JAMAME Journal of Advanced Mathematics and Mathematics Education
2026, VOL. 09, NO. 1, 1-13

New Features for k-Jacobsthal and k-Jacobsthal-Lucas Sequence

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ABSTRACT: Special integer sequences can be generalized by different many ways. The basic form of these ways is to add a parameter to recurrence relation. From special sequences, *k*-Jacobsthal, *k*-Jacobsthal-Lucas sequences are obtained adding a parameter to the recurrence relation of the Jacobsthal and Jacobsthal-Lucas numbers. In literature, there are some papers concerning the properties of *k*-Jacobsthal and *k*-Jacobsthal-Lucas sequences. But, we think they are not sufficient, so we aim to study new properties of these generalized sequences. The features related with *k*-Jacobsthal and *k*-Jacobsthal-Lucas sequences will be acquired through the Binet formulas, recurrence relations, generating functions of these numbers.

Keywords: *Binet formula, Generating function, k-Jacobsthal number, k-Jacobsthal–Lucas number*

k-Jacobsthal ve k-Jacobsthal-Lucas Dizileri için Yeni Özellikler

ÖZET: Özel tamsayı dizileri birçok farklı yolla genelleştirilebilir. Bu yolların temel biçimi, yineleme bağıntısına bir parametre eklemektir. Özel dizilerden, Jacobsthal ve Jacobsthal-Lucas sayılarının yineleme bağıntısına bir parametre eklenerek k-Jacobsthal ve k-Jacobsthal-Lucas dizileri elde edilir. Literatürde, k-Jacobsthal ve k-Jacobsthal-Lucas dizilerinin özellikleriyle ilgili bazı makaleler bulunmaktadır. Ancak, bunların yeterli olmadığını düşünüyoruz, bu nedenle bu genelleştirilmiş dizilerin yeni özelliklerini incelemeyi amaçlıyoruz. k-Jacobsthal ve k-Jacobsthal-Lucas dizileriyle ilgili özellikler, Binet formülleri, yineleme bağıntıları ve bu sayıların üreteç fonksiyonları aracılığıyla elde edilecektir.

Anahtar Kelimeler: *Binet formülü, Üreteç fonksiyonu, k-Jacobsthal sayısı, k-Jacobsthal–Lucas sayısı*

1. Introduction

By changing recurrence relations, starting values or both of them are the most used type of obtaining generalized special integer sequences. Sloane gave a very valuable and popular on-line encyclopedia of integer sequences to the mathematics world in [2]. The Fibonacci sequence was the earliest special integer sequence in literature. Zhang provided some identities about generalized second-order integer sequences in [14]. A generalization of the Fibonacci sequence called k-Fibonacci sequence was studied by Falcon, Plaza, Bolat, and Catarino [2,3,6,8]. The Jacobsthal and Jacobsthal-Lucas numbers are also popular examples of types of the second-order sequence. In recent years, many authors have investigated these numbers. You can see the references [1,4,5] and cited therein. In the year 1996, Horadam first studied in detail the Jacobsthal sequence [1]. We reproduce following Definitions 1.1 and 1.2. by Horadam. Srisawat, Sriprad, and Sthityanak derived fundamental identities for the Jacobsthal and Jacobsthal-Lucas sequences by some properties of the matrices in [10]. Cerin discovered interesting sums formulas of the Jacobsthal and Jacobsthal-Lucas numbers in [4,5]. There are some papers concerning the properties of *k*-Jacobsthal and *k*-Jacobsthal-Lucas sequences in literature as [9,11]. A different generalization of Jacobsthal sequence is encountered in [12].

In this section, we present a generalization of the classical Jacobsthal numbers by mean of a recurrence equation with a parameter *k*. In the sequel, we will prove some properties which generalize the respective properties of the classical Jacobsthal and Jacobsthal-Lucas numbers.

