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EFFECTS OF ELASTIC SUPPORTS ON NONLINEAR VIBRATIONS OF A SLIGHTLY CURVED BEAM

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Abstract: In this study, nonlinear vibrations of a slightly curved beam having arbitrary rising function are handled. The beam is restricted in longitudinal direction using elastic supports on both ends. Sag-to-span ratio of the beam, which is assumed to have sinusoidal curvature function at the beginning, is taken as 1/10. Beam being of Euler-Bernoulli type rests on Winkler elastic foundation and carries an arbitrarily placed concentrated mass. Equations of motion are obtained by using Hamilton Principle. Cubic and quadratic nonlinear terms have been aroused at the mathematical model because of the foundation and the beam's elongation. The Method of Multiple Scales (MMS), a perturbation technique, is used to solve the equations of motion analytically. The primary resonance case is taken into account during steady-state vibrations. The natural frequencies are obtained exactly for different control parameters such as supports' types, locations of the masses and linear coefficient of foundation. Frequency-amplitude and frequency-response graphs are drawn by using amplitude-phase modulation equations.

Keywords: Nonlinear vibrations, Slightly curved beam, Elastic supports, Elastic foundation.

Elastik Mesnetlerin Hafifçe Eğri Bir Kirişin Nonlineer Titreşimlerine Etkileri

Öz: Bu çalışmada, keyfi başlangıç fonksiyonuna sahip hafifçe eğri bir kirişin lineer olmayan titreşimleri ele alınmaktadır. Her iki ucundan elastik mesnetler kullanılarak kiriş, boyuna yönünde kısıtlanmıştır. Başlangıçta sinüsoidal eğrilik fonksiyonuna sahip olduğu varsayılan kiriş için, ulaşılan eğrilik yüksekliğinin izdüşüme oranı 1/10 alınmaktadır. Euler-Bernoulli tipinde olan kiriş Winkler elastik zemini üzerine oturmakta ve üzerinde keyfi olarak yerleştirilmiş kütleler taşımaktadır. Hamilton prensibi kullanılarak hareket denklemleri elde edilmiştir. Zeminden ve kiriş uzamasından dolayı matematiksel modelde kübik ve quadratik lineer olmayan terimler ortaya çıkmaktadır. Hareket denklemlerini analitik olarak çözümlemek için bir Pertürbasyon tekniği olan Çok Ölçekli Metod(MMS) kullanılmaktadır. Geçici-durum titreşimleri süresince baskın rezonans durumu dikkate alınmaktadır. Mesnetlerin tipleri, kütlelerin konumları ve zeminin lineer bileşeni gibi farklı mukayese parametreleri için doğal frekanslar elde edilmektedir. Genlik-faz modülasyon denklemleri kullanılarak frekans-genlik ve frekans-cevap grafikleri çizilmiştir.

Anahtar Kelimeler: Lineer olmayan titreşimler, Hafifçe eğri kiriş, Elastik mesnetler, Elastik zemin.

1. INTRODUCTİON

Beam structures are encountered in many different areas such as defense, aviation and the transportation fields. Due to their intense usage, a considerable amount of text books have been published on static and dynamic analysis of the beams [Ugural(2010); Carrera, Giunta and Petrolo(2011); Leissa and Qatu(2011); Librescu and Song(2006); Rao(2007); Sathyamoorthy(1997)]. Curved beams are preferred in many engineering fields for their improved strength over the straight beam structures. Therefore, many researchers have analyzed

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the dynamic responses of the curved beams by using simple models. Matter considered in these models, which have nonlinear behavior in means of system's response, is the case of being in resonance. If the system comes to the resonance state, the amplitudes of vibration increase dangerously, which is unwanted case. The models' nonlinear problems must be examined in order to prevent these cases which may be occur in any time during vibration. Before browsing the studies on the subject, it sounds good to mention from some research studies; Nayfeh and Mook reviewed and presented relevant works to the field up to 1979 in their book *Nonlinear Oscillations*. Considering in-plane, out-of-plane and coupled vibrations, Chidamparam and Leissa (1993) summarized the published literature on the vibrations of curved bars, beams, rings and arches of arbitrary shape. Focusing on the last two decades of research (1989-2012) done on vibration analysis, Hajianmaleki and Qatu (2013) published a research paper for the static and free vibration behaviors of the straight and curved beams. They reviewed various beam theories such as thin (or classical), thick (or shear deformation), layerwise beam theories, and different methods for solving equations of motion, such as the transfer matrix method and the finite element method.

Following studies can be seen as a background to the beams in means of the beam's curvature, any attachments to the beam and elastic/spring foundation of the beam. Rehfield (1974) derived the equations of motion of a shallow arch with an arbitrary rise function and studied the free vibrations. By adding the effects of transverse shear and rotary inertia, Singh and Ali (1975) studied a moderately thick clamped beam with a sinusoidal rise function. Tien *et al*.(1994) studied the dynamics of a shallow arch subjected to harmonic excitation. They investigated the bifurcation behavior of the shallow arch system in the presence of both external and 1:1 internal resonance. Nayfeh *et al*.(1999) studied to construct the nonlinear normal modes of a fixed-fixed buckled beam about its first post-buckling mode. They used MMS in order to investigate the internal resonances. Under the action of a moving load, Wu and Chiang(2004) investigated the radial (in-plane) bending-vibration responses of a uniform circular arch by means of the arch (curved beam) elements. They discussed influence of the moving speed, centrifugal force and frictional force on the dynamic behaviors of the circular arch. Lacarbonara *et al*.(2005) investigated the non-linear one-to-one interactions excited by an external primary-resonance base acceleration of a hinged-hinged imperfect beam with a torsional spring at one end and possessing veering between the frequencies of the lowest two modes. Lee *et al*.(2006) studied a clamped-clamped curved beam subjected to the transverse sinusoidal loads. Using the equations of motion, they determined the effect of parametric excitation near the symmetric mode resonance frequency. Huang and Chen (2007) studied structures with multiple attachments that were subjected to axial forces and oscillations. They examined the remaining model with the pure buckling problem, the free vibration problem, and the general eigenvalue problem. Ecsedi and Dluhi (2005) studied a non-homogeneous curved beam formulated in cylindrical coordinates and examined the static and dynamic analysis of the beam. Xiuchang *et al*.(2013) proposed a wave approach to investigate the wave propagation in the structural waveguides with the curved beam components. In order to predict the out-of-plane vibration of the horizontally curved beams in the mid- and high-frequency range, Kil *et al*.(2014) used the energy flow models. Reis and Iida (2014) studied on how to design elastic curved beams for stable hopping locomotion and the control method by using an unconventional actuation. By making use of free vibration of an elastic curved beams, a design strategy of hopping robots has been determined. Considering three shapes of beam (circular, parabolic, and sinusoidal) and three kinds of taper type (circular, parabolic, and sinusoidal), Lee *et al*.(2014) investigated the free vibrations of horizontally curved beams. A solid regular polygon cross section has been selected. Bayat *et al*.(2015) studied a laminated curved beam with the embedded magnetostrictive layers under simply-supported boundary conditions. They examined the effects of material properties, radius of the curvature and magnetostrictive layers on the vibration suppression. Wang *et al*.(2016) investigated the in-plane vibrations of a sinusoidal phononic crystal curved beam. They depicted the characteristic curve for wavenumber versus frequency. Kumar and Patel (2016) studied the internal resonances between the first symmetric and anti-symmetric modes of the fixed-fixed curved beams. They conducted some experiments through the direct excitation of the modes under a concentrated harmonic force excitation. Adessi *et al*.(2005) studied the regime of high pre-stressed beams. They examined post-buckling configurations of the beam considering a lumped mass that is rigidly clamped to the beam at an arbitrary point along its span and assuming different boundary conditions. By assuming the sinusoidal rising function, Erdogan *et al*. studied the nonlinear vibrations of the curved beams carrying a concentrated mass (2009) and the multiple concentrated masses (2010). For a general state of non-uniform initial stress, Chen and Shen (1998) derived the virtual work expressions of the initially stressed curved beams. They investigated the influence of the arc segment angles, elastic foundations, and initial stresses on the natural frequencies. Oz *et al*.(1998) examined a simply supported slightly curved beam resting on an elastic foundation. Considering the free-undamped and forced-damped vibrations, they analyzed the effects of elastic foundation, axial stretching and curvature on the vibrations of the beams. Abe (2006) studied the validity of nonlinear vibration analysis of the continuous systems with the quadratic and cubic nonlinearities. He treated the non-linear responses of a hinged-hinged Euler-Bernoulli beam resting on an elastic foundation. Kelly and Srinivas (2009) investigated the elastically connected axially-loaded beams, which may be attached to a Winkler foundation. Motaghian *et al*.(2011) proposed an exact solution to the free vibration problem of the beams having mixed boundary conditions. They solved the governing differential equations of the beams having some underlying elastic springs, which occupied a particular length of the beam. Wang *et al*.(2013) studied the nonlinear interaction of an inextensional beam on an elastic foundation with a three-to-one internal resonance. *Sato et al.(2008*) presented a mathematical hypothesis that a beam on equidistant elastic supports can be considered as a beam on the elastic foundation. They examined the relationship between them in the cases of static and free vibration. Ozkaya *et al*.(2016) investigated the dynamic behavior of a slightly curved beam resting on multiple springs. In simply supported case, the linear and nonlinear frequencies of the system were analyzed in detail. Ozyigit *et al*.(2017) analyzed the free out-of-plane vibrations of the curved beams which are symmetrically and nonsymmetrically tapered. They also investigated the out-of-plane free vibrations of the curved uniform and tapered beams with additional mass.

