization of Tzitzeica Curves by Using q-frame in Euclidean 3-space

Gamze Nur ŞEN[1] , Gül UĞUR KAYMANLI[*2] , Cumali EKİCİ[3]

[1,3] Eskişehir Osmangazi Üniversitesi, Fen Fakültesi, Matematik ve Bilgisayar Bilimleri Bölümü, 26040, Eskişehir, Türkiye

[2] Çankırı Karatekin Üniversitesi, Fen Fakültesi, Matematik Bölümü, 18200, Çankırı, Türkiye

Abstract: Bu çalışmada, q-çatısı kullanılarak 3 boyutlu Öklid uzayında Tzitzeica (T-eğrisi), küresel (s-eğrisi) ve küresel Tzitzeica (sT-eğrisi) eğrileri analiz edilmiştir. Herhangi bir noktasındaki burulmasının, o noktadaki oskülatör düzlem ile orijin arasındaki uzaklığın karesine oranı sıfırdan farklı sabit bir değer olan bir Tzitzeica eğrisi vardır. 3-boyutlu Öklid uzayında Tzitzeica ve küresel eğrileri q-çatısı kullanılarak tanımlayıp ilgili teoremleri inceledikten sonra, küresel Tzitzeica eğrileriyle ilgili teoremler ispatlanmıştır.

3-boyutlu Öklid Uzayında q-çatılı Tzitzeica Eğrilerinin Karakterizasyonu

Öz: Bu çalışmada, q-çatısı kullanılarak 3 boyutlu Öklid uzayında Tzitzeica (T-eğrisi), küresel (s-eğrisi) ve küresel Tzitzeica (sT-eğrisi) eğrileri analiz edilmiştir. Herhangi bir noktasındaki burulmasının, o noktadaki oskülatör düzlem ile orijin arasındaki uzaklığın karesine oranı sıfırdan farklı sabit bir değer olan bir Tzitzeica eğrisi vardır. 3-boyutlu Öklid uzayında Tzitzeica ve küresel eğrileri q-çatısı kullanılarak tanımlayıp ilgili teoremleri inceledikten sonra, küresel Tzitzeica eğrileriyle ilgili teoremler ispatlanmıştır.

1. Introduction

Romanian Mathematician Gheorghe Tzitzeica introduced a class of curves known as Tzitzeica curves in which the ratio between the torsion and the square of the distance from its origin to the plane spanned by tangent and normal vectors at an arbitrary point of the curve, is constant [1,2].

Since the curves have many applications in science and engineering, they have attracted considerable attention as one of the fascinating topics in differential geometry. According to [3], the authors examined that there is no T-curve with constant curvature by using Frenet frame in Euclidean 3-space but proved some statements about the relations between Tzitzeica Salkowski and anti-Salkowski curves. The characterizations of T-curves and surfaces were given as a Ph.D. thesis by Tunc in [4]. Most recently, researchers have studied Tzitzeica curves with several frames in Euclidean space. The authors have characterized the Tzitzeica curves in terms of type 2-positional adapted frames in 3-dimensional Euclidean space and have given some results on the spherical Tzitzeica curves [5]. Eren and Ersoy have used well-known Bishop frame to examine the T-curve and have given the condition for spherical image of a curve to be T-curve in [6] by using both type-1 and type-2 Bishop frame. The authors have focused on Tzitzeica curves with framed curves in their study [7]. In 2022, Sariaydin and Yazla defined the T-curve with q-frame in 3-dimensional Euclidean space in [8]; however, we, in this work, have examined Tzitzeica and spherical Tzitzeica curves and have proved some related theorems in detail by using q-frame used in [9, 10, 11]. Most recently, [12] has worked on Tzitzeica curves according to modified orthogonal frames. We have discussed special theorems and provided information about the spherical Tzitzeica curve in three-dimensional Euclidean space, while in [13] it was considered in Minkowski space.