Definition 1.1 For any natural number n, the Jacobsthal sequence $\{J_n\}_{n=0}^{\infty}$ is defined beginning the values $J_0 = 0$, $J_1 = 1$ with $J_n = J_{n-1} + 2J_{n-2}$, $(n \geq 2)$ [1].

After the German mathematician Ernst Jacobsthal, they are called Jacobsthal numbers.

Definition 1.2 For $n \in \mathbb{N}$ and the Jacobsthal-Lucas sequence $\{C_n\}_{n=0}^{\infty}$is defined recursively by beginning the values $C_0 = 2$, $C_1 = 1$ recursively by $C_n = C_{n-1} + 2C_{n-2}$, $(n \geq 2)$ [1].

The characteristic polynomial for the Jacobsthal and the Jacobsthal-Lucas numbers is denoted by $x^2 - x - 2 = 0$ and the roots are 2, -1. The Jacobsthal numbers are obtained by the Binet formula as $j_n = \frac{x_1^n - x_2^n}{x_1 - x_2} = \frac{2^n - (-1)^n}{3}$. The Jacobsthal-Lucas numbers are obtained by the Binet formula as $c_n = x_1^n + x_2^n = 2^n + (-1)^n$ [1].

Definition 1.3 For $n \in \mathbb{N}$, $k > 0, k \in \mathbb{R}$, the *k*-Jacobsthal sequence $\{J_{k,n}\}_{n=0}^{\infty}$is defined recursively by

$$J_{k,0} = 0, \quad J_{k,1} = 1, \quad J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, \quad (n \geq 2) \quad [9,11].$$

If we take $k = 1$, the classical Jacobsthal sequence is obtained.

Definition 1.4 For $n \in \mathbb{N}$, $k > 0, k \in \mathbb{R}$, the *k*-Jacobsthal Lucas sequence $\{C_n\}_{n=0}^{\infty}$is defined recursively by

$$C_{k,0} = 2, \quad C_{k,1} = 1, \quad C_{k,n} = kC_{k,n-1} + 2C_{k,n-2}, (n \geq 2) \quad [11].$$

If we take $k = 1$, the classical Jacobsthal-Lucas sequence is obtained.

If we consider the given recurrence relations for *k*-Jacobsthal and *k*-Jacobsthal Lucas sequences as a difference equation, we get $r^2 = kr + 2$ with the solutions of the equation are $r_1 = \frac{k + \sqrt{k^2 + 8}}{2}$, $r_2 = \frac{k - \sqrt{k^2 + 8}}{2}$.

For $n \in \mathbb{N}$, $k > 0, k \in \mathbb{R}$, the *k*-Jacobsthal numbers are obtained by the Binet formula as

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}. \tag{1.1}$$

Similarly, the *k*-Jacobsthal Lucas numbers are obtained by the Binet formula as

$$C_{k,n} = r_1^n + r_2^n \tag{1.2}$$

[9, 11].

Lemma 1.5 The following relations are satisfied by k-Jacobsthal and k-Jacobsthal-Lucas numbers

a) $C_{k,n} = kJ_{k,n} + 4J_{k,n-1}$
b) $J_{k,n}C_{k,n} = J_{k,2n}$
c) $J_{k,a+b-2} = J_{k,a}J_{k,b-1} + 2J_{k,a-1}J_{k,b-2}$
d) $J_{k,mn+r} - C_{k,m}J_{k,m(n-1)+r} + (-2)^m J_{k,m(n-2)+r} = 0$
e) $J_{k,n+6} = (k^2 + 4)J_{k,n+4} - 4J_{k,n+2}$ [13].

2. Some Properties on *k*-Jacobsthal Sequence

In the section, we present new features of the k-Jacobsthal sequence.

