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The beta Nadarajah-Haghighi distribution

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Abstract

Recently, there has been a great interest among statisticians and applied researchers in constructing flexible distributions for better modeling non-monotone failure rates. We study a lifetime model of the beta generated family, called the beta Nadarajah-Haghighi distribution, which can be used to model survival data. The proposed model includes as special models some important distributions. The hazard rate function is an important quantity characterizing life phenomena. Its hazard function can be constant, decreasing, increasing, upside-down bathtub and bathtub-shaped depending on the parameters. We provide a comprehensive mathematical treatment of the new distribution and derive explicit expressions for some of its basic mathematical quantities. The method of maximum likelihood is used for estimating the model parameters and a small Monte Carlo simulation is conducted. We fit the proposed model to two real data sets to prove empirically its flexibility as compared to other lifetime distributions.

Keywords: Beta distribution, Moment, Nadarajah-Haghighi distribution, Mean deviation.

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1. Introduction

Extending continuous univariate distributions by introducing a few extra shape parameters is an essential method to explore better the skewness and tail weights and other properties of the generated distributions. Following the latest trend, applied statisticians are now able to construct more generalized distributions, which provide better goodness-of-fit measures when fitted to real data rather than by using the classical distributions.

The exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability and survival analysis. This model was the first lifetime model for which statistical methods were extensively developed in the lifetime literature. A generalization of the exponential distribution was recently proposed by Nadarajah and Haghighi (NH) [20]. Its cumulative distribution function (cdf) is given by

$$G(x) = 1 - \exp[1 - (1 + \lambda x)^{\alpha}], \quad x > 0, \tag{1.1}$$

where $\lambda > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. The probability density function (pdf) and the hazard rate function (hrf) corresponding to (1.1) are given by

$$g(x) = \alpha\,\lambda(1 + \lambda x)^{\alpha-1} \exp[1 - (1 + \lambda x)^{\alpha}], \tag{1.2}$$

and

$$h(x) = \alpha\,\lambda(1 + \lambda x)^{\alpha-1},$$

respectively.

Then, if X follows the NH distribution, we shall denote by $X \sim NH(\alpha, \lambda)$. The exponential distribution is a special case of the NH model when $\alpha = 1$. Nadarajah and Haghighi [20] pointed out that the its hrf can be monotonically increasing for $\alpha > 1$, monotonically decreasing for $\alpha < 1$ and, for $\alpha = 1$, it becomes constant. They also presented some motivations for introducing this distribution.

The first motivation is based on the relationship between the pdf in (1.2) and its hrf. The NH density function can be monotonically decreasing and its hrf can be increasing. The gamma, Weibull and exponentiated exponential (EE) distributions do not allow for an increasing failure function when their corresponding densities are monotonically decreasing. The second motivation is related to the ability of the NH distribution to model data that have their mode fixed at zero. The gamma, Weibull and EE distributions are not suitable for situations of this kind. The third motivation is based on the following mathematical relationship: if Y is a Weibull random variable with shape parameter α and scale parameter λ, then the density function in equation (1.2) is the same as that of the random variable $Z = Y - \lambda - 1$ truncated at zero, that is, the NH distribution can be interpreted as a truncated Weibull distribution.

In this paper, we propose a new model called the *beta Nadarajah-Haghighi* (BNH) distribution, which contains as sub-models the exponential, generalized exponential (GE) [10], beta exponential (BE) [23], NH and exponentiated NH (ENH) [13] distributions. These special cases are given in Table 1. Besides extending these five distributions, the advantage of the new model, in addition to the advantages of the NH distribution, lies in the great flexibility of its pdf and hrf. Thus, the new model provides a good alternative to many existing life distributions in modeling positive real data sets. As we will show later, the hrf of the BNH distribution can exhibit the classical four forms (increasing, decreasing, unimodal and bathtub-shaped) depending on its shape parameters. We obtain some basic mathematical properties and discuss maximum likelihood estimation of the model parameters.

The paper is outlined as follows. In Section 2, we define the BNH distribution and provide plots of the density and hazard rate functions. We derive a useful linear representation in Section 3. In Sections 4, 5 and 6, we obtain explicit expressions for the moments, quantile function and moment generating function (mgf), respectively. Incomplete moments and mean deviations are determined in Sections 7 and 8, respectively. In Section 9, we present the Rényy and Shannon entropies. The BNH order statistics are investigated in Section 10. Maximum likelihood estimation and a small simulation study are addressed in Sections 11 and 12. Two empirical applications to real data are illustrated in Section 13. Finally, Section 14 offers some concluding remarks.

Table 1. Special models of the BNH distribution.

α	a	b	Reduced distribution
1	-	-	BE distribution (Nadarajah and Kotz, 2006)
1	-	1	GE distribution (Kundu and Gupta, 1998)
-	1	-	ENH distribution (Lemonte, 2013)
-	1	1	NH distribution (Nadarajah and Haghighi, 2011)
1	1	1	exponential distribution

2. The BNH distribution

Several ways of generating new distributions from classic ones were developed recently. Eugene et al. [6] proposed the beta family of distributions. They demonstrated that its density function is a generalization of the density function of the order statistics of a random sample from a parent G distribution and studied some general properties. This class of generalized distributions has received considerable attention in recent years. In particular, taking $G(x)$ to be the density function of the normal distribution, they defined and studied the beta normal distribution, highlighting its great flexibility in modeling not only symmetric heavy-tailed distributions, but also skewed and bimodal distributions.

Nadarajah and Gupta [22], Nadarajah and Kotz [21], [23], Lee et al. [16] and Akinsete et al. [1] defined the beta Fréchet, beta Gumbel, beta exponential, beta Weibull and beta Pareto distributions by taking $G(x)$ to be the cdf of the Fréchet, Gumbel, exponential, Weibull and Pareto distributions, respectively. More recently, Barreto-Souza et al. [2], Pescim et al. [24] and Cordeiro and Lemonte [4] proposed the beta generalized exponential, beta generalized half-normal and beta Birnbaum-Saunders distributions respectively.

The generalization of the NH distribution is motivated by the work of Eugene et al. [6]. Let $G(x)$ be the baseline cdf depending on a certain parameter vector. In order to have greater flexibility in modeling observed data, they defined the beta family by the cdf and pdf

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1} dt = I_{G(x)}(a,b), \tag{2.1}$$

and

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1} [1-G(x)]^{b-1} g(x),$$

respectively, where $a > 0$ and $b > 0$ are two additional shape parameters, which control skewness through the relative tail weights, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$is the gamma function.