2. Materials and Methods

In E^3, the metric used is $\langle u, w \rangle = u_1 w_1 + u_2 w_2 + u_3 w_3$ for any vectors $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$. The norm of the vector w is given by

$$\|w\| = \sqrt{|\langle w, w \rangle|}.$$

Corresponding author: gulugurk@karatekin.edu.tr.

A Tzitzeica curve in 3-dimensional Euclidean space is a curve $\alpha = \alpha(s)$ for which the ratio of its torsion τ and the square of the distance d_{osc} from the origin to the osculating plane (osc-plane) at any point $\alpha(s)$ of the curve is constant where $d_{osc} = \langle b, \alpha \rangle$ [1].

Let $\alpha(s)$ be a curve and s be an arc length parameter with a non-vanishing second derivative. The Frenet formulas are given

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where Frenet vectors and curvature functions are

$$t = \frac{\alpha'}{\|\alpha'\|}, \qquad b = \frac{(\alpha' \wedge \alpha'')}{\|\alpha' \wedge \alpha''\|}, \qquad n = b \wedge t,$$
$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2},$$

respectively [14]. Besides Frenet frame, we use the q-frame $\{t, n_q, b_q, k\}$ where k is the projection vector; that is $k_x = (1,0,0), k_y = (0,1,0), k_z = (0,0,1)$. In order to define q-frame, we first need to give relation between Frenet frame and q-frame with the help of angle θ between n and q-normal n_q vectors. The relation matrix takes the form

$$\begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

In this context, the q-frame of α can be given ([15]) in the following form

$$\begin{bmatrix} t' \\ n_q' \\ b_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}$$

where q-curvatures k_1, k_2, k_3 are

$$k_1 = \langle t', n_q \rangle$$
$$k_2 = \langle t', b_q \rangle$$
$$k_3 = \langle n_q', b_q \rangle$$

3.Results

3.1. Characterization of tzitzeica curves with q-frame in 3-dimensional euclidean space E^3

In this section, we characterize the Tzitzeica curves by using q-frame in terms of q-curvatures in 3-dimensional Euclidean space.

For a space curve $\alpha: I \subset \mathbb{R} \to E^3$, the planes at each point of α spanned by $\{t, n_q\}$, $\{n_q, b_q\}$ and $\{t, b_q\}$ are called q_osc-plane, q-normal plane and q-rectifying

plane, respectively. If the position vector of the curve lies in the q_osc-plane, q-normal plane and q-rectifying plane then it is called q_osc curve, q-normal curve and q-rectifying curve, respectively.

Definition 3.1 Let α be a unit speed curve with $k_1 > 0$ and $k_3 \neq 0$ in E^3. If the curve α satisfies the condition

$$\frac{k_3}{d_{qos}^2} = c,$$

then α is called q-Tzitzeica curve such that d_{qos} is the distance from the origin to the q_osc-plane at any point of curve and $c \neq 0$ is a real number [8].
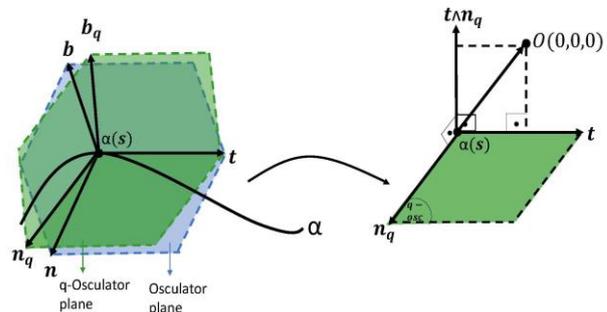

Figure 1. The distance d_{qos} from the origin to the q_osc-plane in E^3

Simple calculations show that

$$d(O, qos) = d_{qos}$$
$$= \left| \frac{\langle \alpha(s), t \times n_q \rangle}{\|t \times n_q\|} \right|$$
$$= |\langle b_q, \alpha \rangle|.$$

Corollary 3.2 Let α be a unit speed curve, $\{t, n, b\}$ be its Frenet frame and $\{t, n_q, b_q\}$ be its q-frame. Then the equation

$$\langle b_q, \alpha \rangle = \frac{sin\theta}{k_1} + cos\theta \langle b, \alpha \rangle$$

is provided where θ is the angle between q-frame vectors and Frenet frame vectors. If the angle θ is vanishing, then there is an equation

$$d_{osc} = d_{qos}$$

between the distance from the origin to the osc-plane and the q_osc-plane.