Some studies considering the effects of the boundary conditions are such that; Ozkaya *et al*.(1997) studied the nonlinear vibrations of a beam-mass system under the different boundary conditions. For different boundary conditions, locations and magnitude of the masses, he examined the effects of the mid-plane stretching on vibrations of the beam. Applying the coupled displacement field method, Rao *et al*.(2006) investigated the large amplitude free vibrations of the uniform shear flexible hinged-hinged and clamped-clamped beams. The effect of the concentrated mass on the vibrations was investigated. Wiedemann (2007) studied the Euler-Bernoulli beams interconnected by arbitrary joints and confined to arbitrary boundary conditions. Then, he presented an analytical solution for natural frequencies, modes shapes and orthogonality conditions on the system. Assuming the beam has a combination condition of clamped, free, pinned, and sliding, Goncalves *et al*.(2007) presented a numerical study on the vibration modes of the beam by means of a compact mode shape. For with various classical (or non-classical) boundary conditions, Wu and Chen (2008) examined the free vibrations of the beams carriying multiple sets of concentrated elements with each set consisting of a point mass, a translational and rotational spring. For arbitrary boundary conditions, Kiani (2010) examined the effects of slenderness ratio of the nanotube, small scale effect, initial axial load and stiffness of the elastic matrix on the natural frequencies of the single-walled nanotubes. Using a systematic theoretical procedure, Lin (1998) presented a static analysis of the extensional circular-curved Timoshenko beams with general nonhomogeneous elastic boundary conditions and found the generalized Green function of the differential equations. Lestari and

Hanagud(2001) found some closed form exact solutions to the problem of nonlinear vibrations of the buckled beams. They assumed their model with axial spring in spite of general supports conditions. Ghayesh (2012) investigated the free and forced vibrations of a Kelving-Voigt viscoelastic beam supported by a nonlinear spring. Linear and nonlinear frequencies of the system were analyzed by considering the nonlinear spring effect. In order to investigate the static pull-in instability of beam-type nano-electromechanical systems, Tadi Beni *et al*.(2011) considered the effect of Casimir attraction and elastic boundary conditions. They utilized through the rotational springs. Sari and Pakdemirli (2013) studied the dynamic behavior of a slightly curved microbeam having nonideal boundary conditions. They also presented references for the choice of resonable resonant conditions, design applications, and industrial applications of such systems. Jin *et al*.(2017) studied the vibration analysis of the 2-D curved beams with variable curvatures and general boundary conditions. They used the 2-D elasticity theory which not requires any assumptions on the deformations and stresses along the thickness direction. Shi *et al.*(2017) presented a unified method for modeling of functionally graded carbon-nanotube-reinforced composite (FG-CNTRC) beams based on first-order shear deformation elasticity theory. Using arbitrary boundary conditions, including various classical boundary conditions and elastic supports, they investigated free-vibration analysis of FG-CNTRC beams. In this work, nonlinear vibrations of curved beams restricted by elastic supports on both ends were investigated. The elastic supports were converted to the translational and rotational springs. The boundary conditions were idealized by means of suitable springs' coefficients. The mathematical models of the system were derived. In order to seek analytical solutions, the Method of Multiple Scales (MMS), a perturbation technique, was used. The amplitude and phase modulation equations weree obtained by considering primary resonance case. Assuming the curvature of the beam was a sinusoidal function, the numerical solutions were obtained for the steady-state phase of the vibrations.

2. DERIVATION OF MATHEMATICAL MODEL

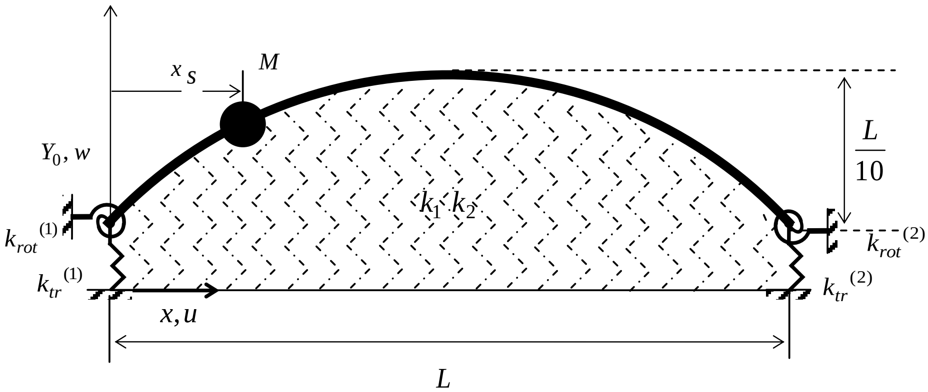

Figure 1

. Curved Beam resting on elastic foundation, which was restricted at both end with elastic supports.

In Fig.(1), a curved beam-mass system is restricted on both ends with the elastic supports. It is assumed that the supports are made of the translational(k_{tr}) and rotational(k_{rot}) springs. For the Winkler elastic foundation, let us assume that the foundation comprises of the springs'coefficients of linear(k_1) and nonlinear(k_2). w_m and u_m denote the transversal and longitudinal displacements, respectively. Assuming that ratio of the maximum amplitude of the beam to its projected lenght L is equal *1/10*, let us keep in mind that the beam's curvature function is of an arbitrarily arising function Y_0. Additionaly, there is a concentrated mass M attached at arbitrarily point($x=x_s$) of the beam.

In order to obtain the equations of motion of the system and its conditions, we use the Hamilton Principle defined as below:

$$\delta\int_{t_1}^{t_2}\left(T-U\right).dt=0 \tag{1}$$

where T is the system's kinetic energy and consists of the tranverse motion of the beam and concentrated mass. U is the system's potential energy and consists of the stretching and bending of the beam, the elastic foundation, the end springs.

In order to analyze the equations of motion within this system, U and T are written as follows;

$$U=\frac{1}{2}.E.A\left\{\int_0^{x_s}\left(u_1'+Y_0'.w_1'+\frac{1}{2}.w_1'^2\right)^2.dx+\int_{x_s}^{L}\left(u_2'+Y_0'.w_2'+\frac{1}{2}.w_2'^2\right)^2.dx\right\}+\frac{1}{2}.E.I.\left\{\int_0^{x_s}w_1''^2.dx+\int_{x_s}^{L}w_2''^2.dx\right\}$$
$$+\frac{1}{2}.\left\{\int_0^{x_s}\left(k_1.w_1^2+\frac{1}{2}.k_2.w_1^4\right).dx+\int_{x_s}^{L}\left(k_1.w_2^2+\frac{1}{2}.k_2.w_2^4\right).dx\right\}$$
$$+\frac{1}{2}.k_{tr}^{(1)}.w_1^2\Big|_{x=0}+\frac{1}{2}.k_{tr}^{(2)}.w_2^2\Big|_{x=L}+\frac{1}{2}.k_{rot}^{(1)}.w_1'^2\Big|_{x=0}+\frac{1}{2}.k_{rot}^{(2)}.w_2'^2\Big|_{x=L}$$
$$T=\frac{1}{2}\rho.A.\left\{\int_0^{x_s}\dot{w}_1^2.dx+\int_{x_s}^{L}\dot{w}_2^2.dx\right\}+\frac{1}{2}.M.\dot{w}_1^2\Big|_{x=x_s} \tag{2}$$

In Eq. (2), E is the young modulus, ρ is the density, A is the cross sectional area of the beam, and I is the moment of inertia of the beam cross-section with respect to the neutral axis. ($\dot{}$) and (') denote differentiations with respect to the time t and the spatial variable x, respectively.