Then, the cdf of the BNH distribution is given by

$$F(x; \theta) = \mathrm{I}_{1-\exp[1-(1+\lambda x)^\alpha]}(a, b), \quad x > 0, \tag{2.2}$$

where $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$ and $\theta = (\alpha, \lambda, a, b)^T$.

The corresponding density and hazard rate functions to (2.2) are given by

$$f(x; \theta) = \frac{\alpha\,\lambda}{\mathrm{B}(a, b)}(1+\lambda x)^{\alpha-1}\{1-\exp[1-(1+\lambda x)^\alpha]\}^{a-1}\{\exp[1-(1+\lambda x)^\alpha]\}^b, \tag{2.3}$$

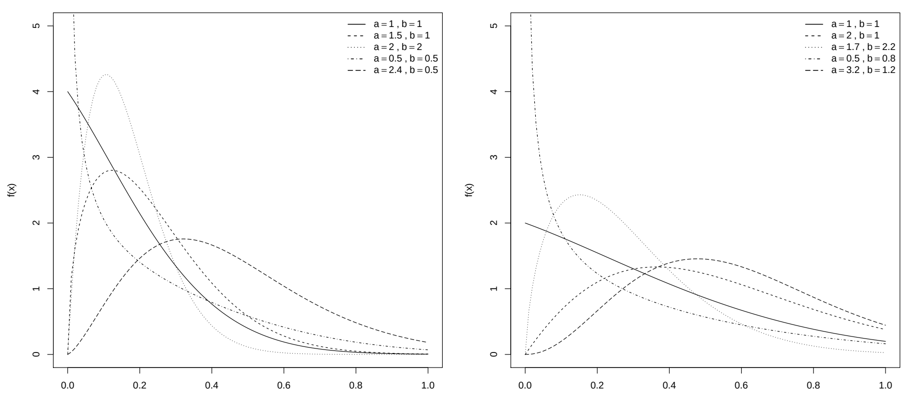
and

$$h(x; \theta) = \frac{\alpha\,\lambda(1+\lambda x)^{\alpha-1}\{1-\exp[1-(1+\lambda x)^\alpha]\}^{a-1}\{\exp[1-(1+\lambda x)^\alpha]\}^b}{\mathrm{B}(a, b) - \mathrm{B}_{1-\exp[1-(1+\lambda x)^\alpha]}(a, b)}. \tag{2.4}$$

A random variable X following (2.2) is denoted by $X \sim \mathrm{BNH}(\theta)$. Simulating the BNH random variable is relatively simple. Let Y be a random variable distributed according to the usual beta distribution given by (2.1) with parameters a and b. Then, using the inverse transformation method, the random variable X can be expressed as

$$X = \frac{1}{\lambda}\left\{[1-\log(1-Y)]^{\frac{1}{\alpha}} - 1\right\}.$$

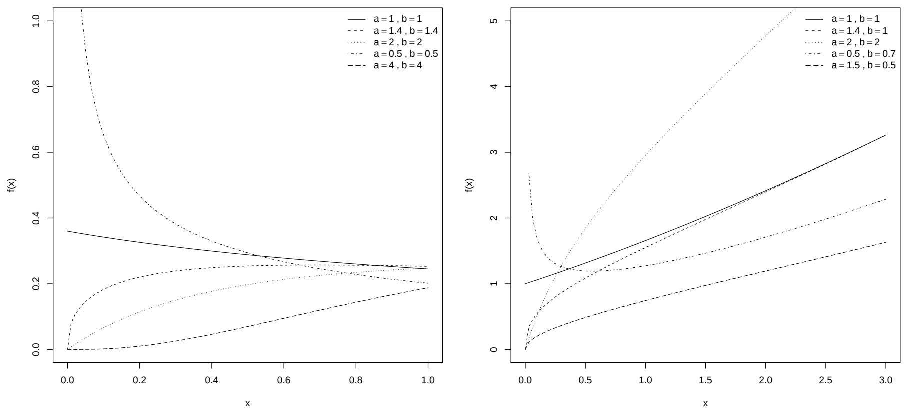
Plots of the density and hazard rate functions for selected parameters a and b, including the special case of the NH distribution, are displayed in Figures 1 and 2, respectively.



(a) $\alpha = 2.0$ and $\lambda = 2.0$ (b) $\alpha = 0.5$ and $\lambda = 0.5$

Figure 1. Plots of the BNH density function for some parameter values.

Proposition 1: The BNH density function is log-convex if $\alpha < 1$ and $a < 1$, and it is log-concave if $\alpha > 1$ and $a > 1$.



(a) $\alpha = 0.4$ and $\lambda = 0.9$ (b) $\alpha = 2.5$ and $\lambda = 0.4$

Figure 2. Plots of the BNH hazard function for some parameter values.

Proof: Let $z = (1 + \lambda\, t)^{\alpha}$. It implies that $z \geq 1$ for $t \geq 0$. We have $t = (z^{\frac{1}{\alpha}} - 1)/\lambda$. Now, rewriting the BNH pdf as a function of z, $\xi(z)$, we obtain

$$\xi(z) = f((z^{\frac{1}{\alpha}} - 1)/\lambda) = z^{\frac{\alpha-1}{\alpha}} \left[1 - \exp\{1 - z\}\right]^{b-1} \exp\{b(1 - z)\}. \tag{2.5}$$

The results follows by noting that the second derivative of $\log\left[\xi(z)\right]$ is

$$\frac{d^2 \, \log\left[\xi(z)\right]}{d\, z^2} = \frac{1 - \alpha}{\alpha\, z^2} + \frac{(1 - a)\, \exp\{1 - z\}}{1 - \exp\{1 - z\}} + \frac{(1 - a)\, \exp\{1 - z\}}{\left[1 - \exp\{1 - z\}\right]^2}. \tag{2.6}$$

Proposition 2: For any $\lambda > 0$ and $b > 0$, the BNH distribution has an increasing hrf if $\alpha < 1$ and $a < 1$ and it has a decreasing hrf if $\alpha > 1$ and $a > 1$. The hrf is constant if $\alpha = 1$, $a = 1$ and $b = 1$.

Proof: The result holds by using the log-convexity of the density function. Figure 2 displays some plots of the hrf for some parameter values. The parameter λ does not change the shape of the hrf since it is a scale parameter. It is evident that the hrf of the proposed distribution can be decreasing, increasing, upside-down bathtub shaped (unimodal) or bathtub-shaped. It is difficult (or even impossible) to determine analytically the parameter spaces corresponding to the upside-down bathtub shaped and bathtub-shaped hrfs for the BNH distribution. However, we can observe from Equation (2.4) that unimodal and bathtub-shaped hrfs can only be obtained when $\alpha < 1$ and $\alpha > 1$ respectively. So, the new three parameter distribution is quite flexible and can be used effectively in analysing real data.