Theorem 2.1 For any $a, b \in \mathbb{N}$, the following equality holds for the *k*-Jacobsthal sequence

$$J_{k,a+b-2} = \frac{J_{k,a}J_{k,b} - 4J_{k,a-2}J_{k,b-2}}{k}.$$

Proof. We use the recurrence relation and Lemma 1.5(c) for the proof;

$$J_{k,a+b-2} = J_{k,a}J_{k,b-1} + 2J_{k,a-1}J_{k,b-2}$$
$$= J_{k,a}\left(\frac{J_{k,b} - 2J_{k,b-2}}{k}\right) + 2\left(\frac{J_{k,a} - 2J_{k,a-2}}{k}\right)J_{k,b-2}$$

$$= \frac{J_{k,a}J_{k,b} - 2J_{k,a}J_{k,b-2} + 2J_{k,a}J_{k,b-2} - 4J_{k,a-2}J_{k,b-2}}{k}$$

$$= \frac{J_{k,a}J_{k,b} - 4J_{k,a-2}J_{k,b-2}}{k}$$

Theorem 2.2 The expression $(k^2+8)J_{k,n}^2 + 4(-2)^n$ is always equal to a perfect square.

Proof. From the Binet formula for *k*-Jacobsthal numbers; we get

$$(k^2+8)J_{k,n}^2 + 4(-2)^n = (r_1 - r_2)^2\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)^2 + 4(r_1r_2)^n$$

$$= r_1^{2n} + r_2^{2n} - 2r_1^n r_2^n + 4r_1^n r_2^n$$

$$= r_1^{2n} + 2r_1^n r_2^n + r_2^{2n}$$

$$= (r_1^n + r_2^n)^2 = C_n^2$$

It follows that, this is also a perfect square.

Theorem 2.3 For any $n \in \mathbb{N},\ n > 0,$ the following equality holds for the *k*-Jacobsthal sequence

$$\frac{J_{k,n+1}^2 - 4J_{k,n-1}^2}{k} = J_{k,2n}$$

Proof. Using the recurrence relation of the *k*-Jacobsthal sequence, we get $J_{k,n} = \frac{J_{k,n+1} - 2J_{k,n-1}}{k}$ and by Lemma 1.5, we hace $C_{k,n} = kJ_{k,n} + 4J_{k,n-1}$ and $J_{k,n}C_{k,n} = J_{k,2n}$. Then,

$$C_{k,n} = kJ_{k,n} + 2J_{k,n-1} + 2J_{k,n-1}$$

$$= J_{k,n+1} + 2J_{k,n-1}$$

Multiplying the equations together, we get the result

$$J_{k,n}C_{k,n} = \frac{J_{k,n+1} - 2J_{k,n-1}}{k}\left(J_{k,n+1} + 2J_{k,n-1}\right) = \frac{J_{k,n+1}^2 - 4J_{k,n-1}^2}{k} = J_{k,2n}.$$

Theorem 2.4 The following equality holds for the *k*-Jacobsthal sequence

$$J_{k,n}^2 + J_{k,n+1}^2 = \frac{C_{k,2n+2} + C_{k,2n} + 2(-2)^n}{k^2+8}$$

Proof . We use (1.1) for the proof where $r_1 = \frac{k + \sqrt{k^2+8}}{2}$, and $r_2 = \frac{k - \sqrt{k^2+8}}{2}$ the roots of difference equation of recurrence relation for *k*-Jacobsthal sequence.

$$J_{k,n}^2 + J_{k,n+1}^2 = \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)^2 + \left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right)^2$$

$$= \frac{r_1^{2n} - 2(r_1r_2)^n + r_2^{2n}}{(r_1 - r_2)^2} + \frac{r_1^{2n+2} - 2(r_1r_2)^{n+1} + r_2^{2n+2}}{(r_1 - r_2)^2}$$

$$= \frac{C_{k,2n} - 2(-2)^n + C_{k,2n+2} - 2(-2)^{n+1}}{(r_1 - r_2)^2}$$

$$= \frac{C_{k,2n} + C_{k,2n+2} + 2(-2)^n}{k^2 + 8}$$

Theorem 2.5 Let $m, n,$ and $r \in \mathbb{N},\ n \geq 2.$ The following equalities are satisfied

a) $J_{k,mn} = J_{k,m(n-1)} C_{k,m} - (-2)^m J_{k,m(n-2)},$
b) $J_{k,mn+r} = J_{k,m(n-1)+r} C_{k,m} - (-2)^m J_{k,m(n-2)+r}.$