For the depicted system, let us derive the mathematical models. Inserting Eq. (1) into Eq. (2), one obtains one and double folded integrals. One folded integrals correspond to the boundary and continuity conditions. Double folded integrals correspond to the equations of motion. By invoking the necessary calculations, the longitudinal displacement term (u_m) could be eliminated from the equations of motion in the tranverse direction. Finally, one obtains following the equations of motion and the boundary and continuity conditions as follows:

$$\rho.A.\ddot{w}_{m+1}+E.I.w_{m+1}^{iv}+k_1.w_{m+1}+k_2.w_{m+1}^3=\frac{E.A}{L}.\left\{\int_0^{x_s}\left\{Y_0'.w_1'+\frac{1}{2}.w_1'^2\right\}.dx+\int_{x_s}^{x_{s+1}}\left\{Y_0'.w_2'+\frac{1}{2}.w_2'^2\right\}.dx\right\}.\left(Y_0''+w_{m+1}''\right)$$
$$\left(E.I.w_1'''+k_{tr}^{(1)}.w_1\right)\Big|_{x=0}=0,\quad\left(E.I.w_1''-k_{rot}^{(1)}.w_1'\right)\Big|_{x=0}=0,\quad\left(E.I.w_2'''-k_{tr}^{(2)}.w_2\right)\Big|_{x=L}=0,\quad\left(E.I.w_2''+k_{rot}^{(2)}.w_2'\right)\Big|_{x=L}=0$$
$$w_1\big|_{x=x_s}=w_2\big|_{x=x_s},\ w_1'\big|_{x=x_s}=w_2'\big|_{x=x_s},\ w_1''\big|_{x=x_s}=w_2''\big|_{x=x_s},\ \left(E.I.w_1'''-E.I.w_2'''\right)\Big|_{x=x_s}=M.\ddot{w}_1\big|_{x=x_s} \tag{3-4}$$

where $x_0=0$ and $x_{s+1}=L$, $m=0,1,..s$.

The equations of the motion and the conditions were dependent on the size of the system and the materials used. In order to make them independent from the dimensional parameters, the following definitions must be made:

$$\hat{w}=\frac{w}{r},\ \hat{Y}_0=\frac{Y_0}{r},\ \hat{x}=\frac{x}{L},\ \hat{t}=\frac{1}{L^2}.\sqrt{\frac{E.I}{\rho.A}}.t,\ I=r^2.A,\ \eta=\frac{x_s}{L},\ \eta_0=0,\ \eta_1=\eta,\ \eta_2=1,$$

$$\alpha=\frac{M}{\rho.A.L},\ \ \gamma_1=\frac{k_1.L^4}{E.I},\ \ \gamma_2=\frac{k_2.L^4}{E.A},\ \ \kappa_{tr}^{(i)}=\frac{k_{tr}^{(i)}.L^3}{E.I},\ \ \kappa_{rot}^{(i)}=\frac{k_{rot}^{(i)}.L}{E.I}\qquad i=1,2 \tag{5}$$

where r is the radius of gyration of the beam's cross section, α is the mass ratio between the concentrated mass and the beam's mass, η is the dimensionless distance of the mass from left hand-side support, γ_1 and γ_2 are the dimensionless linear and nonlinear coefficients of the elastic foundation, κ_{tr} and κ_{rot} are the dimensionless translational and rotational springs'coefficients, respectively.

Making necessary simplifications after non-dimensionalization, the equations of motion via the boundary and continuity conditions can be rewritten as follows:

$$\ddot{\hat{w}}_{m+1}+\hat{w}_{m+1}^{iv}+\gamma_1.\hat{w}_{m+1}+\gamma_2.\hat{w}_{m+1}^3=\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}\left\{\hat{Y}_0'.\hat{w}_{r+1}'+\frac{1}{2}.\hat{w}_{r+1}'^{\,2}\right\}.d\hat{x}.\left(\hat{Y}_0''+\hat{w}_{m+1}''\right)$$
$$\hat{w}_1\Big|_{\hat{x}=\eta}=\hat{w}_2\Big|_{\hat{x}=\eta},\ \ \hat{w}_1'\Big|_{\hat{x}=\eta}=\hat{w}_2'\Big|_{\hat{x}=\eta},\ \ \hat{w}_1''\Big|_{\hat{x}=\eta}=\hat{w}_2''\Big|_{\hat{x}=\eta},\ \ \left(\hat{w}_1'''-\hat{w}_2'''\right)\Big|_{\hat{x}=\eta}=\alpha.\ddot{\hat{w}}_1\Big|_{\hat{x}=\eta}$$
$$\left(\hat{w}_1'''+\kappa_{tr}^{(1)}.\hat{w}_1\right)\Big|_{\hat{x}=0}=0,\ \ \left(\hat{w}_1''-\kappa_{rot}^{(1)}.\hat{w}_1'\right)\Big|_{\hat{x}=0}=0,\ \ \left(\hat{w}_2'''-\kappa_{tr}^{(2)}.\hat{w}_2\right)\Big|_{\hat{x}=1}=0,\ \ \left(\hat{w}_2''+\kappa_{rot}^{(2)}.\hat{w}_2'\right)\Big|_{\hat{x}=1}=0 \tag{6}$$

3. ANALYTICAL SOLUTIONS

In order to search approximate solutions to the problem, the method of multiple scales (MMS)[Nayfeh (1973), (1981)] will be applied to the partial differential equations and the corresponding boundary conditions directly. First of all, the sign of dimensionless (^) must be removed so that the equations and conditions come into a view easily understand. Adding dimensionless damping μ and external forcing $F.cos(\Omega.t)$, where Ω is excitation frequency, one has following equation;

$$\ddot{w}_{m+1}+w_{m+1}^{iv}+\gamma_1.w_{m+1}+\gamma_2.w_{m+1}^3+2.\mu.\dot{w}_{m+1}=\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}\left\{Y_0'.w_{r+1}'+\frac{1}{2}.w_{r+1}'^{\,2}\right\}.dx.\left(Y_0''+w_{m+1}''\right)+F.\cos(\Omega.t) \tag{7}$$

Eq. (7) is assumed to have an expansion solution as follows:

$$w_i(x,t;\varepsilon)=\sum_{j=1}^{3}\varepsilon^j.w_{ij}(x,T_0,T_1,T_2). \tag{8}$$

where ε is a small bookkeeping parameter artificially inserted into the equations, $T_0=t$ is the fast time scale, and $T_1=\varepsilon.t$ and $T_2=\varepsilon^2.t$ are the slow time scales in MMS.

Derivatives with respect to time are defined as $D_n\equiv\partial/\partial T_n$, and written as:

$$\frac{d}{dt}=D_0+\varepsilon.D_1+\varepsilon^2.D_2+...,\ \ \frac{d^2}{dt^2}=D_0^{\,2}+2.\varepsilon.D_0.D_1+\varepsilon^2.(D_1^{\,2}+2.D_0.D_2)+... \tag{9}$$

In this analysis, the beam's curvature function is assumed as $Y_0=\sim 0(1)$. In order to counter the effects of the nonlinear terms via the same order of damping and forcing, the forcing and damping terms are reordered as follows:

$$\mu=\varepsilon^2.\tilde{\mu} \qquad F_{p+1}=\varepsilon^3.\tilde{F}_{p+1} \tag{10}$$

Inserting Eqs. (8-10) into Eq. (7) and separating the terms of each order, one finds the following equations:

<u>*Order*</u> *ε*[1]*:*

$$D_0^2.w_{(m+1)1}+w_{(m+1)1}^{iv}+\gamma_1.w_{(m+1)1}=\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}Y_0'.w_{(r+1)1}'.dx\right\}.Y_0'' \tag{11}$$

<u>*Order*</u> *ε*[2]*:*

$$D_0^2.w_{(m+1)2}+w_{(m+1)2}^{iv}+\gamma_1.w_{(m+1)2}=-2.D_0.D_1.w_{(m+1)1}+\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}Y_0'.w_{(r+1)2}'.dx\right\}.Y_0''+\frac{1}{2}.\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}w_{(r+1)1}'^2.dx\right\}.Y_0''$$
$$+\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}Y_0'.w_{(r+1)1}'.dx\right\}.w_{(m+1)1}'' \tag{12}$$