We now explore the asymptotics behaviors of the cumulative, density and hazard functions. First, as $x \to 0$, equations (2.2), (2.3) and (2.4) are given by

$$F(x) \sim \frac{(\alpha\,\lambda\,x)^a}{a\,B(a,b)} \qquad \text{as} \quad x \to 0,$$
$$f(x) \sim \frac{(\alpha\,\lambda)^a\,x^{a-1}}{B(a,b)} \qquad \text{as} \quad x \to 0,$$
$$h(x) \sim \frac{(\alpha\,\lambda)^a\,x^{a-1}}{B(a,b)} \qquad \text{as} \quad x \to 0.$$

Second, the asymptotics of equations (2.2), (2.3) and (2.4) as $x \to \infty$ are given by

$$1 - F(x) \sim \frac{\exp[-b(\lambda\,x)^\alpha]}{b\,B(a,b)} \qquad \text{as} \quad x \to \infty,$$
$$f(x) \sim \frac{\alpha\,\lambda^\alpha x^{\alpha-1}\exp[-(\lambda\,x)^\alpha]}{B(a,b)} \qquad \text{as} \quad x \to \infty,$$
$$h(x) \sim b\,\alpha\,\lambda^\alpha\,x^{\alpha-1} \qquad \text{as} \quad x \to \infty.$$

3. Linear representations

In this section, we provide linear representations for the cdf and pdf of X. For $|z| < 1$ and $b > 0$ a real non-integer number, the generalized binomial expansion holds

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i\,\Gamma(b)}{i!\,\Gamma(b-i)}\,z^i.$$

Let

$$w_i = w_i(a,b) = \frac{(-1)^i\,\Gamma(a+b)}{(a+i)\,i!\,\Gamma(a)\,\Gamma(b-i)}.$$

Applying this identity in equation (2.1) gives the linear representation

$$F(x) = \sum_{i=0}^{\infty} w_i\,H_{a+i}(x),$$

where $H_a(x) = G(x)^a$ denotes the exponentiated-G (exp-G) cumulative distribution and $G(x)$ is obtained from (1.1). By differentiating the last equation, the BNH pdf can be expressed as a linear representation

$$f(x) = \sum_{i=0}^{\infty} w_i\,h_{a+i}(x), \tag{3.1}$$

where $h_a(x) = aG(x)^{a-1}g(x)$ denotes the exponentiated NH (exp-NH) density function with power parameter $a > 0$. The properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar et al. [18] for exponentiated Weibull, Gupta et al. [9] for exponentiated Pareto, Gupta and Kundu [10] for exponentiated exponential, Nadarajah [19] for exponentiated Gumbel, Lemonte [13] for ENH, among several others. Equation (3.1) allows that some mathematical properties such as ordinary and incomplete moments, generating function and mean deviations of the BNH distribution can be derived from those quantities of the ENH distribution.

4. Moments

Hereafter, let $Y_{a+i} \sim$ ENH(a+i) denotes the ENH random variable with power parameter $a+i$. The s-th integer moment of X follows from (3.1) as

$$E(X^s) = \sum_{i=0}^{\infty} w_i\, E(Y_{a+i}^s),$$

where the sth moment of Y_{a+i} can be obtained from

$$E(Y_{a+i}^s) \;=\; \lambda^{-s}\,(a+i) \sum_{j,k,r=0}^{\infty} \frac{(-1)^{j+r}\binom{s}{j}\binom{\frac{s-j}{\alpha}}{k}\binom{a+i-1}{r}k!}{(1+r)^{k+1}}.$$

Then, the sth integer moment of X can be expressed as

$$E(X^s) \;=\; \lambda^{-s} \sum_{i,j,k,r=0}^{\infty} \frac{(-1)^{i+j+r}\binom{s}{j}\binom{\frac{s-j}{\alpha}}{k}\binom{a+i-1}{r}k!\,\Gamma(a+b)}{(1+r)^{k+1}\,i!\,\Gamma(a)\Gamma(b-i)}.$$

5. Quantile function

The quantile function (qf) of X is given by

$$Q(u) = F^{-1}(u) = \lambda^{-1}[1-\log(1-I_u^{-1}(a,b))]^{1/\alpha} - 1,\ 0<u<1, \qquad (5.1)$$

where $I_u^{-1}(a,b)$ is the inverse of the incomplete beta function. The shortcomings of the classical kurtosis measure are well-known. There are many heavy-tailed distributions for which this quantity is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of classical kurtosis for many generalized distributions. The Bowley's skewness is based on quartiles [12]:

$$B = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)},$$

and the Moors' kurtosis [17] is based on octiles:

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

where $Q(\cdot)$ is obtained from (5.1). Plots of the skewness and kurtosis for some choices of the parameter b as functions of a, and for some choices of a as functions of b, for $\alpha = 2.0$ and $\lambda = 3.0$, are displayed in Figure 3. The inverse of the incomplete beta function $I_u^{-1}(a,b)$ can be expressed as a power series from the Wolfram website
http://functions.wolfram.com/06.23.06.0004.01

