Süleyman Demirel Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 22, Sayı 3, 1169-1174, 2018

Süleyman Demirel University
Journal of Natural and Applied Sciences
Volume 22, Issue 3, 1169-1174, 2018

DOI: 10.19113/sdufenbed.470192

On the Approximation to Complex Matrix-valued Functions by Using Solutions of Partial Complex Differential Equation in Matrix Form

Yeşim SAĞLAM ÖZKAN[*1], **Sezayi HIZLIYEL**[2]

[1,2]Bursa Uludag University, Faculty of Arts and Science, Department of Mathematics, 16059, Bursa, Turkey

(Alınış / Received: 20.11.2017, Kabul / Accepted: 12.10.2018, Online Yayınlanma / Published Online: 13.10.2018)

Keywords
Generalized Beltrami systems,
Generalized Q-holomorphic functions,
Weierstrass-Stone approximation

Abstract: In this work, we seek the solutions of the equation

$$\frac{\partial w}{\partial \bar{\phi}} = Aw + B\overline{w}$$

with linear coefficients

$$\begin{aligned} A &= \alpha^{(0)} + \alpha^{(1)}\phi + \alpha^{(2)}\overline{\phi}, \\ B &= \beta^{(0)} + \beta^{(1)}\phi + \beta^{(2)}\overline{\phi}, \end{aligned}$$

such that using this solutions we approximated to complex matrix valued function which possess the form $w = K^{(0)} + \phi K^{(1)} + \bar{\phi} K^{(2)}$. Here ϕ is a generating solution for Q-holomorphic functions.

Matris Formda Kompleks Değerli Kısmi Diferensiyel Denklemlerin Çözümleri Yardımıyla Kompleks Matris Değerli Fonksiyonlara Yaklaşım Üzerine

Anahtar Kelimeler
Genelleştirilmiş Beltrami sistemleri,
Genelleştirilmiş Q-holomorf fonksiyonlar,
Weierstrass-Stone yaklaşım teoremi

Özet: Bu çalışmada

$$\begin{aligned} A &= \alpha^{(0)} + \alpha^{(1)}\phi + \alpha^{(2)}\overline{\phi}, \\ B &= \beta^{(0)} + \beta^{(1)}\phi + \beta^{(2)}\overline{\phi}, \end{aligned}$$

lineer katsayılara sahip olan

$$\frac{\partial w}{\partial \bar{\phi}} = Aw + B\overline{w}$$

denkleminin çözümleri araştırıldı. Bu çözümler kullanılarak $w = K^{(0)} + \phi K^{(1)} + \bar{\phi} K^{(2)}$ formuna sahip kompleks matris değerli fonksiyonlara yaklaşıldı. Burada ϕ, Q-holomorf fonksiyonlar için bir doğurucu çözümdür.

1. Introduction

A similar theory to analytic function theory was developed by Douglish and Bojarski [1, 2] for elliptic systems which have the form

$$w_{\bar{z}}(z) - Q(z)w_z(z) = 0, \tag{1}$$

where w is a vector of the type $m \times 1$ and Q is a quasidiagonal matrix of the type $m \times m$. Since such systems result from the reductions of first order generalized elliptic systems in the plane to a canonical form, they are investigated. Another generalization was given by Hile [3]. He considered Eq. (1) where w is an $m \times s$ matrix whose elements are complex valued and $Q(z)$ is self commuting complex valued function. If Q doesn't have eigenvalues of magnitude 1 for each z in the domain Ω_0 in the complex plane $\mathbb{C}$, then Hile called the system (1) as generalized Beltrami system. Such a system can not be brought into the quasidiagonal form of Douglish and Bojarski by a similarity transformation [see 3, p. 108]. The solutions of such systems are called as Q-holomophic functions. Hile introduced the notion of generating solution $\phi(z) := \phi_0(z) + N(z)$ to investigate Q-holomophic functions, where N is nilpotent part and ϕ_0 is the main diagonal term of ϕ. ϕ_0 satisfies the Beltrami equation

$$\frac{\partial \phi_0}{\partial \bar{z}} - \lambda \frac{\partial \phi_0}{\partial z} = 0.$$

Using the Beltrami homeomorphism, we may write the generating solution in a more convenient form $\phi = zI +$

* *Corresponding author: ysaglam@uludag.edu.tr*

$N(z)$ [see 4, p. 169], Furthermore, one can write a Q-holomorphic function $\Phi(z)$ as an analytic function of generating solution that is $\Phi(z) \equiv f(\phi(z))$ [3]. So, differentiation with respect to ϕ and $\overline{\phi}$, conjugate to ϕ, can be given as

$$\frac{\partial}{\partial\phi} = (\phi_z\overline{\phi}_{\bar{z}} - \phi_{\bar{z}}\overline{\phi}_z)^{-1}\left[\overline{\phi}_z\frac{\partial}{\partial\bar{z}} - \overline{\phi}_{\bar{z}}\frac{\partial}{\partial z}\right]$$

and

$$\frac{\partial}{\partial\overline{\phi}} = (\phi_z\overline{\phi}_{\bar{z}} - \phi_{\bar{z}}\overline{\phi}_z)^{-1}\phi_z D, \qquad (2)$$

respectively. Here $D := (\partial/\partial\bar{z})I - Q(\partial/\partial z)$. From (2) we can rewrite equation (1) as

$$\frac{\partial w}{\partial\overline{\phi}} = 0.$$

Later in [5, 6], by using the techniques of Vekua and Bers, Hızlıyel and Çağlıyan gave a function theory for the equation

$$\frac{\partial w}{\partial\overline{\phi}} = Aw + B\overline{w}, \qquad (3)$$

where the unknown $w(z) = \{w_{ij}(z)\}$ is an $m \times s$ complex matrix, $Q(z) = \{q_{ij}(z)\}$ is a self commuting complex matrix-valued functions of the type $m \times m$. They assumed $q_{k,k-1} \neq 0$ for $k = 2,\ldots,m$ and $A = \{a_{ij}(z)\}$ and $B = \{b_{ij}(z)\}$ are commuting with Q. Solutions of this equation were called generalized Q-holomorphic functions. The result obtained in the case of Eq. (3) resemble closely those from the classical theory of Vekua [7] and Bers [8]. In [9], Tutschke and Vasudeva considered the Vekua equation in the form