Proof Let α be a unit speed s-curve with radius c. One have

$$c = \|\alpha(s)\| = \sqrt{\langle \alpha(s), \alpha(s) \rangle}.$$

Taking derivative gives us $\langle \alpha, n \rangle = -\frac{1}{\kappa}$. On the other hand using the relation between Frenet frame and q-frame

$$\boldsymbol{b_q} = -sin\theta\ \boldsymbol{n} + cos\theta\ \boldsymbol{b}$$

and the linearity of dot product, one can easily find the equation.

Theorem 3.3 Let α be a unit speed curve in E^3. If α is a T-curve then

$$k_3'(s)\langle\boldsymbol{b_q},\alpha\rangle + 2k_2(s)k_3(s)\langle t,\alpha\rangle + 2k_3^2(s)\langle\boldsymbol{n_q},\alpha\rangle = 0$$

is obtained [8].

Proof Let α be a unit speed curve in E^3. We can write

$$\frac{k_3(s)}{\langle\boldsymbol{b_q},\alpha\rangle^2} = c \neq 0$$

by using the condition of T-curve with q-frame. By taking the derivative of this expression and using the properties of q-frame, the equation

$$\frac{k_3'(s)\langle\boldsymbol{b_q},\alpha\rangle^2 - 2k_3(s)\langle\boldsymbol{b_q},\alpha\rangle\langle\boldsymbol{b_q'},\alpha\rangle}{\langle\boldsymbol{b_q},\alpha\rangle^4} = 0$$

is obtained. If the derivative formula for the $\boldsymbol{b_q}$, and the linearity of dot product are used, we can get

$$\frac{\begin{bmatrix} k_3'(s)\langle\boldsymbol{b_q},\alpha\rangle^2 + 2k_2(s)k_3(s)\langle\boldsymbol{b_q},\alpha\rangle\langle t,\alpha\rangle \\ +2k_3^2(s)\langle\boldsymbol{b_q},\alpha\rangle\langle\boldsymbol{n_q},\alpha\rangle \end{bmatrix}}{\langle\boldsymbol{b_q},\alpha\rangle^4} = 0.$$

When the necessary simplifications are made, we can conclude

$$k_3'(s)\langle\boldsymbol{b_q},\alpha\rangle + 2k_2(s)k_3(s)\langle t,\alpha\rangle + 2k_3^2(s)\langle\boldsymbol{n_q},\alpha\rangle = 0.$$

Now, we need to write the position vector satisfying

$$\alpha(s) = s_1(s)\ t + s_2(s)\ \boldsymbol{n_q} + s_3(s)\ \boldsymbol{b_q}$$

for some differentiable functions $s_i(s), 0 \leq i \leq 2$, and q-frame vectors. In order to find $s_i(s), 0 \leq i \leq 2$, we first need to use the definition of unit speed curve and the property of dot product. With the help of these properties, we have

$$\langle\alpha(s), t\rangle = s_1(s)\langle t,t\rangle + s_2(s)\langle\boldsymbol{n_q},t\rangle + s_3(s)\langle\boldsymbol{b_q},t\rangle$$

and the equation

$$s_1(s) = \langle t,\alpha\rangle$$

is easily found. Using $c = \|\alpha(s)\|$, and taking derivative of this equation, we have $s_1(s) = 0$. Taking derivative of $s_1(s)$ and using derivative formula for the first q-frame vector t, the equation

$$k_1\langle\boldsymbol{n_q},\alpha\rangle+k_2\langle\boldsymbol{b_q},\alpha\rangle = -1 \qquad (1)$$