$$I_u^{-1}(a,b) = \sum_{i=1}^{\infty} q_i\,[a\,B(a,b)\,u]^{i/a}.$$

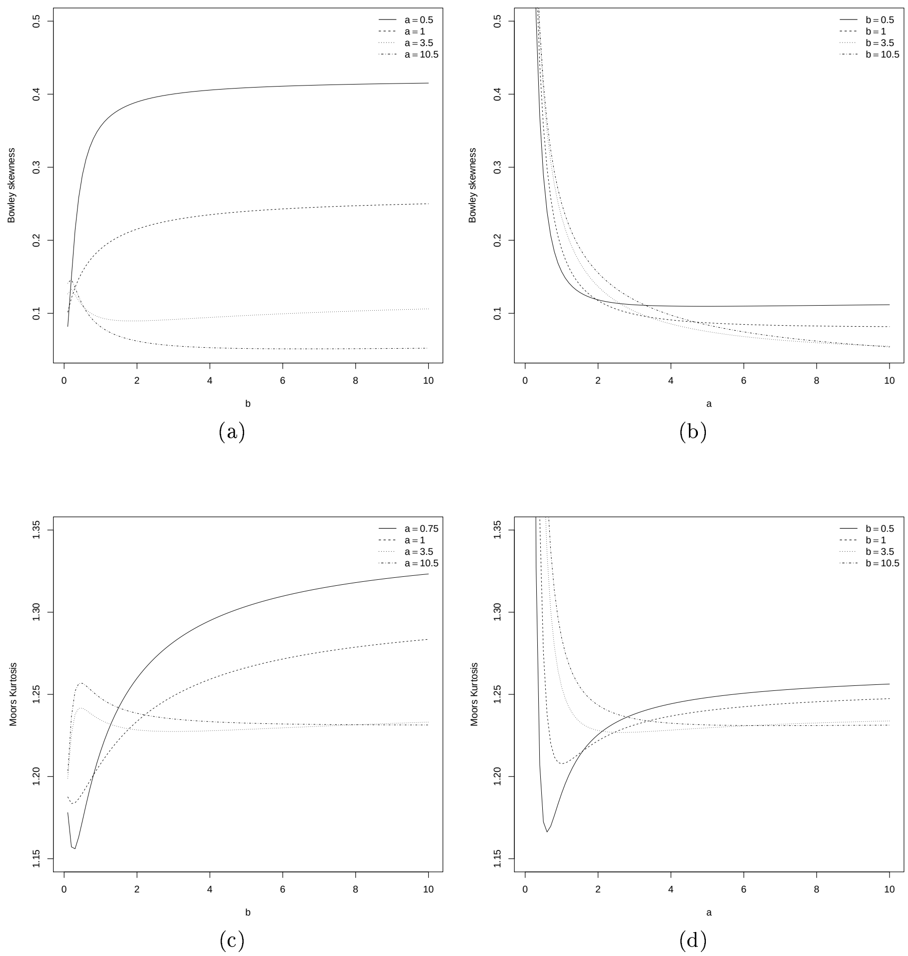


Figure 3. Plots of the Bowley skewness and Moors kurtosis for some parameter values.

Here, $q_1 = 1$ and the other q_i's (for $i \geq 2$) can be obtained from the recurrence equation

$$q_i = \frac{1}{i^2 + (a-2)i + (1-a)} \Big\{ (1-\delta_{i,2}) \sum_{r=2}^{i-1} q_r \, q_{i+1-r} \left[r(1-a)(i-r) - r(r-1) \right] + \sum_{r=1}^{i-1} \sum_{s=1}^{i-r} q_r \, q_s \, q_{i+1-r-s} \left[r(r-a) + s(a+b-2)(i+1-r-s) \right] \Big\},$$

where $\delta_{i,2} = 1$ if $i = 2$ and $\delta_{i,2} = 0$ if $i \neq 2$. Using the generalized power series, we can write

$$Q(u) = \sum_{k=1}^{\infty} \beta_k \, \left[I_u^{-1}(a,b) \right]^k,$$

where $\beta_1 = 1/[\alpha\,\lambda]$, $\beta_2 = 1/[2\,\alpha^2\,\lambda]$, $\beta_3 = (\alpha^2+1)/(6\alpha^3\,\lambda)$, $\beta_4 = (8\,\alpha^4 + 5\,\alpha^3 + 10\,\alpha^2 + 1)/[120\,\alpha^5\,\lambda]$, etc. Then,

$$Q(u) = \sum_{k=1}^{\infty}\beta_k\left(\sum_{i=1}^{\infty} q_i\,[a\,B(a,b)\,u]^{i/a}\right)^k = \sum_{k=1}^{\infty}\beta_k\,u^{\frac{k}{a}}\left(\sum_{i=0}^{\infty} q_{i+1}\,[a\,B(a,b)\,u]^{i/a}\right)^k = \sum_{k=1}^{\infty}\beta_k\,u^{\frac{k}{a}}\left(\sum_{i=0}^{\infty}\lambda_i\,u^{i/a}\right)^k, \tag{5.2}$$

where $\lambda_i = q_{i+1}\,[a\,B(a,b)]^{i/a}$ for $i \geq 0$. We use throughout the paper a result of Gradshteyn and Ryzhik [8] for a power series raised to a positive integer k (for $k \geq 1$)

$$\left(\sum_{i=0}^{\infty}\lambda_i\,v^i\right)^k = \sum_{i=0}^{\infty} c_{k,i}\,v^i, \tag{5.3}$$

where $c_{k,0} = \lambda_0^k$ and the coefficients $c_{k,i}$ (for $i = 1, 2, \ldots$) are obtained from the recurrence equation

$$c_{k,i} = (i\,\lambda_0)^{-1}\sum_{m=1}^{i}[m(k+1) - i]\,\lambda_m\,c_{k,i-m}. \tag{5.4}$$

Clearly, $c_{k,i}$ can be determined from the quantities $\lambda_0, \ldots, \lambda_i$. Based on equation (5.3), we can write

$$Q(u) = \sum_{k=1}^{\infty}\beta_k\,u^{\frac{k}{a}}\sum_{i=0}^{\infty} c_{k,i}\,u^{i/a} = \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\beta_k\,c_{k,i}\,u^{(i+k)/a} = \sum_{l=1}^{\infty} e_l\,u^{l/a}, \tag{5.5}$$

where (for $l \geq 1$) $e_l = \sum_{(k,i)\in I_l}\beta_k\,c_{k,i}$ and

$$I_l = \{(i,k) | l = i + k, k = 1, 2, \cdots,\ i = 0, 1, \cdots\}.$$

Let $W(\cdot)$ be any integrable function in the positive real line. We can write

$$\int_0^{\infty} W(x)\,f(x;\theta)dx = \int_0^1 W\left(\sum_{l=1}^{\infty} e_l\,u^{l/a}\right)du. \tag{5.6}$$

Equations (5.5) and (5.6) are the main results of this section since we can obtain from them various BNH mathematical quantities. Established algebraic expansions to determine these quantities based on equation (5.6) can be more efficient then using numerical integration of the density (2.3), which can be prone to rounding off errors among others. For the majority of these quantities we can substitute ∞ in the sum by a moderate number as twenty. In fact, several of them can follow by using the hight-hand integral for special $W(\cdot)$ functions, which are usually more simple than if they are based on the left-hand integral.