Proof. From (1.1) and (1.2), we have

$$J_{k,m(n-1)+r} C_{k,m} - (-2)^m J_{k,m(n-2)+r}$$

$$= (r_1^m + r_2^m) \frac{r_1^{m(n-1)+r} - r_2^{m(n-1)+r}}{r_1 - r_2} - (r_1 r_2)^m \frac{(r_1^{m(n-2)+r} - r_2^{m(n-2)+r})}{r_1 - r_2}$$

$$= \frac{r_1^{mn+r} - r_1^m r_2^{m(n-1)+r} - r_2^m r_1^{m(n-1)+r} - r_2^{mn+r} + r_2^m r_1^{m(n-1)+r} + r_1^m r_2^{m(n-1)+r}}{r_1 - r_2}$$

$$= \frac{r_1^{mn+r} - r_2^{mn+r}}{r_1 - r_2} = J_{k,mn+r}$$

Theorem 2.6 For any $m, n,$ and $r \in \mathbb{N},$ we get

$$\sum_{n=0}^{\infty} J_{k,mn}\, x^n = \frac{J_{k,m} x}{1 - C_{k,m} x + (-2)^m x^2}$$

$$\sum_{n=0}^{\infty} J_{k,mn+r}\, x^n = \frac{J_{k,r} + J_{k,m+r} x - C_{k,m} J_{k,r} x}{1 - C_{k,m} x + (-2)^m x^2}$$

Proof. Let $T = \sum_{n=0}^{\infty} J_{k,mn+r}\, x^n = J_{k,r} + J_{k,m+r} x + \sum_{n=2}^{\infty} J_{k,mn+r}\, x^n.$

We first multiply T with $-C_{k,m} x$, then we get

$$-C_{k,m} x T = -\sum_{n=0}^{\infty} C_{k,m} J_{k,mn+r}\, x^{n+1} = -C_{k,m} J_{k,r} x - \sum_{n=1}^{\infty} C_{k,m} J_{k,mn+r}\, x^{n+1}$$

$$= -C_{k,m} J_{k,r} x - \sum_{n=2}^{\infty} C_{k,m} J_{k,m(n-1)+r}\, x^n.$$

Similarly, multiplying T with $(-2)^m x^2$ then we have

$$(-2)^m x^2 T = \sum_{n=0}^{\infty} (-2)^m J_{k,mn+r} x^{n+2} = \sum_{n=2}^{\infty} (-2)^m J_{k,m(n-2)+r} x^n.$$

From these equations, we get

$$T(1 - C_{k,m} x + (-2)^m x^2) = J_{k,r} + J_{k,m+r} x - C_{k,m} J_{k,r} x + \sum_{n=2}^{\infty} [(J_{k,mn+r} - C_{k,m} J_{k,m(n-1)+r} + (-2)^m J_{k,m(n-2)+r})]\, x^n.$$

By Theorem 2.5, we know that $(J_{k,mn+r} - C_{k,m} J_{k,m(n-1)+r} + (-2)^m J_{k,m(n-2)+r} = 0)$

Thus, $T = \sum_{n=0}^{\infty} J_{k,mn+r}\, x^n = \frac{J_{k,r}+J_{k,m+r}x - C_{k,m}J_{k,r}x}{1-C_{k,m}x+(-2)^m x^2}$.