is obtained, and then

$$k_1'\langle\boldsymbol{n_q},\alpha\rangle + k_1\big(\langle\boldsymbol{n_q'},\alpha\rangle + \langle\boldsymbol{n_q},\alpha'\rangle\big) + k_2'\langle\boldsymbol{b_q},\alpha\rangle \\ + k_2\big(\langle\boldsymbol{b_q'},\alpha\rangle + \langle\boldsymbol{b_q},\alpha'\rangle\big) = 0$$

is calculated. After using $t = \alpha'$ and the derivative formula for the both $\boldsymbol{n_q}$ and $\boldsymbol{b_q}$, and the orthogonality property of frame, respectively, the equation

$$\langle\boldsymbol{n_q},\alpha\rangle(k_1' - k_2k_3) + \langle\boldsymbol{b_q},\alpha\rangle(k_2' + k_1k_3) = 0 \quad (2)$$

is clear. With the help of (1) and (2), that is, if the equation (1) is multiplied by $-(k_2' + k_1k_3)$ and the equation (2) is multiplied by k_2 and added side by side,

$$\big(-k_1(k_2' + k_1k_3) + k_2(k_1' - k_2k_3)\big)\langle\boldsymbol{n_q},\alpha\rangle \\ = k_2' + k_1k_3$$

is obtained. Using last equation, we can have

$$\langle\boldsymbol{n_q},\alpha\rangle = \frac{1}{-k_1 + \frac{k_2(k_1' - k_2k_3)}{k_2' + k_1k_3}}.$$

Doing basic calculations gives us

$$s_2(s) = \langle\boldsymbol{n_q},\alpha\rangle = \frac{k_2' + k_1k_3}{k_1k_2' + k_1^2k_3 - k_2k_1' + k_2^2k_3}.$$

On the other side, if the equation (1) is multiplied by $-k_1' + k_2k_3$ and the equation (2) is multiplied by k_1 and added side by side,

$$\big(-k_2(k_1' - k_2k_3) + k_1(k_2' + k_1k_3)\big)\langle\boldsymbol{b_q},\alpha\rangle \\ = k_1' - k_2k_3$$

is obtained. Using last equation, we can have

$$\langle\boldsymbol{b_q},\alpha\rangle = \frac{1}{-k_2 + \frac{k_1(k_2' + k_1k_3)}{k_1' - k_2k_3}}.$$

Derivation of $s_2(s) = \langle\boldsymbol{n_q},\alpha\rangle$ gives us

$$\langle\boldsymbol{n_q'},\alpha\rangle + \langle\boldsymbol{n_q},t\rangle = s_2'(s),$$

and using $\boldsymbol{b_q'} = -k_1t + k_3\boldsymbol{b_q}$ and $\langle t,\alpha\rangle = 0$, we obtain

$$\langle\boldsymbol{b_q},\alpha\rangle = \frac{s_2'(s)}{k_3} = s_3(s).$$

Taking derivative of $s_2(s)$ and dividing by k_3, we conclude with

$$s_3(s) = \frac{\begin{bmatrix} k_1'\left(k_3(k_2^2 - k_1^2) - k_1 k_2' - \frac{k_1 k_2 k_3'}{k_3} - \frac{k_2 k_2''}{k_3}\right) \\ +k_2' k_2\left(-2k_1 k_3 + \frac{k_1''}{k_3} - \frac{k_2 k_3'}{k_3}\right) \\ -k_2((k_1')^2 + 2(k_2')^2 - k_1'' k_1 - k_2'' k_2) \end{bmatrix}}{(k_1 k_2' + k_1^2 k_3 - k_2 k_1' + k_2^2 k_3)^2}.$$

Corollary 3.4 Let α be a unit speed curve in E^3. α is a sT-curve if and only if the equation

$$k_3'\langle \boldsymbol{b_q}, \alpha \rangle + 2k_3^2\langle \boldsymbol{n_q}, \alpha \rangle = 0$$

is satisfied.