6. Moment generating function

Here, we provide a formula for the mgf $M(t) = \mathrm{E}(\mathrm{e}^{\mathrm{tX}})$ of X. Thus, $M(t)$ is given by

$$M_X(t) = \int_0^1 \exp\left[t\,Q(u)\right]\,du = \sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^1 Q(u)^n\,du.$$

However

$$Q(u)^n = \left(\sum_{l=1}^{\infty} e_l\, u^{l/a}\right)^n = u^{n/a}\left(\sum_{l=0}^{\infty} e_{l+1}\, u^{l/a}\right)^n = \sum_{l=0}^{\infty} p_{n,l}\, u^{(n+l)/a},$$

where $p_{n,0} = e_1{}^n$ and the coefficients $p_{n,i}$ (for $i = 1, 2, \ldots$) come from the recurrence equation

$$p_{n,i} = (i\, e_1)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, e_{m+1}\, p_{n,i-m}.$$

Finally, we obtain

$$M_X(t) = a \sum_{n,l=0}^{\infty} \frac{p_{n,l}\ t^n}{n!\,(n+l+a)}.$$

An alternative expression for $E(X^n)$ follows as

$$E(X^n) = a \sum_{l=0}^{\infty} \frac{p_{n,l}}{(n+l+a)}.$$

7. Incomplete moments

The nth incomplete moment of X is defined as $m_r(y) = E(X^r|X > y) = \int_y^{\infty} x^r f(x)dx$. It can be immediately derived from the moments of Y having the ENH distribution. Thus, from equation (3.1), we can write $m_r(y)$ as

$$\begin{aligned} m_r(y) &= \sum_{i,j=0}^{\infty}\sum_{k=0}^{r} w_i \frac{(-1)^{s+j-k}\,\mathrm{e}^{(j+1)}}{(j+1)^{k/\alpha}+1}\binom{a+i-1}{j}\binom{r}{k} \\ &\times\ \Gamma\left(\frac{k}{\alpha}+1, (j+1)(1+\lambda\, y)^{\alpha}\right). \end{aligned}$$

An alternative expression for $m_r(y)$ takes the form

$$\begin{aligned} m_r(y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{r} \frac{(-1)^{r+i-j}\,\mathrm{e}^{(b+i)}}{\lambda^r\, B(a,b)\,(b+i)^{j/\alpha+1}}\binom{a-1}{i}\binom{r}{j} \\ &\times\ \Gamma\left(\frac{j}{\alpha}+1, (b+i)(1+\lambda\, x)^{\alpha}\right). \end{aligned}$$

8. Mean deviations

The deviations from the mean and the median are usually used as measures of spread in a population. Let $\mu = E(X)$ and M be the median of the BNH distribution, respectively. The mean deviations about the mean and about the median of X can be calculated as

$$\delta_1 = E(|x-\mu|) = 2\mu\, F(\mu) - 2\, m_1(\mu) \quad \text{and} \quad \delta_2 = E(|x-\theta|) = \mu - 2\, m_1(M)$$

respectively, $F(\mu)$ follows (2.2) and $m_1(q) = \int_0^q t\, f(t)dt$. The function $m_1(q)$ can be expressed as

$$\begin{aligned} m_1(q) &= \sum_{i=0}^{\infty}\sum_{j=0}^{1} \frac{(-1)^{1+i-j}\,\mathrm{e}^{(b+i)}}{\lambda\, B(a,b)(b+i)^{j/\alpha+1}}\binom{a-1}{i}\binom{1}{j} \\ &\times\ \left[\Gamma\left(\frac{j}{\alpha}+1\right) - \Gamma\left(\frac{j}{\alpha}+1, (b+i)(1+\lambda\, x)^{\alpha}\right)\right]. \end{aligned}$$

Based on the mean deviations, we can construct Lorenz and Bonferroni curves, which are important in several areas such as economics, reliability, demography and actuary. For a given probability π, the Bonferroni and Lorenz curves are defined by $B(\pi) = m_1(q)/(\pi\,\mu)$ and $L(\pi) = m_1(q)/\mu$, respectively, where $q = F^{-1}(\pi) = Q(\pi)$ can be obtained from (2.5).

9. Rényi entropy and Shannon entropy

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi [25] and Shannon [27] entropies. The Rényi entropy of a random variable with pdf $f(x)$ is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left(\int_0^\infty f^\gamma(x)dx\right),$$

for $\gamma > 0$ and $\gamma \neq 1$. The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is a special case of the Rényi entropy when $\gamma \uparrow 1$. For the BNH model direct calculation yields

$$\begin{aligned} &\mathrm{E}\{-\log[f(X)]\} = -\log(\alpha\lambda) + \log[B(a,b)] + (1-\alpha)\,\mathrm{E}\{\log[1+\lambda X]\} \\ -\;& b\,\mathrm{E}\{1-(1+\lambda X)^\alpha\} + (1-a)\,\mathrm{E}\left\{\log\left[1-\mathrm{e}^{1-(1+\lambda \mathrm{X})^\alpha}\right]\right\}. \end{aligned}$$

First, we define and compute

$$A(a_1,a_2;\alpha,\lambda,b) = \int_0^\infty (1+\lambda x)^{a_1}\mathrm{e}^{\mathrm{b}[1-(1+\lambda\mathrm{x})^\alpha]}\left[1-\mathrm{e}^{1-(1+\lambda\mathrm{x})^\alpha}\right]^{\mathrm{a}_2}\mathrm{dx}.$$

Using the binomial expansion, we have

$$A(a_1,a_2;\alpha,\lambda,b) \;=\; \frac{\lambda}{\alpha}\sum_{i,j=0}^\infty (-1)^i\binom{a_2}{i}\binom{-1+\frac{a_2+1}{\alpha}}{j}\frac{\Gamma(j+1)}{(b+i)^{j+1}}.$$

Proposition:

Let X be a random variable with pdf (2.3). Then,

$$\mathrm{E}\{\log[1+\lambda X]\} \;=\; \frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha+t-1,a-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0},$$

$$\mathrm{E}\{1-(1+\lambda X)^\alpha\} \;=\; 1-\frac{\alpha\lambda}{B(a,b)}\,A(2\alpha-1,a-1,;\alpha,\lambda,b),$$

and

$$\mathrm{E}\left\{\log\left[1-\mathrm{e}^{1-(1+\lambda\mathrm{X})^\alpha}\right]\right\} \;=\; \frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha-1,a+t-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0}.$$

The simplest formula for the Shannon entropy of X is given by

$$\begin{aligned} \mathrm{E}\{-\log[f(X)]\} \;&=\; -\log(\alpha\lambda)+\log[B(a,b)]+(1-\alpha) \\ &+\; (1-\alpha)\frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha+t-1,a-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0} \\ &-\; b\left[1-\frac{\alpha\lambda}{B(a,b)}\,A(2\alpha-1,a-1,;\alpha,\lambda,b)\right] \\ &+\; (1-a)\frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha-1,a+t-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0}. \end{aligned}$$

After some algebraic developments, we obtain an alternative expression for $I_R(\gamma)$

$$\begin{aligned} I_R(\gamma) &= \frac{\gamma}{1-\gamma}\log(\alpha\lambda) - \frac{\gamma}{1-\gamma}\log\left[B(a,b)\right] \\ &+ \frac{1}{1-\gamma}\log\left[\frac{\lambda}{\alpha}\sum_{i,j=0}^{\infty}(-1)^i\binom{\gamma(\alpha-1)}{i}\binom{(\gamma-1)(1-\frac{1}{\alpha})}{j}\frac{\Gamma(j+1)}{(b+i)^{j+1}}\right]. \end{aligned}$$