$$w_{\bar{z}} = a(z)w + b(z)\bar{w}$$

with $a(z) = a_0 + a_1 z + a_2\bar{z}$ and $b(z) = b_0 + b_1 z + b_2\bar{z}$ and introduced necessary conditions to have a solution of the linear form $w = k_0 + k_1 z + k_2\bar{z}$ where a_i, b_i, k_i $(i = 0,1,2)$ are complex constants. Also they stated that every complex-valued continuous function can uniformly be approximated on a compact set by polynomials of these solutions. The aim of this study is to show that an approximation to complex matrix valued functions can be obtained using solutions of the equation (3) by the help of Weierstrass-Stone approximation theorem.

2. Vekua Type Equation in Matrix Form with Linear Solutions

Let $w = K^{(0)} + \phi K^{(1)} + \bar{\phi}K^{(2)}$ be a linear generalized Q-holomorphic function [5]. We investigate equations in form (3) with linear coefficients

$$\begin{aligned} A &= \alpha^{(0)} + \alpha^{(1)}\phi + \alpha^{(2)}\overline{\phi}, \\ B &= \beta^{(0)} + \beta^{(1)}\phi + \beta^{(2)}\overline{\phi}, \end{aligned}$$

such that $w = K^{(0)} + \phi K^{(1)} + \bar{\phi}K^{(2)}$ is a solution of this equation where $\alpha^{(i)}, \beta^{(i)}, i = 0,1,2$ are constant matrix commuting with Q and $K^{(i)}, i = 0,1,2$, are constant matrix with $m \times s$ type. Since the matrices A and B are commuting with Q, they can be written as $A = A_0 I + N_A$ and $B = B_0 I + N_B$ respectively [see 5, p.439]. Where $A_0 = \alpha^{(0)}_{11} + \alpha^{(1)}_{11}z + \alpha^{(2)}_{11}\bar{z}$ and $B_0 = \beta^{(0)}_{11} + \beta^{(1)}_{11}z + \beta^{(2)}_{11}\bar{z}$ are the main diagonal term of A, B and N_A, N_B are the nilpotent part of A, B respectively. Substituting $w = K^{(0)} + \phi K^{(1)} + \bar{\phi}K^{(2)}$ into Eq. (3), we obtain

$$\begin{aligned} K^{(2)} &= \left(\alpha^{(0)} + \alpha^{(1)}\phi + \alpha^{(2)}\overline{\phi}\right)\left(K^{(0)} + \phi K^{(1)} + \overline{\phi}K^{(2)}\right) \\ &\quad + \left(\beta^{(0)} + \beta^{(1)}\phi + \beta^{(2)}\overline{\phi}\right)\left(\overline{K}^{(0)} + \overline{\phi}\,\overline{K}^{(1)} + \phi\overline{K}^{(2)}\right). \end{aligned}$$

By equating the coefficients of $1, \phi, \overline{\phi}, \phi\phi, \phi\overline{\phi}, \overline{\phi}\overline{\phi}$, we obtain the following system

$$\begin{cases} \alpha^{(0)}K^{(0)} + \beta^{(0)}\overline{K}^{(0)} = K^{(2)}, \\ \alpha^{(0)}K^{(1)} + \alpha^{(1)}K^{(0)} + \beta^{(0)}\overline{K}^{(2)} + \beta^{(1)}\overline{K}^{(0)} = 0, \\ \alpha^{(0)}K^{(2)} + \alpha^{(2)}K^{(0)} + \beta^{(0)}\overline{K}^{(1)} + \beta^{(2)}\overline{K}^{(0)} = 0, \\ \alpha^{(1)}K^{(1)} + \beta^{(1)}\overline{K}^{(2)} = 0, \\ \alpha^{(1)}K^{(2)} + \alpha^{(2)}K^{(1)} + \beta^{(1)}\overline{K}^{(1)} + \beta^{(2)}\overline{K}^{(2)} = 0, \\ \alpha^{(2)}K^{(2)} + \beta^{(2)}\overline{K}^{(1)} = 0, \end{cases} \qquad (4)$$

where $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \beta^{(0)}, \beta^{(1)}, \beta^{(2)}$ are coefficients that should be found. Equation (3) can be written as

$$\sum_{i=1}^{m}\sum_{l=1}^{s}\left(\frac{\partial w}{\partial\overline{\phi}}\right)_{il} e^{il} = \sum_{i=1}^{m}\sum_{l=1}^{s}\sum_{j=1}^{i}\left(a_{ij}w_{jl} + b_{ij}\overline{w}_{jl}\right)e^{il}$$

where $(\ldots)_{il}$ means the ith row and jth column elements of $(\ldots)$. For $i = 1$ and for fixed l, $1 \leq l \leq s$, we have

$$\frac{\partial w_{1l}}{\partial\bar{z}} = A_0 w_{1l} + B_0\overline{w}_{1l} \qquad (5)$$

and for $2 \leq i \leq m$ and for fixed l, $1 \leq l \leq s$,

$$\left(\frac{\partial w}{\partial\overline{\phi}}\right)_{il} - A_0 w_{il} - B_0\overline{w_{il}} - \sum_{j=1}^{i-1}\left(a_{ij}w_{jl} + b_{ij}\overline{w}_{jl}\right) = 0,$$