Theorem 3.5 Let $M \subset E^3$ be a curve with coordinate neighborhood (I, α). The geometric location of the centers of the spheres, which are the three common points of M and infinity, is

$$c(s) = \alpha(s) + s_2(s)\,\boldsymbol{n_q} + \lambda\,\boldsymbol{b_q},$$

so that the q-vectors at the point $c(s)$ corresponding to the point $s \in I$ are $\{\mathbf{t}, \boldsymbol{n_q}, \boldsymbol{b_q}\}$ where $\lambda \in \mathbb{R}$ and $s_2 : I \to \mathbb{R}$ is the same as the coefficient of $\boldsymbol{n_q}$ in the equation of spherical curve.

Proof The equation of the sphere with center $c(s)$ and radius r is given $f : I \to \mathbb{R}$, $f(s) = \langle \alpha(s) - c(s), \alpha(s) - c(s) \rangle - r^2$. The equation

$$f(s) = f'(s) = f''(s) = 0$$

is satisfied since there are three common points with the spheres of the point $c(s)$ of M, that is,
$$S^2 = \{x \mid x \in E^3, \langle x - c(s), x - c(s) \rangle = r^2\}.$$
Since $f(s) = 0$, we get

$$\langle \alpha(s) - c(s), \alpha(s) - c(s) \rangle = r^2.$$

Taking derivative of this equation and using the equation $f'(s) = 0$, one can get

$$\langle \alpha'(s), \alpha(s) - c(s) \rangle + \langle \alpha(s) - c(s), \alpha'(s) \rangle = 0,$$

and then

$$\langle \boldsymbol{t}, \alpha(s) - c(s) \rangle = 0. \tag{3}$$

On the other hand, we have

$$\alpha(s) - a(s) = s_1(s)\boldsymbol{t}(s) + s_2(s)\boldsymbol{n_q}(s) + s_3(s)\boldsymbol{b_q}(s) \tag{4}$$

for q-frame vectors $\boldsymbol{t}, \boldsymbol{n_q}, \boldsymbol{b_q}$. If we use linearity of dot product and the properties of base, respectively if the dot product of the equation (4) with $\boldsymbol{t}(s)$ is taken and the equation (3) is used, we can get
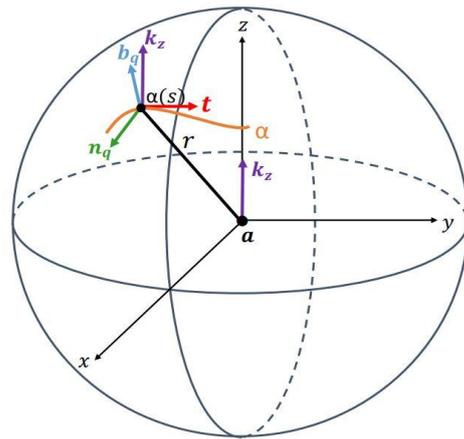
$$s_1(s) = 0.$$

Therefore, the proof is completed.

Figure 2. Spherical curve with the center $a(s)$ and radius r

Theorem 3.6 Let $M \subset E^3$ be a curve with coordinate neighborhood (I, α). For any $s \in I$, when $k_2 = 0$ at the point $\alpha(s)$, the radius of osc-sphere is constant if and only if the centers of osc-spheres are the same such that $s_3 \neq 0$, $k_3 \neq 0$.