10. Order statistics

Let $X_1,\ldots,X_n$ be a random sample of size n from BNH(a,b,α,λ). Then, the pdf and cdf of the ith order statistic, say $X_{i:n}$, are given by

$$\begin{aligned} f_{i:n}(x) &= \frac{f(x)}{B(i,n-i+1)}F^{i-1}(x)(1-F(x))^{n-i} \\ &= \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}(-1)^m\binom{n-i}{m}f(x)F^{i+m-1}(x), \end{aligned} \tag{10.1}$$

and

$$\begin{aligned} F_{i:n}(x) &= \int_0^x f_{i:n}(t)dt \\ &= \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\frac{(-1)^m}{m+i}\binom{n-i}{m}F^{i+m}(x), \end{aligned} \tag{10.2}$$

where $F^{i+m}(x) = \left[\sum_{r=0}^{\infty} b_r\, G(x)^r\right]^{i+m}$. Using (5.3) and (5.4), equations (10.1) and (10.2) can be written as

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\sum_{r=1}^{\infty}\frac{(-1)^m r c_{i+m,r}}{m+i}g(x)G^{r-1}(x),$$

and

$$F_{i:n}(x) = \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\sum_{r=0}^{\infty}\frac{(-1)^m c_{i+m,r}}{m+i}G^r(x).$$

Therefore, the sth moment of $X_{i:n}$ follows as

$$\begin{aligned} E(X_{i:n}^s) &= \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\sum_{r=1}^{\infty}\frac{(-1)^m c_{i+m,r}}{m+i}\int_0^{+\infty}t^s g(t)G^{r-1}(t)dt \\ &= \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\sum_{r=1}^{\infty}\frac{(-1)^m c_{i+m,r}}{m+i} \\ &\times\ \alpha\sum_{l=0}^{r-1}\sum_{i_1=0}^{s}\sum_{i_2=0}^{\infty}(-1)^{l+s-i_1}\lambda^{-s+1}\binom{r-1}{l}\binom{s}{i_1}\binom{\frac{i_1}{\alpha}}{i_2}\frac{\Gamma(1+i_2)}{(1+i_2)^{l+1}}. \end{aligned}$$

11. Estimation

Here, we determine the maximum likelihood estimates (MLEs) of the model parameters of the new family from complete samples only. Let $x_1,\ldots,x_n$ be observed values from the BNH distribution with parameters a,b,α and λ. Let $\Theta = (a,b,\alpha,\lambda)^\top$ be the parameter vector. The total log-likelihood function for Θ is given by

$$(11.1)\qquad \ell = \ell(\Theta) = n\log(\alpha\lambda) - n\log\left[B(a,b)\right] + (1-\frac{1}{\alpha}) \times \sum_{i=1}^{n}\log(1-t_i) + (a-1)\sum_{i=1}^{n}\log(1-\mathrm{e}^{\mathrm{t_i}}) + \mathrm{b}\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{t_i},$$

where $t_i = 1-(1+\lambda x_i)^{\alpha}$.

Numerical maximization of (11.1) can be performed by using the RS method [26], which is available in the gamlss package of the R, SAS (Proc NLMixed) or the Ox program, sub-routine MaxBFGS [5] or by solving the nonlinear likelihood equations obtained by differentiating (11.1).

The components of the score function $U_n(\Theta) = (\partial\ell_n/\partial a, \partial\ell_n/\partial b, \partial\ell_n/\partial\alpha, \partial\ell_n/\partial\lambda)$ are

$$U_a = \frac{\partial\ell}{\partial a} = -n\psi(a) + n\psi(a+b) + \sum_{i=1}^{n}\log(1-\mathrm{e}^{\mathrm{t_i}}),$$

$$U_b = \frac{\partial\ell}{\partial b} = -n\psi(b) + n\psi(a+b) + \sum_{i=1}^{n} t_i,$$

$$U_\alpha = \frac{n}{\alpha} - \frac{1}{\alpha^2}\sum_{i=1}^{n} log(1-t_i) - (1-\frac{1}{\alpha})\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{1-t_i} + (1-a)\sum_{i=1}^{n}\frac{t_i^{(\alpha)}\mathrm{e}^{\mathrm{t_i}}}{1-\mathrm{e}^{\mathrm{t_i}}} + b\sum_{i=1}^{n} t_i^{(\alpha)}$$

and

$$U_\lambda = \frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - (1-\frac{1}{\alpha})\sum_{i=1}^{n}\frac{t_i^{(\lambda)}}{1-t_i} + (1-a)\sum_{i=1}^{n}\frac{t_i^{(\lambda)}\mathrm{e}^{\mathrm{t_i}}}{1-\mathrm{e}^{\mathrm{t_i}}} + b\sum_{i=1}^{n} t_i^{(\lambda)},$$

where $t_i^{(\alpha)} = -(1+\lambda x_i)^{\alpha}\log\left[1+\lambda x_i\right]$ and $t_i^{(\lambda)} = -\alpha x_i(1+\lambda x_i)^{\alpha-1}$. Setting these equations to zero, $U_a = U_b = U_\alpha = U_\lambda = 0$, and solving them simultaneously yields the MLE $\widehat{\Theta}$ of Θ.

For interval estimation on the model parameters, we require the observed information matrix whose elements $U_{rs} = \partial^2\ell/\partial r\partial s$ (for $r,s = a,b,\alpha,\lambda$) can be obtained from the authors upon request. Under standard regularity conditions that are fulfilled for the proposed model whenever the parameters are in the interior of the parameter space, we can approximate the distribution of $(\widehat{\Theta}-\Theta)$ by the multivariate normal $N_{r+3}(0, J(\Theta)^{-1})$ distribution, where r is the number of parameters of the baseline distribution.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct likelihood ratio (LR) statistics for testing some sub-models of the BNH distribution. For example, we may use LR statistics to check if the fit using the BNH distribution is statistically "superior" to the fits using the ENH, NH, E, GE, BE distributions for a given data set. In any case, considering the partition $\Theta = (\Theta_1^T, \Theta_2^T)^T$, tests of hypotheses of the type $H_0: \Theta_1 = \Theta_1^{(0)}$ versus $H_A: \Theta_1 \neq \Theta_1^{(0)}$ can be performed using the LR statistic $w = 2\{\ell(\widehat{\Theta}) - \ell(\widetilde{\Theta})\}$, where $\widehat{\Theta}$ and $\widetilde{\Theta}$ are the estimates of Θ under H_A and H_0, respectively. Under the null hypothesis H_0, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the vector Θ_1 of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_q^2 distribution. Often with lifetime data and reliability studies, one encounters censoring. A very simple random censoring mechanism very often realistic is one in which each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i = \min(X_i, C_i)$ and $\delta_i = I(X_i \leq C_i)$ is such that $\delta_i = 1$ if X_i is a time to event and $\delta_i = 0$ if it is right censored for $i = 1,\ldots,n$. The censored likelihood $L(\Theta)$ for the model parameters is

$$L(\Theta) \propto \prod_{i=1}^{n} [f(x_i; a, b, \alpha, \lambda)]^{\delta_i} \; [S(x_i; a, b, \alpha, \lambda)]^{1-\delta_i}, \tag{11.2}$$

where $S(x; a, b, \alpha, \lambda) = 1 - F(x; a, b, \alpha, \lambda)$ is the survival function obtained from (2.2) and $f(x; a, b, \alpha, \lambda)$ is given by (2.3). We maximize the log-likelihood (11.2) in the same way as described before.