Equations in (4) can be rewritten for $i = 1$ and for fixed l, $1 \leq l \leq s$ in component form, respectively:

$$\begin{aligned} \alpha^{(0)}_{11}K^{(0)}_{1l} + \beta^{(0)}_{11}\overline{K}^{(0)}_{1l} &= K^{(2)}_{1l} && (6)\\ \alpha^{(0)}_{11}K^{(1)}_{1l} + \alpha^{(1)}_{11}K^{(0)}_{1l} + \beta^{(0)}_{11}\overline{K}^{(2)}_{1l} + \beta^{(1)}_{11}\overline{K}^{(0)}_{1l} &= 0, && (7)\\ \alpha^{(0)}_{11}K^{(2)}_{1l} + \alpha^{(2)}_{11}K^{(0)}_{1l} + \beta^{(0)}_{11}\overline{K}^{(1)}_{1l} + \beta^{(2)}_{11}\overline{K}^{(0)}_{1l} &= 0, && (8)\\ \alpha^{(1)}_{11}K^{(1)}_{1l} + \beta^{(1)}_{11}\overline{K}^{(2)}_{1l} &= 0, && (9)\\ \alpha^{(2)}_{11}K^{(1)}_{1l} + \alpha^{(1)}_{11}K^{(2)}_{1l} + \beta^{(1)}_{11}\overline{K}^{(1)}_{1l} + \beta^{(2)}_{11}\overline{K}^{(2)}_{1l} &= 0, && (10)\\ \alpha^{(2)}_{11}K^{(2)}_{1l} + \beta^{(2)}_{11}\overline{K}^{(1)}_{1l} &= 0. && (11) \end{aligned}$$

Note that the determinant of coefficients $\alpha^{(2)}_{11}, \alpha^{(1)}_{11}, \alpha^{(0)}_{11}, \beta^{(2)}_{11}, \beta^{(1)}_{11}, \beta^{(0)}_{11}$ vanishes. Since Cramer's rule from linear algebra fails for solving this system, we should have to look for the largest non-vanishing minor. If we omit the last column and the fifth row, we get a 5×5 determinant whose value is $-K^{(2)}_{1l}|d_{1l}|^2$ where $d_{1l} = \overline{K}^{(1)}_{1l}K^{(0)}_{1l} - \overline{K}^{(0)}_{1l}K^{(2)}_{1l}$. Hence if we suppose $K^{(2)}_{1l} \neq 0$ and $d_{1l} \neq 0$, and choose $\beta^{(2)}_{11}$ arbitrarily, other coefficients are then uniquely determined:

$$\begin{aligned}
\alpha_{11}^{(2)} &= -\beta_{11}^{(2)}\overline{K}_{1l}^{(1)}(K_{1l}^{(2)})^{-1}, \\
\alpha_{11}^{(1)} &= -\beta_{11}^{(2)}\overline{K}_{1l}^{(2)}(K_{1l}^{(2)})^{-1} - \left(|K_{1l}^{(2)}|^2 - |K_{1l}^{(1)}|^2\right)|d_{1l}|^{-2}|K_{1l}^{(2)}|^2, \\
\alpha_{11}^{(0)} &= K_{1l}^{(2)}\overline{K}_{1l}^{(1)}d_{1l}^{-1} - \beta_{11}^{(2)}\overline{K}_{1l}^{(0)}(K_{1l}^{(2)})^{-1}, \\
\beta_{11}^{(1)} &= \beta_{11}^{(2)}(K_{1l}^{(2)})^{-1}K_{1l}^{(1)} + \left(|K_{1l}^{(2)}|^2 - |K_{1l}^{(1)}|^2\right)|d_{1l}|^{-2}K_{1l}^{(2)}K_{1l}^{(1)}, \\
\beta_{11}^{(0)} &= \beta_{11}^{(2)}K_{1l}^{(0)}(K_{1l}^{(2)})^{-1} - (K_{1l}^{(2)})^2 d_{1l}^{-1}.
\end{aligned}$$

We substitute these obtained values into Eq. (10). It can be seen that, equation (10) is satisfied if and only if $K_{1l}^{(2)}\overline{K}_{1l}^{(2)} = K_{1l}^{(1)}\overline{K}_{1l}^{(1)}$ for each choice of $\beta_{11}^{(2)}$. For $1 < i \leq m$ and $1 \leq l \leq s$ one can obtain following system:

$$\alpha_{i1}^{(0)}K_{1l}^{(0)} + \beta_{i1}^{(0)}\overline{K}_{1l}^{(0)} = K_{il}^{(2)} - \sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(0)}), \tag{12}$$

$$\alpha_{i1}^{(0)}K_{1l}^{(1)} + \alpha_{i1}^{(1)}K_{1l}^{(0)} + \beta_{i1}^{(0)}\overline{K}_{1l}^{(2)} + \beta_{i1}^{(1)}\overline{K}_{1l}^{(0)} = -\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(1)} + \alpha_{ij}^{(1)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(2)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(0)}), \tag{13}$$

$$\alpha_{i1}^{(0)}K_{1l}^{(2)} + \alpha_{i1}^{(2)}K_{1l}^{(0)} + \beta_{i1}^{(0)}\overline{K}_{1l}^{(1)} + \beta_{i1}^{(2)}\overline{K}_{1l}^{(0)} = -\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(2)} + \alpha_{ij}^{(2)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(0)}), \tag{14}$$

$$\alpha_{i1}^{(1)}K_{1l}^{(1)} + \beta_{i1}^{(1)}\overline{K}_{1l}^{(2)} = -\sum_{j=2}^{i}(\alpha_{ij}^{(1)}K_{jl}^{(1)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(2)}), \tag{15}$$

$$\alpha_{i1}^{(2)}K_{1l}^{(1)} + \alpha_{i1}^{(1)}K_{1l}^{(2)} + \beta_{i1}^{(1)}\overline{K}_{1l}^{(1)} + \beta_{i1}^{(2)}\overline{K}_{1l}^{(2)} = -\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(1)} + \alpha_{ij}^{(1)}K_{jl}^{(2)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(2)}), \tag{16}$$

$$\alpha_{i1}^{(2)}K_{1l}^{(2)} + \beta_{i1}^{(2)}\overline{K}_{1l}^{(1)} = -\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(2)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(1)}), \tag{17}$$

This system can be solved similarly to system (6)-(11) and the coefficients $\alpha_{il}^{(0)}$, $\alpha_{il}^{(1)}$, $\alpha_{il}^{(2)}$, $\beta_{il}^{(0)}$, $\beta_{il}^{(1)}$, $\beta_{il}^{(2)}$ can be determined succesively.