Proof Let r be the radius and the centers of osc-sphere at the point $\alpha(s) \in M$ are

$$c(s) = \alpha(s) + s_2(s)\boldsymbol{n_q} + s_3(s)\boldsymbol{b_q}.$$

The radius is found as

$$\begin{aligned} r &= \|\overrightarrow{\alpha c}\| \\ &= \|c(s) - \alpha(s)\| \\ &= \|s_2(s)\boldsymbol{n_q} + s_3(s)\boldsymbol{b_q}\| \\ &= \sqrt{s_2^2(s) + s_3^2(s)}. \end{aligned}$$

Since r is constant, when we take derivation, we get

$$s_2(s)s_2'(s) + s_3(s)s_3'(s) = 0.$$

Doing some basic calculations and using $s_3(s) = \frac{s_2'(s)}{k_3}$ gives us

$$s_3'(s) + s_2(s)k_3 = 0.$$

On the other hand, since $c(s) = \alpha(s) + s_2(s)\boldsymbol{n_q} + s_3(s)\boldsymbol{b_q}$, the equation

$$c'(s) = (1 - s_2(s)k_1 - s_3(s)k_2)\boldsymbol{t} + (s_2(s)k_3 + s_3'(s))\boldsymbol{b_q},$$

holds. Putting $s_2(s) = \frac{k_2' + k_1 k_3}{k_1 k_2' + k_1^2 k_3 - k_2 k_1' + k_2^2 k_3}$ and $s_3(s) = \frac{s_2'(s)}{k_3}$ and $k_2 = 0$ into the last equation gives

$$c'(s) = (s_2(s)k_3 + s_3'(s))\boldsymbol{b_q}.$$

Since $s_2(s)k_3 + s_3'(s) = 0$, we conclude $c'(s) = 0$. Then one can say $c(s)$ is constant for any $s \in I$.

In order to show the necessary condition of the proof, assume $c(s)$ is constant for any $s \in I$. Since $r = \|\overrightarrow{\alpha c}\|$, we have

$$\langle \alpha(s) - c(s), \alpha(s) - c(s) \rangle = r^2.$$

Taking derivative of the last equation with respect to parameter s helps us to find

$$\langle c'(s), c(s) - \alpha(s) \rangle = r(s) \frac{dr}{ds}|_s.$$

Since $c(s)$ is constant, we have $r(s) \frac{dr}{ds}|_s = 0$. We can find $\frac{dr}{ds}|_s = 0$ and therefore, $r(s)$ is constant for any $s \in I$. Otherwise, we have a contradiction with $s_2(s) \neq 0$ and $s_3(s) \neq 0$. Therefore, the statement is proven.

Theorem 3.7 Let $M \subset E^3$ be a space curve with coordinate neighborhood (I, α). Suppose $k_3 \neq 0, k_2 = 0$ and $s_3(s) \neq 0$ for any $s \in I$. In this case, M is a s-curve if and only if the centers of osc-spheres at the point $\alpha(s)$ are the same for any $s \in I$.

Proof If α be a s-curve, then it's obvious $\|\overrightarrow{O\alpha}\| = r$. On the other hand, assume that the centers of osc-spheres at the point $\alpha(s)$ are b for any $s \in I$. Since the radius of the osculating spheres are also constant and equal to r, for all $\alpha(s) \in M$, we have

$$d(\alpha(s), b) = r.$$

If $b = c$, then

$$r = \|\alpha(s) - c\| = d(\alpha(s), b).$$

Since r is constant, we have $d(\alpha(s), b)$ constant.

Theorem 3.8 Given the arc-lengh parameter $s \in I$ and $M \subset E^3$ with coordinate neighborhood (I, α) with $k_3 \neq 0, k_2 = 0$, M be a s-curve if and only if

$$s_3'(s) + s_2(s)k_3 = 0$$

when $s_3(s) \neq 0$ for any $s \in I$.

Proof Let the curve M be spherical. Since the center $c(s)$ of the osculating spheres is constant for all $s \in I$, taking the derivative of $c(s)$, we obtain

$$(s_3'(s) + s_2(s)k_3(s)) b_q(s) = 0$$

for all $s \in I$. Hence, we find

$$s_3'(s) + s_2(s)k_3(s) = 0.$$

On the other hand, suppose that $s_3'(s) + s_2(s)k_3(s) = 0$. Then we get

$$c'(s) = 0 \Rightarrow c(s) = \text{constant}$$

for all $s \in I$. Therefore the curve M is spherical.