12. Applications

In this section, we present two applications of the new distribution for two real data sets to illustrate its potentiality. We compared the fits of the BNH distribution with some of its special cases and other models such as beta Weibull (BW) [7], exponetiated Weibull (EW) [18], Weibull (W), generalized exponential (GE) [10] and exponetial (E) distributions given by:

- BW: $f(x) = \frac{c\lambda^c}{B(a,b)} x^{c-1} \exp[-b(\lambda x)^c]\{1 - \exp[-(\lambda x)^c]\}^{a-1}$;
- EW: $f(x) = c\, a\lambda(\lambda x)^{c-1} \exp[-(\lambda x)^c]\{1 - \exp[-(\lambda x)^c]\}^{a-1}$;
- W: $f(x) = c\lambda^c x^{c-1} \exp[-(\lambda x)^c]$;
- GE: $f(x) = \alpha\lambda \exp[-\lambda x](1 - \exp[-\lambda x])^{\alpha-1}$;
- E: $f(x) = \lambda \exp[-\lambda x]$.

First, we consider an uncensored data set corresponding to the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [15]. The second data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 [14]. The required numerical evaluations are carried out using the AdequacyModel package of the R software. Table 2 provides some descriptive measures for the two data sets.

Table 2. Descriptives statistics

Statistics	Real data sets	
	Data Set 1	Data Set 2
Mean	9.3656	46.3289
Median	6.3950	40.0000
Variance	110.4250	1244.4644
Skewness	3.2866	1.0432
Kurtosis	18.4831	3.4021
Minimum	0.0800	0.3000
Maximum	79.0500	154.0000

The MLEs of the model parameters for the fitted distributions and the Cramér-von Mises (W^*) and Arderson-Darling (A^*) statistics are given in Table 3. These test statistics are described in [3]. They are used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

We note that the BNH model fits the first data set better than the others models according to these statistics A^* and W^*. On the other hand, the second data set is better fitted by the BNH and Weibull distributions according to these statistics. Therefore, for both data sets, the BNH distribution can be chosen as the best distribution.

Plots of the estimated pdf and cdf of the BNH, ENH, NH and GE models fitted are given in Figure 4. The QQ-Plots are presented in Figure 5.

Table 3. MLEs, (standard errors in parentheses), A* and W* statistic.

Data Set	Distribution	Estimates				A*	W*
1(n=128)	BNH (α, λ, a, b)	0.1643	0.0649	1.5848	21.6176	**0.2022**	**0.0302**
		(0.3236	0.05939	0.2859	56.0143)		
	BW (α, β, a, b)	2.7346	0.9074	0.6662	0.3219	0.2882	0.0436
		(1.599	1.5103	0.2450	0.4363)		
	ENH(α, λ, c)	0.6372	0.3444	1.6885		0.2779	0.0421
		(0.1173	0.1752	0.3646)			
	EW (α, β, c)	2.7964	0.2989	0.6544		0.2884	0.0437
		(1.2631	0.1687	0.1346)			
	NH (α, λ)	0.9226	0.1217			0.6138	0.1017
		(0.1514	0.0344)				
	W (α, β)	0.1046	1.0478			0.7864	0.1314
		(0.0093	0.0675)				
	GE (λ, c)	0.1212	1.2179			0.6741	0.1122
		(0.0135	0.1488)				
	E (λ)	0.1068				0.7159	0.1192
		(0.0094)					
2(n=121)	BNH (α, λ, a, b)	1.4783	0.0090	1.3645	1.7799	**0.3985**	0.0537
		(1.1933	0.00580	0.4293	3.3848)		
	BW (α, β, a, b)	0.8184	2.0818	1.4783	0.0104	0.4418	0.05887
		(0.3705	2.2577	0.4305	0.0077)		
	ENH(α, λ, c)	1.6630	0.0119	1.2657		0.4251	0.0567
		(0.5787	0.0062	0.1877)			
	EW (α, β, c)	0.8131	0.0174	1.4761		0.4491	0.0599
		(0.3345	0.0045	0.3820)			
	NH (α, λ)	2.5705	0.0061			0.5443	0.0748
		(0.7452	0.0021)				
	W (α, β)	0.0199	1.3053			0.4013	**0.0536**
		(0.0014	0.0934)				
	GE (λ, c)	0.0278	1.5179			0.4288	0.0615
		(0.0029	0.1927)				
	E (λ)	0.0216				0.4146	0.0585
		(0.0019)					

13. Simulation

In this section, we conduct Monte Carlo simulation studies to assess on the finite sample behavior of the MLEs of α, λ, a and b. All results were obtained from 5000 Monte Carlo replications and the simulations were carried out the `R` programming language. In each replication, a random sample of size n is drawn from the BNH(α, λ, a, b) distribution and the BFGS method has been used by the authors for maximizing the total log-likelihood function $l(\theta)$. The BNH random number generation was performed using the inversion method. The true parameter values used in the data generating processes are $\alpha = 1.5, \lambda = 2, a = 0.5$ and $b = 2.5$. The Table 4 reports the empirical means, bias the mean squared errors (MSE) of the corresponding estimators for sample sizes $n = 25, 50, 100, 200$ and 400. From these figures in this table, we note that, as the sample size increases, the empirical biases and mean squared errors decrease in all the cases analyzed, as expected.