$$\begin{aligned}
\alpha_{i1}^{(2)} &= [-\beta_{i1}^{(2)}\overline{K}_{1l}^{(1)} - \sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(2)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(1)})](K_{1l}^{(2)})^{-1}, \\
\alpha_{i1}^{(1)} &= \overline{d}_{1l}^{-1}[-\overline{K}_{1l}^{(0)}\sum_{j=2}^{i}(\alpha_{ij}^{(1)}K_{jl}^{(1)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(2)}) \\
&\quad +\overline{K}_{1l}^{(2)}\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(1)} + \alpha_{ij}^{(1)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(2)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(0)})] \\
&\quad +|d_{1l}|^{-2}\overline{K}_{1l}^{(2)}\left(|K_{1l}^{(1)}|^2 - |K_{1l}^{(2)}|^2\right)[K_{il}^{(2)} - \sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(0)})] \\
&\quad +d_{1l}^{-1}\overline{K}_{1l}^{(2)}(K_{1l}^{(2)})^{-1}[(K_{1l}^{(2)})\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(2)} + \alpha_{ij}^{(2)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(0)}) \\
&\quad -K_{1l}^{(0)}\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(2)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(1)})] - \beta_{i1}^{(2)}\overline{K}_{1l}^{(2)}(K_{1l}^{(2)})^{-1}, \\
\alpha_{i1}^{(0)} &= d_{1l}^{-1}[-\overline{K}_{1l}^{(1)}\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(0)}) \\
&\quad +\overline{K}_{1l}^{(0)}\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(2)} + \alpha_{ij}^{(2)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(0)}) \\
&\quad -(K_{1l}^{(2)})^{-1}|K_{1l}^{(0)}|^2\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(2)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(1)}) + \overline{K}_{1l}^{(1)}K_{il}^{(2)}] - \beta_{i1}^{(2)}\overline{K}_{1l}^{(0)}(K_{1l}^{(2)})^{-1},
\end{aligned}$$

$$\begin{aligned}
\beta_{i1}^{(1)} &= \overline{d}_{1l}^{-1}[-K_{1l}^{(1)}\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(1)} + \alpha_{ij}^{(1)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(2)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(0)}) \\
&\quad +K_{1l}^{(0)}\sum_{j=2}^{i}(\alpha_{ij}^{(1)}K_{jl}^{(1)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(2)}] \\
&\quad +|d_{1l}|^{-2}K_{1l}^{(1)}\left(|K_{1l}^{(2)}|^2 - |K_{1l}^{(1)}|^2\right)[K_{il}^{(2)} - \sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(0)})] \\
&\quad -d_{1l}^{-1}K_{1l}^{(1)}(K_{1l}^{(2)})^{-1}[K_{1l}^{(2)}\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(2)} + \alpha_{ij}^{(2)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(0)}) \\
&\quad -K_{1l}^{(0)}\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(2)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(1)})] + \beta_{i1}^{(2)}K_{1l}^{(1)}(K_{1l}^{(2)})^{-1} \\
\beta_{i1}^{(0)} &= d_{1l}^{-1}[-K_{1l}^{(0)}\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(2)} + \alpha_{ij}^{(2)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(0)}) \\
&\quad +(K_{1l}^{(0)})^2(K_{1l}^{(2)})^{-1}\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(2)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(1)}) - K_{1l}^{(2)}K_{il}^{(2)} \\
&\quad +K_{1l}^{(2)}\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(0)})] + \beta_{i1}^{(2)}K_{1l}^{(0)}(K_{1l}^{(2)})^{-1}
\end{aligned}$$

where $\alpha_{i1}^{(0)}, \alpha_{i1}^{(1)}, \alpha_{i1}^{(2)}, \beta_{i1}^{(0)}, \beta_{i1}^{(1)}$ come from Eqs.(13) -(15) and (17). Let put values which obtained above into Eq. (16). It can be seen that Eq. (16) is satisfied if and only if $K_{jl}^{(2)}K_{1l}^{(1)} = K_{jl}^{(1)}K_{1l}^{(2)}$ and $\overline{K}_{jl}^{(1)}K_{1l}^{(1)} = \overline{K}_{jl}^{(2)}K_{1l}^{(2)}$ for each choice of $\beta_{i1}^{(2)}$. The following theorem has been proved after above calculations:

Theorem 2.1. *The linear function* $w = K^{(0)} + \phi K^{(1)} + \overline{\phi}K^{(2)}$ *is a solution of following equation*

$$\frac{\partial w}{\partial \overline{\phi}} = \left(\alpha^{(0)} + \alpha^{(1)}\phi + \alpha^{(2)}\overline{\phi}\right)w + \left(\beta^{(0)} + \beta^{(1)}\phi + \beta^{(2)}\overline{\phi}\right)\overline{w}, \tag{18}$$

providing that $K_{1l}^{(2)} \neq 0$, $\overline{K}_{1l}^{(0)}K_{1l}^{(2)} - \overline{K}_{1l}^{(1)}K_{1l}^{(0)} \neq 0$ *and* $K_{jl}^{(2)}K_{1l}^{(1)} = K_{jl}^{(1)}K_{1l}^{(2)}$, $\overline{K}_{jl}^{(1)}K_{1l}^{(1)} = \overline{K}_{jl}^{(2)}K_{1l}^{(2)}$, $1 \leq j \leq i$. *Arbitrarily choosing* $\beta_{i1}^{(2)}$, *the remaining coefficients are then uniquely determined.*

Remark 2.2. Let us consider the case where the conditions of the Theorem 2.1 is not provided. The following expressions can be proved after a few calculations which are similar to previous ones:

(1) Let $K_{1l}^{(2)} = 0$ and $K_{1l}^{(1)} \neq 0$, for $i = 1$ and fixed l, then $w_{1l} = K_{1l}^{(0)} + zK_{1l}^{(1)}$ is solution of (5) if

$$\alpha_{11}^{(0)} = \beta_{11}^{(1)}\overline{K}_{1l}^{(0)}(K_{1l}^{(1)})^{-1}, \quad \alpha_{11}^{(2)} = -\beta_{11}^{(1)}\overline{K}_{1l}^{(1)}(K_{1l}^{(1)})^{-1},$$

$$\beta_{11}^{(0)} = -\beta_{11}^{(1)}K_{1l}^{(0)}(K_{1l}^{(1)})^{-1}, \quad \beta_{11}^{(2)} = 0, \alpha_{11}^{(1)} = 0,$$

where $\beta_{11}^{(1)}$ can be choosen arbitrarily. For $1 < i \leq m$ and fixed l

$$\begin{aligned}
\alpha_{i1}^{(0)} &= (P_2 - P_4K_{1l}^{(0)}(K_{1l}^{(1)})^{-1} - \beta_{i1}^{(1)}\overline{K}_{1l}^{(0)})(K_{1l}^{(1)})^{-1}, \\
\alpha_{i1}^{(1)} &= P_4(K_{1l}^{(1)})^{-1}, \beta_{i1}^{(2)} = P_6(\overline{K}_{1l}^{(1)})^{-1} \\
\alpha_{i1}^{(2)} &= (P_3 - P_6\overline{K}_{1l}^{(0)}(\overline{K}_{1l}^{(1)})^{-1} - \beta_{i1}^{(0)}\overline{K}_{1l}^{(1)})(K_{1l}^{(0)})^{-1} \\
\beta_{i1}^{(0)} &= (K_{il}^{(2)} + P_1 - \alpha_{i1}^{(0)}K_{1l}^{(0)})(\overline{K}_{1l}^{(0)})^{-1}
\end{aligned}$$

where $\beta_{i1}^{(1)}$ can be choosen arbitrarily and

$$P_1 = -\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(0)}),$$

$$P_2 = -\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(1)} + \alpha_{ij}^{(1)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(2)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(0)}),$$

$$P_3 = -\sum_{j=2}^{i}(\alpha_{ij}^{(0)}K_{jl}^{(2)} + \alpha_{ij}^{(2)}K_{jl}^{(0)} + \beta_{ij}^{(0)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(0)}),$$

$$P_4 = -\sum_{j=2}^{i}(\alpha_{ij}^{(1)}K_{jl}^{(1)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(2)})$$

$$P_5 = -\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(1)} + \alpha_{ij}^{(1)}K_{jl}^{(2)} + \beta_{ij}^{(1)}\overline{K}_{jl}^{(1)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(2)})$$

$$P_6 = -\sum_{j=2}^{i}(\alpha_{ij}^{(2)}K_{jl}^{(2)} + \beta_{ij}^{(2)}\overline{K}_{jl}^{(1)}).$$

Then the solution of (5) is $w_{il} = K_{il}^{(0)} + \sum_{j=1}^{i}(\phi_{ij}K_{jl}^{(1)} + \overline{\phi}_{ij}K_{jl}^{(2)})$.

(2) Let show that the case $K_{1l}^{(2)} \neq 0$ and $\overline{K}_{1l}^{(0)}K_{1l}^{(2)} - \overline{K}_{1l}^{(1)}K_{1l}^{(0)} = 0$ is not possible. Indeed, let multiply Eq. (6) by $\overline{K}_{1l}^{(1)}$ and adding Eq. (8) multiplied by $-\overline{K}_{1l}^{(0)}$. Replacing $\alpha_{11}^{(2)}$ by $\beta_{11}^{(2)}$ in view of (11), it follows

$$\alpha_{11}^{(0)}d_{1l} = K_{1l}^{(2)}\overline{K}_{1l}^{(1)} - \beta_{11}^{(2)}\overline{K}_{1l}^{(0)}(K_{1l}^{(2)})^{-1}d_{1l}.$$

From hypothesis since $\overline{K}_{1l}^{(0)}K_{1l}^{(2)} = \overline{K}_{1l}^{(1)}K_{1l}^{(0)}$, the last equation implies $K_{1l}^{(1)} = 0$. Taking into consideration hypothesis with $K_{1l}^{(1)} = 0$, we get $K_{1l}^{(0)} = 0$. But, this contradicts Eq. (6). Thus, no system (18) exists, since contradiction is obtained in the case of $i = 1$ and this system is going to solve successively for $1 < i \leq m$.

(3) If $K_{1l}^{(2)} = 0$ and $d_{1l} = 0$, then $K_{1l}^{(0)}$ or $K_{1l}^{(1)}$ must be zero. Let examine this situation in three cases:

(3a) If $K_{1l}^{(0)} = 0$ and $K_{1l}^{(1)} \neq 0$, then $w_{1l} = zK_{1l}^{(1)}$ is solution of (5) with

$$\alpha_{11}^{(0)} = 0, \ \ \alpha_{11}^{(1)} = 0, \ \ \beta_{11}^{(0)} = 0,$$

$$\beta_{11}^{(1)} = -\alpha_{11}^{(2)}K_{1l}^{(1)}(\overline{K}_{1l}^{(1)})^{-1}, \ \ \beta_{11}^{(2)} = 0,$$

where $\alpha_{11}^{(2)}$ is arbitrary. For $1 < i \leq m$ and fixed l

$$\alpha_{i1}^{(0)} = P_2(K_{1l}^{(1)})^{-1}, \ \ \alpha_{i1}^{(1)} = P_4(K_{1l}^{(1)})^{-1}$$

$$\beta_{i1}^{(0)} = P_3(\overline{K}_{1l}^{(1)})^{-1}, \ \beta_{i1}^{(1)} = (P_5 - \alpha_{i1}^{(2)}K_{1l}^{(1)})(\overline{K}_{1l}^{(1)})^{-1}$$

$$\beta_{i1}^{(2)} = P_6(\overline{K}_{1l}^{(1)})^{-1}, K_{il}^{(2)} = -P_1$$

where $\alpha_{i1}^{(2)}$ is arbitrary.