Theorem 3.9 Let $\alpha: I \subset \mathbb{R} \to E^3$ be an arc-length s-curve with q-frame in E^3. If the curve α be a T-curve then there is an equation

$$\frac{k_3'}{2k_3^2} = \frac{k_2' + k_1 k_3}{k_1' - k_2 k_3}$$

between the curvatures of the curve α.

Proof The proof is completed by combining the Theorem 3.3. with the condition $k_2 = 0$.

Theorem 3.10 Let $\alpha: I \subset \mathbb{R} \to E^3$ be a unit speed sT-curve with q-frame in E^3 such that $k_3 \neq 0, k_2 = 0$ and $s_3(s) \neq 0$. The third curvature of the curve α, k_3 is equal to

$$k_3 = \sqrt{\frac{k_1' k_3'}{k_1 k_3}}$$

Proof This follows directly from the equation $s_3'(s) + s_2(s)k_3(s) = 0$ and Theorem 3.9.

Acknowledgment

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

References

[1] Tzitzeica, G. 1911. Sur certaines courbes gauches. Ann. De I'Ec, Normale Sup., 28, 9-32.

[2] Tzitzeica, G. 1925. Sur certaines courbes gauches. Ann. De I'Ec, Normale Sup., 42, 379-390.

[3] Bayram, B., Tunç, E., Arslan, K., Öztürk, G. 2018. On Tzitzeica Curves in Euclidean 3-Space $E^3.$ Facta Universitatis, Series: Mathematics and Informatics, 33(3), 409-416.

[4] Tunç, E. 2021. Tzitzeica eğrilerinin ve yüzeylerinin bir karakterizasyonu, Balıkesir Üniversitesi, Fen Bilimleri Enstitüsü, Doktora Tezi, 108s, Balıkesir.

[5] Özen, K.E., Tosun, M., Avcı, K. 2022. Type 2-Positional Adapted Frame and Its Application to

Tzitzeica and Smarandache Curves. Karatekin University Journal of Science, 1(1), 42–53.

[6] Eren, K., Ersoy, S. 2022. Characterizations of Tzitzeica Curves Using Bishop Frames. Mathematical Methods in the Applied Sciences, 45(18), 12046-12059.

[7] Yazıcı, B.D., Karakuş, S.Ö., Tosun, M. 2022. On Framed Tzitzeica Curves in Euclidean Space. Facta Universitatis, Series: Mathematics and Informatics, 37(2), 307-319.

[8] Sarıaydın, M.T., Yazla, A. 2022. On Tzitzeica Curves According to Q-Frame in Euclidean 3-Space. The Eurasia Proceedings of Science Technology Engineering and Mathematics, 17, 19-25.

[9] Ekici, C., Kaymanlı, G.U., Okur, S. 2021. A New Characterization of Ruled Surfaces According to q-frame Vectors in Euclidean 3-space. International Journal of Mathematical Combinatorics, 3, 20-31.

[10] Kaymanli, G.U., Ekici, C., Koçak, M. 2022. On The Hasimoto Surfaces in Euclidean 3-space. Journal of Science and Arts, 22(4), 883-890.

[11] Kaymanli, G.U., Korpinar, T. 2020. A Study on the Harmonic Evolute Surfaces of Quasi Binormal Surfaces. Journal of Science and Arts, 4(53), 881-892.

[12] Damar, E. 2025. Tzitzeica Curves According to Modified Orthogonal Frames. Cumhuriyet Science Journal, 46(3), 595-600.

[13] Kaymanlı, G.U., Şen, G.N., Ekici, C. 2024. Tzitzeica Curves with q-frame in Three-dimensional Minkowski Space. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 73(4), 957-968.

[14] Gray, A., Salamon, S., Abbena, E. 2006. Modern Differential Geometry of Curves and Surfaces with Mathematica, Chapman and Hall/CRC, 1016s,

[15] Dede, M., Ekici, C., Görgülü, A. 2015. Directional q-frame along a Space Curve. IJARCSSE, 5(12), 775-780.