<u>*Order*</u> *ε*[3]*:*

$$D_0^2.w_{(m+1)3}+w_{(m+1)3}^{iv}+\gamma_1.w_{(m+1)3}=-2.D_0.D_1.w_{(m+1)2}-\left(D_1^2+2.D_0.D_2\right).w_{(m+1)1}-\gamma_2.w_{(m+1)1}^3-2.\tilde{\mu}.D_0.w_{(m+1)1}$$
$$+\tilde{F}.cos(\Omega.t)+\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}Y_0'.w_{(r+1)3}'.dx\right\}.Y_0''+\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}w_{(r+1)1}'.w_{(r+1)2}'.dx\right\}.Y_0''+\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}Y_0'.w_{(r+1)2}'.dx\right\}.w_{(m+1)1}''$$
$$+\frac{1}{2}.\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}w_{(r+1)1}'^2.dx\right\}.w_{(m+1)1}''+\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}}Y_0'.w_{(r+1)1}'.dx\right\}.w_{(m+1)2}'' \tag{13}$$

One requires some conditions for solving the Eqs. (11) to (13). They are given for j=1, 2, 3 as below:

$$\left(w_{1j}'''+\kappa_{tr}^{(1)}.w_{1j}\right)\Big|_{x=0}=0,\quad \left(w_{1j}''-\kappa_{rot}^{(1)}.w_{1j}'\right)\Big|_{x=0}=0,\quad \left(w_{2j}'''-\kappa_{tr}^{(2)}.w_{2j}\right)\Big|_{x=1}=0,\quad \left(w_{2j}''+\kappa_{rot}^{(2)}.w_{2j}'\right)\Big|_{x=1}=0,$$
$$w_{1j}\Big|_{x=\eta}=w_{2j}\Big|_{x=\eta},\quad w_{1j}'\Big|_{x=\eta}=w_{2j}'\Big|_{x=\eta},\quad w_{1j}''\Big|_{x=\eta}=w_{2j}''\Big|_{x=\eta}, \tag{14}$$
$$\left(w_{1j}'''-w_{2j}'''\right)\Big|_{x=\eta}=\alpha.\left(D_0^2.w_{1j}+2.D_0.D_1.w_{1(j-1)}+\left(D_1^2+2.D_0.D_2\right).w_{1(j-2)}\right)\Big|_{x=\eta}$$

Eq. (11) from order ε^1 corresponds to the linear problem of the system and other orders in Eqs.(12) and (13) to the nonlinear problem. These cases are investigated seperately while solving these equations. Let us assume that the linear problem accepts the following solution:

$$w_{(m+1)1}(x,T_0,T_1,T_2)=\left[A(T_1,T_2).e^{i.\omega.T_0}+cc\right].Y_{m+1}(x) \tag{15}$$

In Eq. (15), ω is the natural frequency, *cc* is the complex conjugate of the preceding terms, and Y_{m+1} is the function describing the mode shape. By inserting Eq. (15) into Eqs. (11) and (14) while taking as j=1, one obtains the following differential equations and corresponding conditions:

$$Y_{m+1}^{iv} - \beta^4 . Y_{m+1} = \left\{ \sum_{r=0}^{1} \int_{\eta_r}^{\eta_{r+1}} Y_0' . Y_{r+1}' . dx \right\} . Y_0'' , \quad \beta = \sqrt[4]{\omega^2 - \gamma_1}$$

$$\left(Y_1''' + \kappa_{tr}^{(1)} . Y_1 \right)\Big|_{x=0} = 0, \quad \left(Y_1'' - \kappa_{rot}^{(1)} . Y_1' \right)\Big|_{x=0} = 0, \quad \left(Y_2''' - \kappa_{tr}^{(2)} . Y_2 \right)\Big|_{x=1} = 0, \quad \left(Y_2'' + \kappa_{rot}^{(2)} . Y_2' \right)\Big|_{x=1} = 0$$

$$Y_1\big|_{x=\eta} = Y_2\big|_{x=\eta}, \quad Y_1'\big|_{x=\eta} = Y_2'\big|_{x=\eta}, \quad Y_1''\big|_{x=\eta} = Y_2''\big|_{x=\eta}, \quad \left(Y_1''' - Y_2''' + \alpha . \omega^2 . Y_1 \right)\Big|_{x=\eta_p} = 0 \qquad (16)$$

To be able to find the solutions at the order ε^2 of the perturbation series, an assumption of $A=A(T_2)$ must be provided. This means that there is no dependence of this order on T_1. By inserting Eq. (15) into Eq. (12), the following solution is suitable at this order:

$$w_{(m+1)2} = \left[A^2 . e^{2.\mathrm{i}.\omega.T_0} + cc \right] . \phi_{(m+1)1}(x) + 2. A . \bar{A} . \phi_{(m+1)2}(x) \qquad (17)$$

Substituting Eq.(17) into both Eq.(12) and (14) while keeping in mind that j=2, yields the following equations and conditions:

$$\phi_{(m+1)1}^{iv} - \tau^4 . \phi_{(m+1)1} - \left\{ \sum_{r=0}^{1} \int_{\eta_r}^{\eta_{r+1}} Y_0' . \phi_{(r+1)1}' . dx \right\} . Y_0'' = \frac{1}{2} . \left\{ \sum_{r=0}^{1} \int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'^2 . dx \right\} . Y_0'' + \left\{ \sum_{r=0}^{1} \int_{\eta_r}^{\eta_{r+1}} Y_0' . Y_{r+1}' . dx \right\} . Y_{m+1}''$$

$$\phi_{(m+1)2}^{iv} + \psi^4 . \phi_{(m+1)2} - \left\{ \sum_{r=0}^{1} \int_{\eta_r}^{\eta_{r+1}} Y_0' . \phi_{(r+1)2}' . dx \right\} . Y_0'' = \frac{1}{2} . \left\{ \sum_{r=0}^{1} \int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'^2 . dx \right\} . Y_0'' + \left\{ \sum_{r=0}^{1} \int_{\eta_r}^{\eta_{r+1}} Y_0' . Y_{r+1}' . dx \right\} . Y_{m+1}''$$

$$\left(\phi_{1g}''' + \kappa_{tr}^{(1)} . \phi_{1g} \right)\Big|_{x=0} = 0, \quad \left(\phi_{1g}'' - \kappa_{rot}^{(1)} . \phi_{1g}' \right)\Big|_{x=0} = 0, \quad \left(\phi_{2g}''' - \kappa_{tr}^{(2)} . \phi_{2g} \right)\Big|_{x=1} = 0, \quad \left(\phi_{2g}'' + \kappa_{rot}^{(2)} . \phi_{2g}' \right)\Big|_{x=1} = 0$$

$$\phi_{1g}\big|_{x=\eta} = \phi_{2g}\big|_{x=\eta}, \quad \phi_{1g}'\big|_{x=\eta} = \phi_{2g}'\big|_{x=\eta}, \quad \phi_{1g}''\big|_{x=\eta} = \phi_{2g}''\big|_{x=\eta}, \qquad (g=1,2.)$$

$$\left(\phi_{11}''' - \phi_{21}''' + 4. \alpha . \omega^2 . \phi_{11} \right)\Big|_{x=\eta} = 0 ; \quad (g=1), \qquad \left(\phi_{12}''' - \phi_{22}''' \right)\Big|_{x=\eta} = 0 \ (g=2) \qquad (18)$$

where τ and ψ are defined as follows:

$$\tau = \sqrt[4]{4.\omega^2 - \gamma_1} , \quad \psi = \sqrt[4]{\gamma_1} \qquad (19)$$

By substituting Eqs. (15) and (17) into both Eqs. (13) and (14) while keeping in mind j=3, the solutions at the last order of the perturbation series are assumed in the following form;

$$w_{(m+1)3}(x, T_0, T_2) = \varphi_{m+1}(x, T_2) . e^{\mathrm{i}.\omega.T_0} + W_{m+1}(x, T_2) + cc \qquad (20)$$

where $W_{m+1}(x,T_2)$ corresponds to the solutions for the non-secular terms, and *cc* is referred to the complex conjugate of the preceding terms.