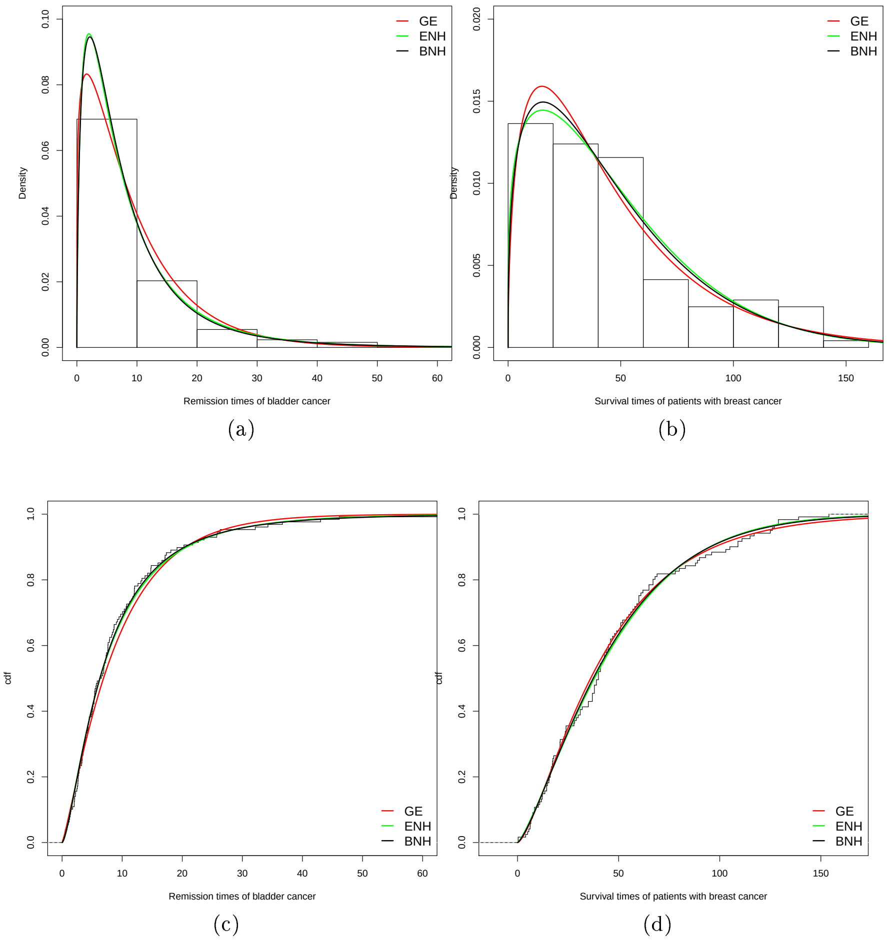
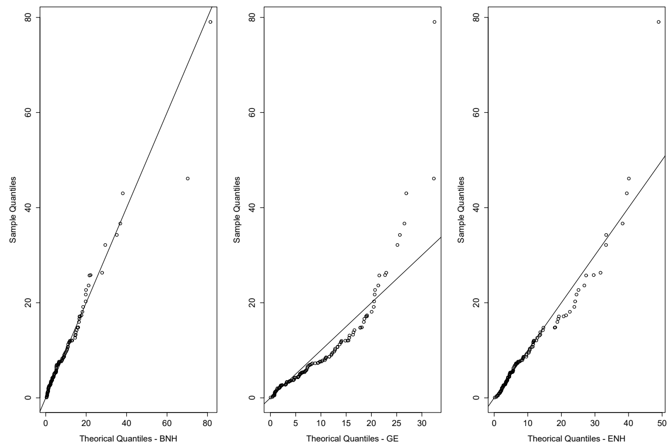


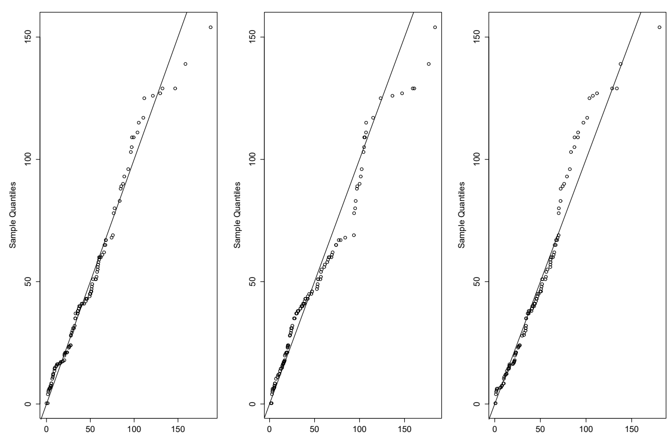
Figure 4. Plots of the estimated pdf and cdf of the BNH, ENH and GE models.

14. Concluding remarks

Continuous univariate distributions have been extensively used over the past decades for modeling data in several fields such as environmental and medical sciences, engineering, demography, biological studies, actuarial, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. In this paper, we proposed a new distribution called the beta Nadarajah-Haghighi (BNH) distribution, which generalizes the Nadarajah-Haghighi distribution. Further, the proposed distribution includes as special models other well-known distributions in the statistical literature. We studied some of its mathematical and statistical properties. We provided explicit expressions for the moments, incomplet moments, mean deviations, Rényi and Shannon entropies. The new



(a) Data set 1



(b) Data set 2

Figure 5. The QQ-plot for the the BNH, ENH and GE models.

model provide a good alternative to many existing life distributions in modeling positive real data sets. The hazard rate function of the BNH model can be constant, decreasing, increasing, upside-down bathtub and bathtub-shaped. The model parameters are estimated by maximum likelihood and the expected information matrix is derived. The usefulness of the new model is illustrated in two applications to real data using goodness-of-fit tests. Both applications have shown that the new model is superior to other fitted models. Therefore, the BNH distribution is an alternative model to the beta Weibull, exponentiated Weibull, Weibull, generalized exponential, extended exponential distributions and exponentiated Nadarajah-Haghighi distributions. We hope that the proposed model may be interesting for a wider range of statistical applications.

Table 4. Empirical means, bias and mean squared errors

n	Parameter	Mean	Bias	MSE
25	α	2,1882	0,6882	1,7969
	λ	2,4169	0,4169	0,8079
	a	0,5192	0,0192	0,0201
	b	2,1575	-0,3425	1,6367
50	α	1,8886	0,3886	0,8238
	λ	2,2029	0,2029	0,3293
	a	0,5116	0,0116	0,0089
	b	2,2749	-0,2251	0,8922
100	α	1,7211	0,2211	0,4281
	λ	2,1077	0,1077	0,1942
	a	0,5056	0,0056	0,004
	b	2,3704	-0,1296	0,5281
200	α	1,6029	0,1029	0,1628
	λ	2,0396	0,0396	0,074
	a	0,5040	0,004	0,0018
	b	2,4378	-0,0622	0,2087
400	α	1,5676	0,0676	0,1229
	λ	2,0329	0,0329	0,0614
	a	0,5020	0,002	0,0009
	b	2,4656	-0,0344	0,1636

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