(3b) If $K_{1l}^{(1)} = 0$ and $K_{1l}^{(0)} \neq 0$, then we can obtain

$$\beta_{11}^{(j)} = -\alpha_{11}^{(j)}K_{1l}^{(0)}(\overline{K}_{1l}^{0})^{-1}, \ j = 0,1,2,$$

where $\alpha_{11}^{(0)}, \alpha_{11}^{(1)}, \alpha_{11}^{(2)}$ are arbitrary. For $1 < i \leq m$ and fixed l, we get

$$\beta_{i1}^{(0)} = (K_{il}^{(2)} + P_1 - \alpha_{i1}^{(0)}K_{1l}^{0})(\overline{K}_{1l}^{(0)})^{-1}$$

$$\beta_{i1}^{(1)} = (P_2 - \alpha_{i1}^{(1)}K_{1l}^{0})(\overline{K}_{1l}^{(0)})^{-1}$$

$$\beta_{i1}^{(2)} = (P_3 - \alpha_{i1}^{(2)}K_{1l}^{0})(\overline{K}_{1l}^{(0)})^{-1}$$

where $\alpha_{i1}^{(s)}$ $(s = 0,1,2)$ are arbitrary.

(3c) If $K_{1l}^{(0)} = K_{1l}^{(1)} = 0$, then one has $w_{1l} = 0$ and all of coefficients can be chosen arbitrarily. For $1 < i \leq m$ and fixed l, the coefficients can be chosen to provide $K_{il}^{(2)} = -P_1$ and $P_s = 0$ $(s = 2,3,4,5,6)$.

3. Co-associated Vekua Type Equations in Matrix Form

Let $lw = 0$ is identical with equation (3) that is

$$lw = \frac{\partial w}{\partial \bar{\phi}} - Aw - B\overline{w}$$

and an operator $\pounds$ has the following form

$$\pounds w = \frac{\partial w}{\partial \phi} + \tilde{A}(\phi)w + \tilde{B}(\phi)\overline{w}. \tag{19}$$

For given operator l, we search for sufficient conditions on coefficients $\tilde{A}, \tilde{B}$ so that $\pounds$ transforms the space of solutions to equation (3) into itself. In this case, operator l is said to be associated to an operator $\pounds$. For finding these coefficients, let us take into account the expression

$$l(\pounds w) = \tfrac{\partial w}{\partial \bar{\phi}}(\tfrac{\partial w}{\partial \phi} + \tilde{A}w + \tilde{B}\overline{w})$$

$$-A(\tfrac{\partial w}{\partial \phi} + \tilde{A}w + \tilde{B}\overline{w}) - B(\overline{\tfrac{\partial w}{\partial \phi} + \tilde{A}w + \tilde{B}\overline{w}})$$

and $lw = 0$. Therefore, we obtain that $l(\pounds w) = 0$ is a linear combination of $\frac{\partial w}{\partial \phi}, \overline{\frac{\partial w}{\partial \phi}}, w, \overline{w}$. Equating coefficients of these terms to zero, we get

$$B = \tilde{B} \tag{20}$$

$$\frac{\partial A}{\partial \phi} + \frac{\partial \tilde{A}}{\partial \bar{\phi}} = 0 \tag{21}$$

$$\frac{\partial B}{\partial \phi} + \frac{\partial \tilde{B}}{\partial \bar{\phi}} + B\left[(\overline{A} - A) + \left(\tilde{A} - \overline{\tilde{A}}\right)\right] = 0. \tag{22}$$

Since $A = \alpha^{(0)} + \alpha^{(1)}\phi + \alpha^{(2)}\bar{\phi}$, Eq. (21) gives $\frac{\partial \tilde{A}}{\partial \bar{\phi}} = -\alpha^{(1)}$. The solution of this equation can be written as following form

$$\tilde{A} = \Phi + J[-\alpha^{(1)}]$$

by the Corollary 3.4 in [5], this means $\tilde{A}$ can be determine uniquely up to an arbitrary Q-holomorphic function where

J is the Pompeiu operator defined in [5, p. 433]. Therefore, we see that $\tilde{A}(\phi) = -\alpha^{(1)}\overline{\phi} + \Phi(\phi)$ where Φ is a $Q-$holomorphic function. On condition that $\Phi(\phi)$ is a linear function, as $\Phi(\phi) = \gamma^{(0)} + \gamma^{(1)}\phi$, on equating the coefficients of $1, \phi, \bar{\phi}, \phi\phi, \phi\bar{\phi}, \bar{\phi}\bar{\phi}$, equation (22) leads to the relations below

$$\begin{aligned} \beta^{(1)} + \beta^{(2)} + \beta^{(0)}\lambda &= 0,\ \beta^{(1)}\lambda - \beta^{(0)}\bar{\sigma} = 0, \\ \beta^{(2)}\lambda + \beta^{(0)}\sigma &= 0, \beta^{(1)}\bar{\sigma} = 0, \\ \beta^{(1)}\sigma - \beta^{(2)}\bar{\sigma} &= 0,\ \beta^{(2)}\sigma = 0, \end{aligned}$$

where $\lambda = \bar{\alpha}^{(0)} - \alpha^{(0)} + \gamma^{(0)} - \bar{\gamma}^{(0)}$ and $\sigma = \bar{\alpha}^{(1)} - \alpha^{(1)} - \alpha^{(2)} - \bar{\gamma}^{(1)}$.

Let $A(\phi) = \alpha^{(0)}$ and $B(\phi) = \beta^{(0)}$ be constant matrices for ease. Therefore, we obtain that $\tilde{A}(\phi)$ must be $Q-$holomorphic from (21) and $\tilde{B} = \beta^{(0)}$ from Eq. (20). If we assume that $\tilde{A}(\phi)$ is also constant, condition (22) yields

$$\bar{\alpha}^{(0)} - \alpha^{(0)} = \bar{\gamma}^{(0)} - \gamma^{(0)},$$

means that $Im\alpha^{(0)} = Im\gamma^{((0))}$. If two Vekua type equations are associated to the same operator $\pounds$, then they are said to be co-associated. Consequently, if $\pounds = \partial w/\partial\phi + \gamma^{(0)}w + \beta^{(0)}\overline{w}$, then $\pounds$ maps the solutions of

$$\frac{\partial w}{\partial\bar{\phi}} - (\mu + i\upsilon)w - \beta^{(0)}\bar{w} = 0 \tag{23}$$

type equations into itself, where μ is an arbitrary real matrix $\left(\upsilon = Im\gamma^{(0)}\right)$.