By means of a detuning parameter σ showing closeness of the natural frequency to the external excitation frequency, one take the excitation frequency as below:

$$\Omega = \omega + \varepsilon^2.\sigma \tag{21}$$

By inserting Eqs. (20) and (21) into Eqs. (13) and (14), taking in mind j=3, and eliminating the secular terms, one obtains the following differential equations and conditions:

$$\begin{aligned}
&\varphi_{m+1}^{iv} - \beta^4.\varphi_{m+1} - \left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_0'.\varphi_{r+1}'.dx\right\}.Y_0'' = -2\mathrm{i}.\omega.\left(\dot{A}+\tilde{\mu}.A\right)Y_{m+1} + \frac{\tilde{F}_{m+1}}{2}.e^{\mathrm{i}.\sigma.T_2} \\
&+A^2.\bar{A}.\Bigg[-3.\gamma_2.Y_{m+1}{}^3 + \left\{\frac{3}{2}.\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'{}^2.dx + \sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_0'.\phi_{(r+1)1}'.dx + 2.\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_0'.\phi_{(r+1)2}'.dx\right\}.Y_{m+1}'' \\
&+\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_0'.Y_{r+1}'.dx.\left(\phi_{(m+1)1}'' + 2.\phi_{(m+1)2}''\right) + \left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'.\phi_{(r+1)1}'.dx + 2.\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'.\phi_{(r+1)2}'.dx\right\}.Y_0''\Bigg] \\
&\left(\varphi_1''' + \kappa_{tr}^{(1)}.\varphi_1\right)\Big|_{x=0} = 0, \quad \left(\varphi_1'' - \kappa_{rot}^{(1)}.\varphi_1'\right)\Big|_{x=0} = 0, \quad \left(\varphi_2''' - \kappa_{tr}^{(2)}.\varphi_2\right)\Big|_{x=1} = 0, \quad \left(\varphi_2'' + \kappa_{rot}^{(2)}.\varphi_2'\right)\Big|_{x=1} = 0 \\
&\varphi_1\big|_{x=\eta} = \varphi_2\big|_{x=\eta}, \quad \varphi_1'\big|_{x=\eta} = \varphi_2'\big|_{x=\eta}, \quad \varphi_1''\big|_{x=\eta} = \varphi_2''\big|_{x=\eta}, \quad \left(\varphi_1''' - \varphi_2''' + \alpha.\omega^2.\varphi_1\right)\Big|_{x=\eta} = 2.\alpha.\mathrm{i}.\omega.\dot{A}Y_1\big|_{x=\eta}
\end{aligned} \tag{22}$$

In order to have a solution for this nonhomogenous equation, that is Eq.(22), a solvability condition must be satisfied (see details in Refs. Nayfeh (1973), (1981)). Applying the solvability conditions for Eq. (22), one obtains following equations:

$$2.\mathrm{i}.\alpha.\omega.\dot{A}.Y_1{}^2\big|_{x=\eta} = -2.\mathrm{i}.\omega.(\dot{A}+\mu.A).\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}{}^2.dx + \frac{e^{\mathrm{i}.\sigma.T_2}}{2}.\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} F_{r+1}'.Y_{r+1}'.dx + A^2.\bar{A}.\Lambda \tag{23}$$

where the term Λ is defined in the following simplification;

$$\Lambda = \sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}.\Psi_{r+1}.dx$$

$$\begin{aligned}
\Psi_{m+1} = &-3.\gamma_2.Y_{m+1}^3 + \left\{\frac{3}{2}.\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'{}^2.dx + \sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_0'.\phi_{(r+1)1}'.dx + 2.\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_0'.\phi_{(r+1)2}'.dx\right\}.Y_{m+1}'' \\
&+\left\{\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'.\phi_{(r+1)1}'.dx + 2.\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'.\phi_{(r+1)2}'.dx\right\}.Y_0'' + \sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_0'.Y_{r+1}'.dx.\left(\phi_{(m+1)1}'' + 2.\phi_{(m+1)2}''\right)
\end{aligned} \tag{24}$$

The solvability condition for Eq.(23) can be written in the simplest form as;

$$2.\mathrm{i}.\omega.\left(\tilde{\mu}.A + \dot{A}.k\right) + \Lambda.A^2.\bar{A} = \frac{f}{2}.e^{\mathrm{i}.\sigma.T_2} \tag{25}$$

where following assumption and definitions have been done;

$$\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}^2.dx=1, \quad f=\sum_{r=0}^{1}\int_{\eta_r}^{\eta_{r+1}} \tilde{F}'_{r+1}.Y'_{r+1}.dx, \quad k=1+\alpha.Y_1^2\Big|_{x=\eta}. \tag{26}$$

Table 1. First five frequencies for different η values and boundary conditions($\alpha=1$; $\gamma_1=10$)

Support Status	η	ω_1	ω_2	ω_3	ω_4	ω_5
Fixed - Fixed	0.1	23.1680	53.9122	89.8957	151.9954	243.1009
	0.2	19.2681	41.1794	93.3816	177.8843	290.2148
	0.3	15.1251	44.5907	112.6047	195.5009	254.3961
	0.4	13.0624	53.7090	114.6737	167.6918	297.2954
	0.5	12.4611	61.7538	95.9110	199.8845	253.7448
Simple - Simple	0.1	11.3480	30.0731	66.1445	127.2535	213.3666
	0.2	9.3262	27.3386	73.5943	149.4332	246.7603
	0.3	8.0173	30.1287	86.7908	143.2702	209.4037
	0.4	7.3696	35.4771	80.0713	132.7141	246.7604
	0.5	7.1759	39.6049	68.0582	157.9453	206.8256
Slide - Slide	0.1	7.6883	34.8236	83.6314	155.4432	246.7604
	0.2	8.1917	39.0212	85.7896	130.1341	211.7512
	0.3	8.9790	38.2560	68.2106	141.5396	246.7604
	0.4	9.8897	30.6979	76.2983	154.2519	207.5597
	0.5	10.3638	27.5815	88.8827	127.7736	246.7604
Fixed - Simple	0.1	16.9934	45.6683	79.3775	133.5016	217.8755
	0.3	12.5730	33.2960	92.3016	178.1188	234.6121
	0.5	9.6613	47.3951	84.8363	172.7774	236.1612
	0.7	9.8902	41.5105	104.1212	153.4841	237.9464
	0.9	14.6967	37.2315	78.9924	146.4840	238.6910
Fixed - Slide	0.1	6.5371	29.9657	62.0504	102.3515	172.2847
	0.3	6.1365	18.9717	58.1058	134.4783	208.7653
	0.5	4.9853	21.1236	71.6700	115.2555	216.3558
	0.7	3.9972	30.8668	58.2935	118.8226	220.5116
	0.9	3.4641	26.4638	69.1366	135.4792	222.4510
Simple - Slide	0.1	4.8464	18.8487	45.4439	93.4789	167.2645
	0.3	4.2425	14.3552	54.7628	120.1071	166.6047
	0.5	3.5620	18.2264	53.0809	107.6990	181.7691
	0.7	3.0557	22.9572	50.9188	98.7196	193.1136
	0.9	2.7610	19.2220	56.1067	116.8304	199.0220

The polar forms are written as follows:

$$A(T_2)=\frac{1}{2}.a.e^{i.\theta}, \quad \bar{A}(T_2)=\frac{1}{2}.a.e^{-i.\theta}, \quad \theta=\theta(T_2) \tag{27}$$

where the complex amplitude A consists of the real amplitude a and phase θ. Inserting Eq. (27) into Eq.(26), and separating the real and imaginary parts, one obtains following amplitude-phase modulation equations:

$$\omega.\tilde{\mu}.a+\omega.k.\dot{a}=\frac{f}{2}.sin\gamma, \; -\omega.k.a.\dot{\theta}+\frac{\Lambda}{8}.a^3=\frac{f}{2}.cos\gamma, \; \gamma=\sigma.T_2-\theta \tag{28}$$

where γ denotes the phase between forcing and responce of the system.

Table 2. Nonlinearity values(λ) for different η, γ_2 and boundary conditions(α =1; γ_1=10)

Support Status	η	γ_2=0	γ_2=10	γ_2=50	γ_2=100
Fixed-Fixed	0.1	0.8833	1.1684	2.3088	3.7343
	0.2	0.5610	0.7732	1.6221	2.6832
	0.3	0.5520	0.7227	1.4057	2.2595
	0.4	0.5256	0.6857	1.3260	2.1263
	0.5	0.5168	0.6747	1.3062	2.0956
Simple-Simple	0.1	-0.8102	-0.4131	1.1756	3.1615
	0.2	-0.7744	-0.4608	0.7933	2.3610
	0.3	-0.7510	-0.4746	0.6312	2.0133
	0.4	-0.7641	-0.5005	0.5539	1.8720
	0.5	-0.7730	-0.5125	0.5297	1.8324
Slide - Slide	0.1	0.7490	1.5679	4.8437	8.9385
	0.2	0.9508	1.7303	4.8481	8.7454
	0.3	1.2474	1.9540	4.7804	8.3134
	0.4	1.5819	2.1876	4.6104	7.6389
	0.5	1.6825	2.1993	4.2665	6.8506
Fixed-Simple	0.1	0.6728	1.0372	2.4948	4.3168
	0.3	0.3182	0.5582	1.5181	2.7180
	0.5	0.3312	0.5382	1.3664	2.4016
	0.7	0.4114	0.6221	1.4649	2.5184
	0.9	0.6037	0.9083	2.1268	3.6499
Fixed- Slide	0.1	0.2032	1.2640	5.5075	10.8118
	0.3	0.1780	1.1145	4.8602	9.5424
	0.5	0.1325	0.8616	3.7781	7.4237
	0.7	0.1089	0.7487	3.3079	6.5068
	0.9	0.1072	0.6979	3.0610	6.0147
Simple- Slide	0.1	-0.1263	0.9857	5.4334	10.9931
	0.3	-0.3962	0.4844	4.0067	8.4096
	0.5	-0.8222	-0.0623	2.9770	6.7762
	0.7	-1.0980	-0.3776	2.5041	6.1063
	0.9	-1.1666	-0.4704	1.9129	5.5278