Theorem 2.7 If $n \in \mathbb{N}$ and $n \geq 2$, then the following identity is satisfied by the k-Jacobsthal and k-Jacobsthal-Lucas sequences

$$J_{k,2n+3}J_{k,2n-3} = \frac{C_{k,4n} - (-2)^{2n-3}C_{k,6}}{k^2+8}.$$

Proof. We use (1.1) and (1.2):

$$J_{k,2n+3}J_{k,2n-3} = \left(\frac{r_1^{2n+3} - r_2^{2n+3}}{r_1 - r_2}\right)\left(\frac{r_1^{2n-3} - r_2^{2n-3}}{r_1 - r_2}\right)$$

$$= \frac{r_1^{4n} - r_1^{2n+3}r_2^{2n-3} - r_2^{2n+3}r_1^{2n-3} + r_2^{4n}}{(r_1 - r_2)^2}$$

$$= \frac{r_1^{4n} + r_2^{4n} - (r_1r_2)^{2n}(r_1^3r_2^{-3} + r_2^3r_1^{-3})}{(r_1 - r_2)^2}$$

$$= \frac{r_1^{4n} + r_2^{4n} - (r_1r_2)^{2n}\frac{(r_1^6 + r_2^6)}{(r_1r_2)^3}}{(r_1 - r_2)^2}$$

$$= \frac{C_{k,4n} - (-2)^{2n-3}C_{k,6}}{k^2+8}$$

Theorem 2.8: If $n \in \mathbb{N}$ and $n \geq 2$, then the following property is satisfied by the k-Jacobsthal sequence

$$J_{k,n-2}J_{k,n-1}J_{k,n+1}J_{k,n+2} - J_{k,n}^4 = \frac{2^{2n+3} - (-2)^{n-2}C_{k,2n}\left(kC_{k,3} + 16\right)}{(k^2+8)^2}.$$

Proof. The proof will be conducted using the Binet formulas associated with the sequences:

$$J_{k,n-2}J_{k,n-1}J_{k,n+1}J_{k,n+2} - J_{k,n}^4$$

$$= \frac{(r_1^{n-2} - r_2^{n-2})(r_1^{n-1} - r_2^{n-1})(r_1^{n+1} - r_2^{n+1})(r_1^{n+2} - r_2^{n+2}) - (r_1^n - r_2^n)^4}{(r_1 - r_2)^4}$$

$$= \frac{[r_1^{2n} + r_2^{2n} - (r_1r_2)^{n-2}(r_1^4 + r_2^4)][r_1^{2n} + r_2^{2n} - (r_1r_2)^{n-1}(r_1^2 + r_2^2)]}{(r_1 - r_2)^4}$$

$$- \frac{r_1^{4n} - 4r_1^{3n}r_2^n + 6r_1^{2n}r_2^{2n} - 4r_1^nr_2^{3n} + r_2^{4n}}{(r_1 - r_2)^4}$$

$$= \frac{r_1^{4n} + r_2^{4n} + 2r_1^{2n}r_2^{2n} + (r_1^{2n} + r_2^{2n})[-(r_1r_2)^{n-1}(r_1^2 + r_2^2) - (r_1r_2)^{n-2}(r_1^4 + r_2^4)]}{(r_1 - r_2)^4}$$

$$- \frac{r_1^{4n} - 4r_1^{3n}r_2^n + 6r_1^{2n}r_2^{2n} - 4r_1^nr_2^{3n} + r_2^{4n}}{(r_1 - r_2)^4}$$

After some basic algebraic operations, we get

$$= \frac{8r_1^{2n}r_2^{2n} + (r_1^{2n} + r_2^{2n})(r_1r_2)^{n-2}[-r_1{}^3r_2 - r_2{}^3r_1 - r_1^4 - r_2^4] - 4(r_1r_2)^n(r_1^{2n} + r_2^{2n})}{(r_1 - r_2)^4}$$

$$= \frac{8r_1^{2n}r_2^{2n} + (r_1^{2n} + r_2^{2n})(r_1r_2)^{n-2}[-r_1{}^3r_2 - r_2{}^3r_1 - r_1^4 - r_2^4 - 4r_1^2r_2^2}{(r_1 - r_2)^4}$$