4. Linear Solutions of Co-associated Equations

In this section, we determine the coefficients of a linear function $w = K^{(0)} + \phi K^{(1)} + \bar{\phi}K^{(2)}$ in order for this function to be a solution of Eq. (23). Substituting the linear function into (23), we obtain the following system of equations:

$$\begin{aligned} (\mu + i\upsilon)K^{(0)} + \beta^{(0)}\overline{K}^{(0)} &= K^{(2)}, & (24) \\ (\mu + i\upsilon)K^{(1)} + \beta^{(0)}\overline{K}^{(2)} &= 0, & (25) \\ (\mu + i\upsilon)K^{(2)} + \beta^{(0)}\overline{K}^{(1)} &= 0. & (26) \end{aligned}$$

For determining $K^{(0)}, K^{(1)}, K^{(2)}$, we should solve above system. For this, the statements below on linear algebraic equations can be used.

Proposition 4.1. *Provided* $C_{11}^{(1)}\overline{C}_{ij}^{(1)} = \overline{C}_{11}^{(2)}C_{ij}^{(2)}$ $(j = 1, 2, \ldots, i)$*, the equation*

$$C^{(1)}P + C^{(2)}\overline{P} = C^{(3)} \tag{27}$$

is solvable if $C_{11}^{(1)}\overline{C}_{il}^{(3)} = \overline{C}_{11}^{(2)}C_{il}^{(3)}$*, where* $P = \{P_{ij}\}$, $C^{(3)} = \left\{C_{ij}^{(3)}\right\}$ *are* $m \times s$ *matrices and* $C^{(1)} = \left\{C_{ij}^{(1)}\right\}, C^{(2)} = \left\{C_{ij}^{(2)}\right\}$ *are commuting with* Q*.*

Proof. Equation (27) can be written in component form, for $i = 1$ and for fixed l, $1 \le l \le s$, as

$$C_{11}^{(1)}P_{1l} + C_{11}^{(2)}\overline{P}_{1l} = C_{1l}^{(3)} \tag{28}$$

and for $2 \le i \le m$ and for fixed l, $1 \le l \le s$, as

$$C_{11}^{(1)}P_{il} + C_{11}^{(2)}\overline{P}_{il} = C_{il}^{(3)} - \sum_{j=1}^{i-1}(C_{ij}^{(1)}P_{jl} + C_{ij}^{(2)}\overline{P}_{jl}). \tag{29}$$

In the case of $i = 1$, provided $C_{11}^{(1)}\overline{C}_{11}^{(1)} = \overline{C}_{11}^{(2)}C_{11}^{(2)}$, equation (28) is solvable if $C_{11}^{(1)}\overline{C}_{1l}^{(3)} = \overline{C}_{11}^{(2)}C_{1l}^{(3)}$ [see 9, p. 723]. It can be explicitly seen from complex case that provided $C_{11}^{(1)}\overline{C}_{ij}^{(1)} = \overline{C}_{11}^{(2)}C_{ij}^{(2)}$ for $2 \le i \le m$, $j = 1, 2, \ldots, i$, (29) is solvable successively, if $C_{11}^{(1)}\overline{C}_{il}^{(3)} = \overline{C}_{11}^{(2)}C_{il}^{(3)}$. □

Proposition 4.2. *The system*

$$\begin{cases} C^{(4)}P + C^{(5)}\overline{R} = 0, \\ C^{(4)}R + C^{(5)}\overline{P} = 0, \end{cases} \tag{30}$$

has non-trivial solutions if and only if $C_{11}^{(4)}\overline{C}_{ij}^{(4)} = C_{ij}^{(5)}\overline{C}_{11}^{(5)}, (i = 1, \ldots, m, j = 1, \ldots i)$*, where* $C^{(4)}, C^{(5)}$ *are commuting with* Q *and* P, R *are* $m \times s$ *matrices.*

Proof. The system (30) can be written in component form, for $i = 1$ and for fixed l, $1 \le l \le s$, as

$$\begin{cases} C_{11}^{(4)}P_{1l} + C_{11}^{(5)}\overline{R}_{1l} = 0, \\ C_{11}^{(4)}R_{1l} + C_{11}^{(5)}\overline{P}_{1l} = 0. \end{cases} \tag{31}$$

For $2 \le i \le m$ and for fixed l, $1 \le l \le s$, (30) can be written in component form as

$$\begin{cases} C_{11}^{(4)}P_{il} + C_{11}^{(5)}\overline{R}_{il} = -\sum\limits_{j=1}^{i-1}(C_{ij}^{(4)}P_{jl} + C_{ij}^{(5)}\overline{R}_{jl}), \\ C_{11}^{(4)}R_{il} + C_{11}^{(5)}\overline{P}_{il} = -\sum\limits_{j=1}^{i-1}(C_{ij}^{(4)}R_{jl} + C_{ij}^{(5)}\overline{P}_{jl}). \end{cases} \tag{32}$$

For $i = 1$, the system (31) has non trivial solutions if and only if $C_{11}^{(4)}\overline{C}_{11}^{(4)} = C_{11}^{(5)}\overline{C}_{11}^{(5)}$ [see 9, p. 723]. By using obvious statements on linear algebraic equations, if the condition $C_{11}^{(4)}\overline{C}_{ij}^{(4)} = C_{ij}^{(5)}\overline{C}_{11}^{(5)}$ for $2 \le i \le m$, $1 \le j \le i-1$ is considered, it can be seen that the system (32) have solutions. So this completes the proof. □