In undamped free vibrations, the terms f, $\tilde{\mu}$, and σ were taken as zero. For the steady-state solutions, it is assumed that $\dot{a}=0$. This indicates that the amplitude of vibration is constant, that is $a=a_0$. In this case, the following definition can be derived from Eq. (28)

$$\dot{\theta}=\lambda.a_0^{\;2} \quad , \quad \lambda=\frac{\Lambda}{8.\omega.\mathrm{k}} \tag{29}$$

where λ is defined as nonlinearity effects and gives corrections from the natural frequencies to the nonlinear frequencies. Thus, the nonlinear frequency is defined as:

$$\omega_{nl}=\omega+\dot{\theta}=\omega+\lambda.a_0^{\;2}\,. \tag{30}$$

In damped-forced vibrations for the steady-state region, $\dot{a}$ and $\dot{\gamma}$ can be taken as zero and denote no change in amplitude and phase with time. Thus, eliminating γ from Eq. (28), one can obtain following detuning parameter (σ).

$$\sigma=\lambda.a_0{}^2 \pm \sqrt{\left(\frac{f}{2.a.\omega.k}\right)^2-\left(\frac{\tilde{\mu}}{k}\right)^2} \quad . \tag{31}$$

4. NUMERICAL RESULTS

In order to find the approximate solutions to the mathematical model, the rise function of the curved beam are assumed to have sinusoidal variation as Y_0=sin(π.x). The limit values for the translational and rotational springs' coefficients can be kept as zero or infinity in the calculations. For example, some supporting cases are arranged as fixed support; $\kappa_{tr}\to\infty$, $\kappa_{rot}\to\infty$, simply support; $\kappa_{tr}\to\infty$, $\kappa_{rot}\to 0$, sliding support; $\kappa_{tr}\to 0$, $\kappa_{rot}\to\infty$. Thus, the nonlinear problem arising from supports can be taken into consideration. For this purpose, Eq. (16), which corresponds to the linear part of the problem, was solved first. Then, nonlinearity coefficients (λ) were obtained by using Eq. (29).

In order to reach a good conclusion on the supporting, two cases have been presented symmetric and asymmetric in means of geometric constraints. For obtaining the case of symmetric supporting, the beam's both ends are restricted with the same supports, that is, fixed-fixed, simple-simple, slide-slide. Because of having similar properties according to its middle point throughout beam length, the mass positionings (η) are regarded as 0.1, 0.2, 0.3, 0.4, 0.5. On the other hand, the cases of asymmetrical supportings are fixed-simple, fixed-slide, simple-slide. Thus, mass locations (η) are regarded as 0.1, 0.3, 0.5, 0.7, 0.9 from left hand support to the right. Additionally, the dimensionless mass ratio and the linear coefficient of elastic foundation are selected as α=1 and γ_1=10, respectively. For the different mass positions (η) and the supporting cases in every configurations, the first five natural frequencies are given in Table 1. The first mode frequencies seems to decrease with increasing η that is, positioning the mass close to the beam's mid points instead of end points. However, for the supporting case of slide-slide, the frequencies seem to increase with increasing η. For the non-symmetric supporting cases, the mass is arbitrarily positioned from left hand side end to right throughout the beam length. In the supporting cases of fixed-slide and simple-slide, the first mode frequencies decrease with increasing η. These frequencies first decrease and then increase while ηs increase for the case of fixed-simple supporting.

Nonlinearities (λ) of the first mode in the cases of nonlinear foundation coefficients are given in Table 2. As seen on these tables for all the fixed parameters, the nonlinearities have positive and negative signs according to the locations of the mass and the supporting cases. If so, by means of these parameters, the hardening or softening behaviors of the system may occur. The nonlinearities (λ) seem to increase with increasing γ_2. If the mass locations are replaced from the left simple(or fixed) support to the right slide support, the nonlinearities decrease. But the nonlinearities firstly decrease and then increase for the case of fixed-simple.

Undamped-free vibration behavior of the system is best seen in the nonlinear frequency-amplitude curves. The nonlinear frequencies have a parabolic relationship with the maximum amplitude of the vibration as given in Eq. (30). These relations are drawn using curves for the first mode of vibrations in Figs. (2) to (5). The effects of elastic supports and masses' locations to the vibrations are determined through these curves. While doing so, some system parameters are fixed as α=1, γ_1=10 and γ_2=10.

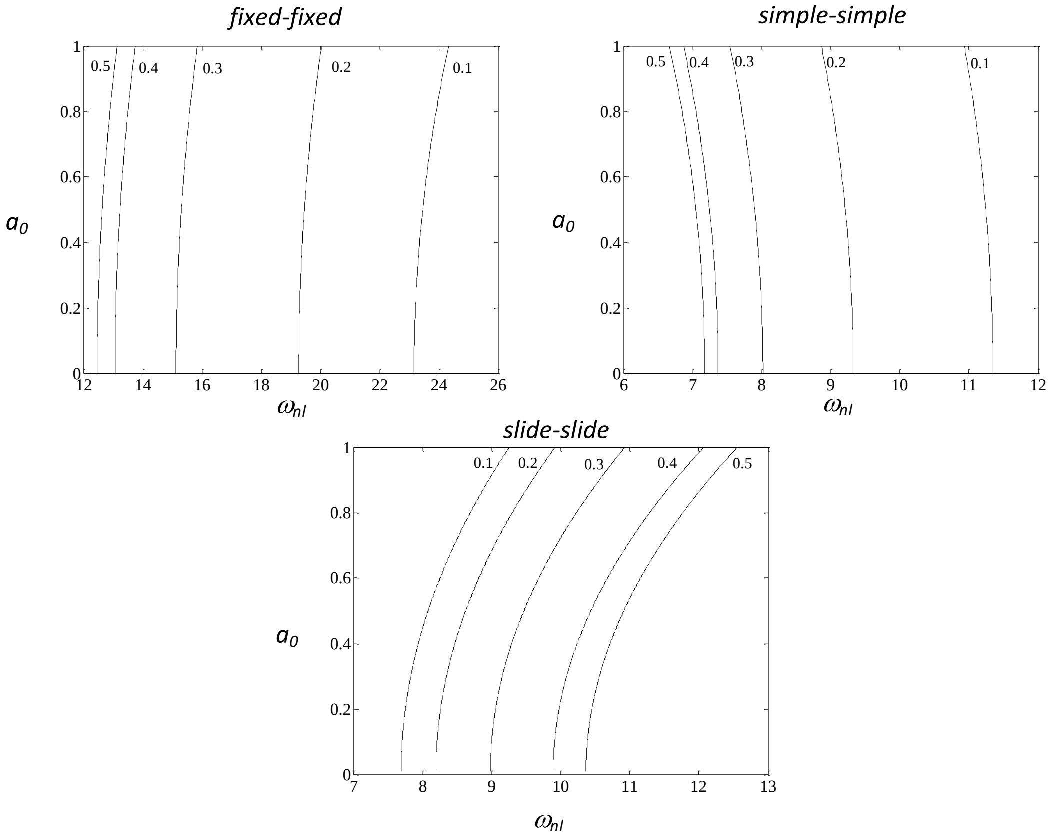

Figure 2.
Nonlinear frequency-amplitude curves for symmetric supporting cases.

In Fig. (2), the nonlinear frequency-amplitude curves are drawn for the beams restricted symmetrically. By placing the mass in different locations from the support of left hand side to the beam's middle point, the effects of masses'locations on the nonlinear frequency-amplitude curves are drawn. For the fixed-fixed and simple-simple supporting cases, the increasing in η results in a system of lower linear frequencies. Additionally, it is seen that the frequencies become much more close to each other with the increasing η. In the case of slide-slide supporting, the increasing in η results in the higher linear frequencies. As seen from the fixed-fixed and slide-slide supporting cases, the nonlinear frequencies increase with amplitude for all the configurations of the mass's location. However, the nonlinear frequencies decrease for the case of simple-simple supporting.

For the curved beams restricted asymmetrically, the nonlinear frequency-amplitude curves are drawn in Fig. (3). In the case of fixed-simple supporting, the nonlinear frequencies increase with increasing amplitude for each curves. Replacing the mass in the beam's middle point, the smallest frequencies can be obtained. Furthermore, higher natural frequencies are obtained in the case of mass being placed in neighborhood of the beam's fixed end instead of the simple supported end. The more close the mass to the slide support is, the smallest values of frequencies can be obtained for the fixed-slide supporting case. For the case of simple-slide supporting, the frequencies increase with the increasing amplitudes in η's 0.1, 0.3 values and decreases in η's 0.5, 0.7, 0.9 values.

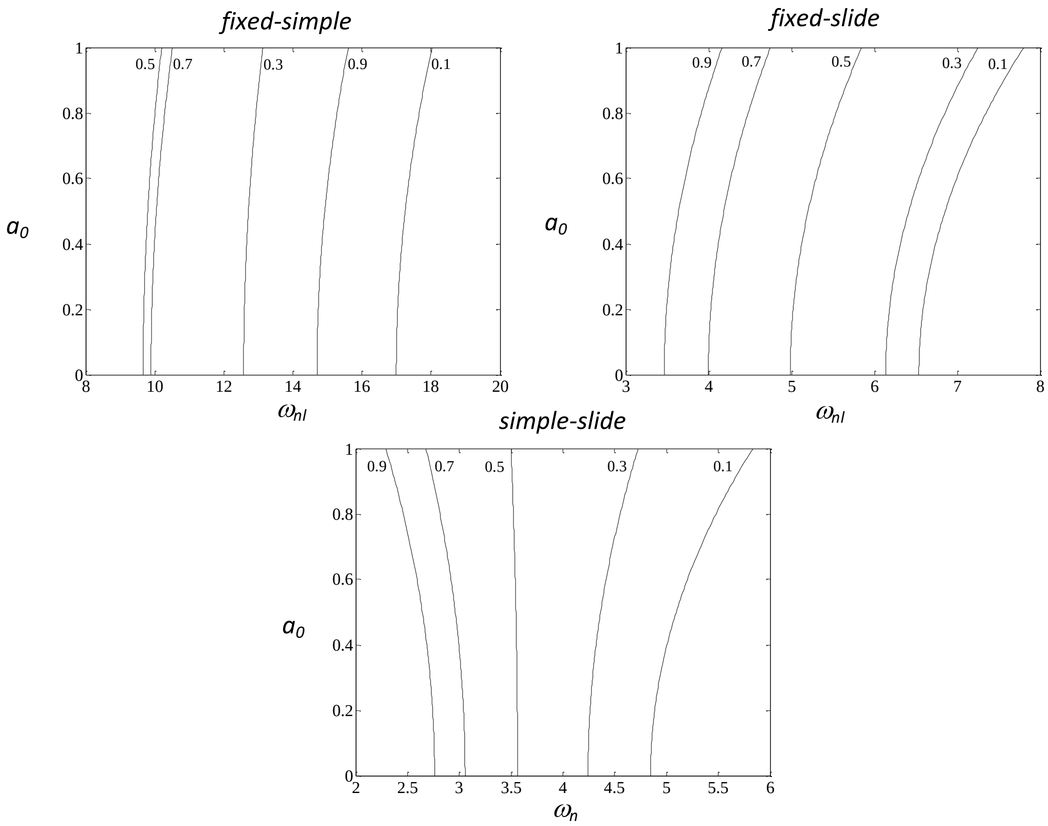

Figure 3.
Nonlinear frequency-amplitude curves for asymmetric supporting cases.

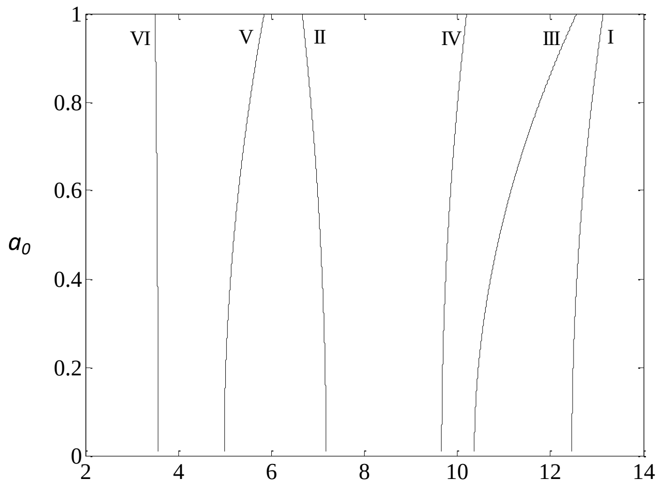

Figure 4.
Nonlinear frequency-amplitude curves for different boundary conditions (α=1, γ_1=10 , γ_2=10, η =0.5).
(I-Fixed-Fixed, II-Simple-Simple, III-Slide-Slide, IV- Fixed-Simple, V- Fixed-Slide, VI- Simple-Slide).

By taking as η=0.5, the nonlinear frequency-amplitude curves for the different supporting cases; fixed-fixed(I), simple-simple(II), slide-slide(III), fixed-simple(IV), fixed-slide(V), simple-slide(VI) are plotted in Fig.4. As seen in these curves, the frequencies corresponding to the cases of I, III, IV and V increase as the amplitudes increase. However, the frequencies corresponding to the cases of II and VI decrease. Also, the highest values of the frequencies are observed in case of I and the smallest in case of VI.

Considering the case where there is damping and external excitation, the nonlinear vibration behavior of the system can be understood via forcing frequency-response curves. When f=1 and $\tilde{\mu}$ =0.2, some curves in Figs. (5) to (7) are drawn by means of Eq.(31). In these figures, only first modes of the tranverse vibrations are dealed. Some parameters of the system are fixed as γ_1=10, γ_2=10, α=1 for the detailed investigations to be done on the supports. By assigning different values of the springs, the symmetrical and asymmetrical cases are obtained.

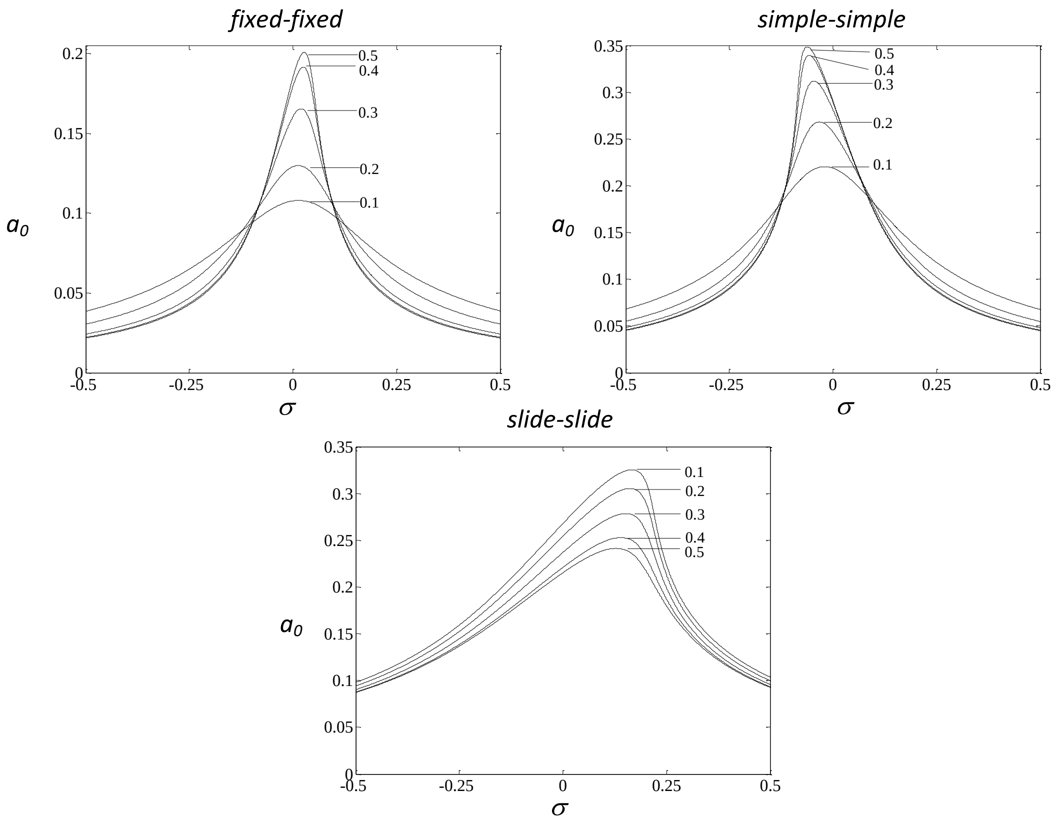

Figure 5.
Forcing frequency-response curves for symmetric supporting cases.