Let us consider Proposition 4.1 and Proposition 4.2 in order to find non-constant solutions of (23). Firstly, if we apply the second one to (25), (26), then we get

$$(\mu + i\upsilon)_{11}(\mu - i\upsilon)_{kj} = \overline{\beta}_{11}^{(0)}\beta_{kj}^{(0)}, \tag{33}$$

where $1 \le k \le m$, $1 \le j \le k$. According to the Proposition 4.1, provided (33) Eq. (24) is solvable if

$$(\mu + i\upsilon)_{11}\overline{K}_{kl}^{(2)} = \overline{\beta}_{11}^{(0)}K_{kl}^{(2)}. \tag{34}$$

For $k = 1$, Eq. (33) can be written as

$$(\mu + i\upsilon)_{11}(\mu - i\upsilon)_{11} = \beta_{11}^{(0)}\overline{\beta}_{11}^{(0)}, \tag{35}$$

where from $|(\mu+i\upsilon)_{11}| = |\beta_{11}^{(0)}|$. Let us suppose that $|\upsilon_{11}| < |\beta_{11}^{(0)}|$, then there exist two real solutions of (35). For $k = 1$, we obtain from Eq. (34) $(\mu+i\upsilon)_{11}\overline{K}_{1l}^{(2)} = \overline{\beta}_{11}^{(0)}K_{1l}^{(2)}$. Assume, without any loss of generality, that $|K_{1l}^{(2)}| = 1$. Let ψ_{ij} be the polar angle of $\alpha_{ij}^{(0)}$ and denote the polar angel of $\beta_{ij}^{(0)}$ with φ_{ij}. By using of Proposition 4.1, similar to complex case [9] , one can obtain

$$K_{1l}^{(2)} = \exp\left(\frac{i}{2}(\psi_{11}+\varphi_{11})\right).$$

In view of (25) in component form, using the value of $K_{1l}^{(2)}$, we find

$$K_{1l}^{(1)} = -\exp\left(\frac{i}{2}(\varphi_{11}-3\psi_{11})\right).$$

A similar calculation for negative value of μ_{11} leads to

$$K_{1l}^{(2)} = i\exp\left(\frac{i}{2}(\varphi_{11}-\psi_{11})\right),$$

$$K_{1l}^{(1)} = -i\exp\left(\frac{i}{2}(\varphi_{11}+3\psi_{11})\right).$$

For $2 \le k \le m$ similar calculations can be done successively. From Eq. (24), permissible values of $K^{(0)}$ can be found by using $K^{(1)}$ and $K^{(2)}$. In consequence of Weierstrass-Stone approximation theorem, utilizing the two possibilities for $K^{(0)}, K^{(1)}, K^{(2)}$ which has been constructed above, two functions w_1 and w_2 can be written in the form $K^{(0)} + \phi K^{(1)} + \overline{\phi}K^{(2)}$. Thus the following statement has been proved:

Theorem 4.3. *Suppose $|\upsilon_{11}| < |\beta_{11}^{(0)}|$. Then there exist two linear generalized Q-holomorphic functions w_1, w_2 which are solutions with the form $K^{(0)} + \phi K^{(1)} + \overline{\phi}K^{(2)}$ of*

$$\partial w/\partial\overline{\phi} = (\mu+i\upsilon)w + \beta^{(0)}\overline{w},$$

such that every complex matrix-valued continuous function on a compact set in the complex plane can uniformly be approximated by w_1, w_2.

Associated equations have been presented for the purpose of solving initial value problems having type $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial z} + Aw + B\overline{w}$, $w(z,0) = \varphi(z)$ [see 10]. On the condition that the initial function is a solution of associated equation, the solution of this initial value problem exists. Moreover, one can construct this solution by using of contraction mapping principle. In previous section, we obtained the sufficient conditions for $\pounds$ and l to be associated. Let us regard the initial value problem

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial \phi} + \tilde{A}w + \tilde{B}\overline{w} \\ w(z,0) &= w_0(z). \end{aligned}$$

Suppose that the initial function be a solution of associated equation and $\tilde{A}$ and $\tilde{B}$ be continuous in $\{t : 0 \le t \le t_0\} \times \bar{D}$ and for every t belong to $C^{\alpha}(\bar{D})$. Hence this initial value problem can be solved and the solution of this problem is constructable by successive approximation method [see 11].

5. Acknowledgment

The authors declare that there is no conflict of interests regarding the publication of this paper and thank the referee for remarks that improved this note.

References

[1] Douglis, A. 1953. A Function theoretic approach to elliptic systems of equations in two variables. Comm. Pure Appl. Math. 6(1953), 259-289.

[2] Bojarskiĭ, B. V. 1966. Theory of generalized analytic vectors (in Russian), Ann. Polon. Math. 17(1966), 281-320.

[3] Hile, G. N. 1982. Function Theory for Generalized Beltrami Systems. Comp. Math. 11(1982), 101-125.

[4] Hızlıyel, S., Çağlıyan, M. 2010. The Riemann Hilbert problem for generalized Q-holomorphix functions. Turk J. Math. 34(2010), 167-180.

[5] Hızlıyel, S.,Çağlıyan M. 2004. Generalized *Q*-holomorphic functions. Complex Var. Theory Appl. 49(2004), 427-447.

[6] Hızlıyel, S., Çağlıyan, M. 2004. Pseudo *Q*-holomorphic functions. Complex Var. Theory Appl. 49(2004), 941-955.

[7] Vekua, I. N. 1962. Generalized Analytic Functions. Pergamon; Oxford. 668p.

[8] Bers, L. 1953. Theory of Pseudo-analytic Functions. New York.

[9] Tutschke, W., Vasudeva, H. 2002. Linear generalized analytic functions. Complex Var. Theory Appl. 47(2002), 719-725.

[10] Tutschke, W. 1989. Solution of initial value problems in classes of generalized analytic functions. Teunber Leibzig and Springer-Verlag. 188p.

[11] Hızlıyel, S., Saglam Özkan, Y. 2016. The Cauchy-Kowalewski theorem in the space of pseudo Q-holomorphic functions, Complex Anal. Oper. Theory. 10(2016), 953-963.