The effects of these cases on the curves are investigated in the figures. The cases of symmetric supporting are taken into account in Fig. (5). In these graphs, different masses' locations have been considered for the comparison. As seen in the curves of the fixed-fixed and simple-simple supporting cases, the maximum amplitudes of the vibrations increase with increasing η; in other means, one obtains the maximum amplitudes for this mass configuration taking place in the beam's middle point of instead of its ends. For the case of slide-slide supporting, the maximum amplitudes decrease with the increasing η.

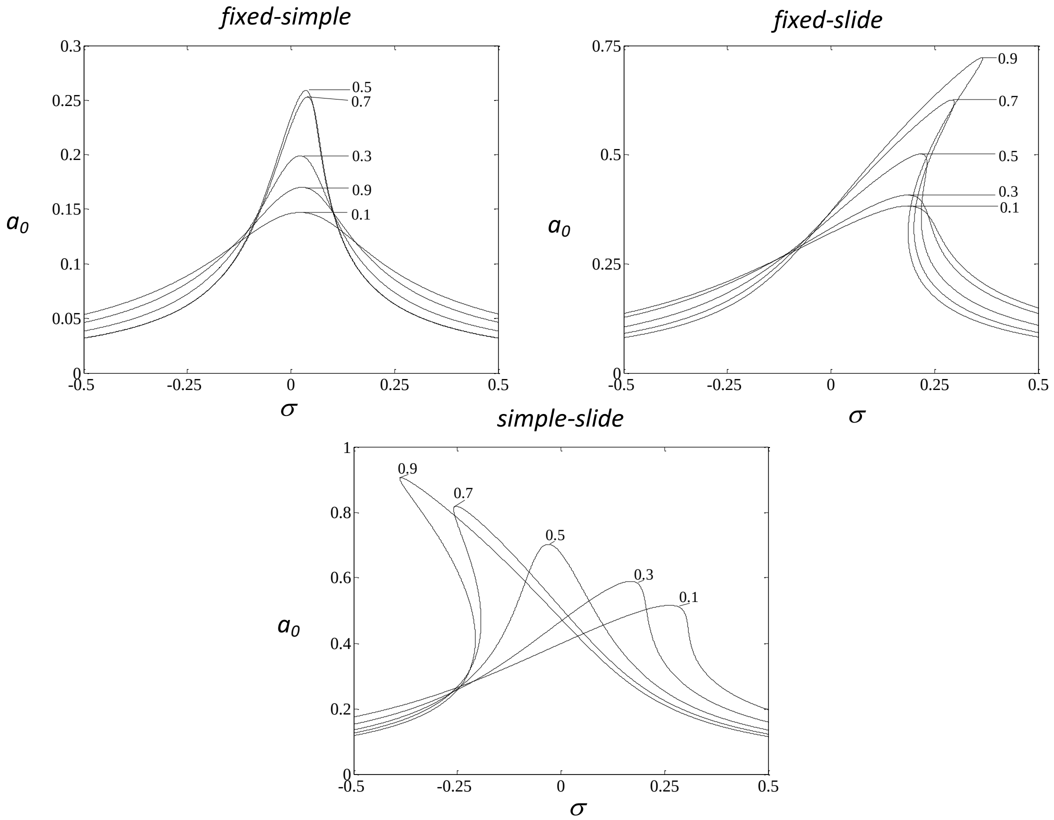

Figure 6.
Forcing frequency-response curves for asymmetric supporting cases.

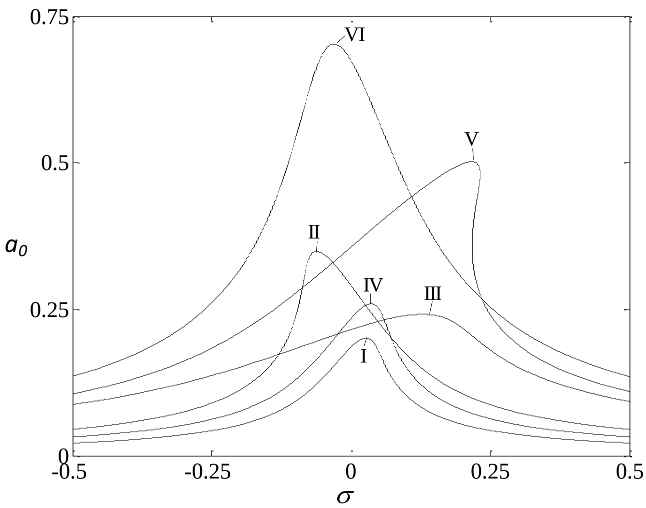

Figure 7.
Forcing frequency-response curves for different boundary conditions at α=1, γ_1=10,γ_2=10, η =0.5.

The cases of asymmetric supporting are taken into account in Fig. (6). In these graphs, closeness of the mass to the supports have been considered. As seen in the case of fixed-simple supporting, the amplitudes are more higher while being in close to the fixed support than the case of simple supporting. As of the fixed-slide supporting case, the maximum amplitudes increase and the jumping regions expand with the increasing η. For the case of simple-slide supporting, somethings seem to happen in the unsimilar way as η increase. A transition occurs from the hardening behavior to softening one. The maximum amplitudes increase as η increase.

By taking as η =0.5, the forcing frequency-response curves are plotted in Fig. (7). The supporting cases have been considered as fixed-fixed(I), simple-simple(II), slide-slide(III), fixed-simple(IV), fixed-slide(V), simple-slide(VI). In these curves, the maximum amplitudes can be ranked from the smallest ones to the big ones as; fixed-fixed(I), slide-slide(III), fixed-simple(IV), simple-simple(II), fixed-slide(V), simple-slide(VI). The softening behaviors are seen for the cases of II and VI.

5. CONCLUSIONS

In this study, nonlinear vibrations of a curved beam restricted by elastic supports on both ends are investigated. Beam carrying one concentrated mass rests on Winkler elastic foundation. The beam's ends are immovable at horizontal direction. Elastic supports have been changed to linear springs by using suitable parameters. To solve the equations of motion, the method of multiple scales is used. For the solutions being thought as perturbation series, first order of the solutions is defined as a linear problem. Assumption of the beam's curvature being occured at the series' first orders has been done in analytical solutions. For numerical analysis, the rise function of the beam has been accepted as sinusoidal type. In numerical calculations, some parameters are fixed such as; the dimensionless mass ratio is α=1, the dimensionless coefficients of the linear and nonlinear elastic foundation are γ_1=10 , γ_2=10, respectively.

Natural frequencies are obtained for the different supporting cases as well as the masses' locations. For every mass positioning η, only one result is calculated. For the symmetric supporting cases such as fixed-fixed and simple-simple, the mass is replaced at different locations between the beam's mid and end points. The first mode frequencies seems to decrease with increasing η that is, positioning the mass close to the beam's mid points instead of its end points. However, for the supporting case of slide-slide, frequencies seem to increase with increasing η. For the non-symmetric supporting cases, the mass is arbitrarily positioned from left hand side end to right throughout the beam length. In the supporting cases of fixed-slide and simple-slide, the first mode frequencies decrease with increasing η. For the supporting case of fixed-simple, these frequencies first decrease with increasing η, and then increase. In the case of primary resonance, the nonlinearity effects of the curved beam-mass system have both positive and negative signs. Adjusting the springs' magnitude and types restricting the beam, one can make the system to have softening behavior. For different supporting cases and locations of the mass, the nonlinear frequencies-amplitude curves and the forcing frequency-response curves are drawn for comparisons. From the frequencies-amplitude curves, one can conclude that both the slide and fixed supporting cases have some raising effects on the nonlinear frequencies; while the simple supporting cases tend to reduce the frequencies. Also, for mass positionings at regions close to the slide supports, lower frequencies occurs. From the forcing frequency-response curves, it is seen that the simple supports have softening behavior while the slides have hardening behavior. The fixed supports have neither hardening behavior nor softening. For mass positioning at regions close to the slide supports, the maximum amplitudes tend to increase with increasing detuning parameter. If the mass is positioned at close regions to the beam's mid point instead of its end points, the maximum amplitudes tend to increase with increasing detuning parameter for the cases of simple and fixed supportings.

In future, new studies on the cases of internal resonance will contribute further distinctions to the current studies. By means of an adjustment in the translational and rotational springs' coefficients, possible internal resonances may be analyzed.

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