$$= \frac{(r_1r_2)^{n-2}[8(r_1r_2)^{n+2} + C_{k,2n}[-r_1{}^3(r_1+r_2) - r_2{}^3(r_1+r_2) - 4r_1^2r_2^2]}{(r_1 - r_2)^4}$$

$$= \frac{(-2)^{n-2}[8(-2)^{n+2} + C_{k,2n}[-kC_{k,3} - 4(-2)^2]}{(r_1 - r_2)^4}$$

$$= \frac{2^{2n+3} - (-2)^{n-2}C_{k,2n}(kC_{k,3} + 16)}{(k^2 + 8)^2} .$$

3. Some Properties on k-Jacobsthal-Lucas Sequence

In the section, we present novel features of the *k*-Jacobsthal-Lucas sequence.

Theorem 3.1 The following property is satisfied by the *k*-Jacobsthal-Lucas sequence

$$C_{k,n}C_{k,n+1} = C_{k,2n+1} + k(-2)^n .$$

Proof. We use (1.2) for the proof:

$$C_{k,n}C_{k,n+1} = (r_1^n + r_2^n)(r_1^{n+1} + r_2^{n+1})$$

$$= r_1^{2n+1} + r_1^n r_2^{n+1} + r_2^n r_1^{n+1} + r_2^{2n+1}$$

$$= C_{k,2n+1} + (r_1r_2)^n(r_1 + r_2)$$

$$= C_{k,2n+1} + k(-2)^n.$$

Theorem 3.2 The following property is satisfied by the *k*-Jacobsthal-Lucas sequence

$$C_{k,n+4} = (k^2 + 4)C_{k,n+2} - 4C_{k,n}.$$

Proof. The proof will be conducted by employing the recurrence relation of the k-Jacobsthal-Lucas sequence:

$$C_{k,n+4} = kC_{k,n+3} + 2C_{k,n+2}$$

$$= k^2C_{k,n+2} + 2kC_{k,n+1} + 2C_{k,n+2}$$

$$= k^2C_{k,n+2} + 2C_{k,n+2} - 4C_{k,n} + 2C_{k,n+2}$$

$$= k^2C_{k,n+2} + 4C_{k,n+2} - 4C_{k,n}$$

$$= (k^2 + 4)C_{k,n+2} - 4C_{k,n}.$$

Theorem 3.3 The following identity is satisfied by the *k*-Jacobsthal-Lucas sequence

$$C_{n-1}C_{n+1} + C_{n-2}C_{n-2} = 2C_{k,2n} + (-2)^{n-1}C_{k,2} + (-2)^{n-2}C_{k,4}.$$

Proof. From (1.2), it is obtained that

$$C_{n-1}C_{n+1} + C_{n-2}C_{n+2} = (r_1^{n-1} + r_2^{n-1})(r_1^{n+1} + r_2^{n+1}) + (r_1^{n-2} + r_2^{n-2})(r_1^{n+2} + r_2^{n+2})$$

$$= r_1^{2n} + r_1^{n-1}r_2^{n+1} + r_2^{n-1}r_1^{n+1} + r_2^{2n} + r_1^{2n} + r_1^{n-2}r_2^{n+2} + r_2^{n-2}r_1^{n+2} + r_2^{2n}$$

$$= 2r_1^{2n} + 2r_2^{2n} + (r_1r_2)^n\left(\frac{r_2}{r_1} + \frac{r_1}{r_2}\right) + (r_1r_2)^n\left(\frac{r_2^2}{r_1^2} + \frac{r_1^2}{r_2^2}\right)$$

$$= 2C_{k,2n} + (-2)^{n-1}C_{k,2} + (-2)^{n-2}\ C_{k,4}\ .$$

Theorem 3.4 The following equality holds for the *k*-Jacobsthal-Lucas sequence

$$C_{k,n+1}^2 - 2C_{k,n}^2 = k\ C_{k,2n+1} + (-2)^{n+3}.$$

Proof. We use (1.2) for the proof:

$$C_{k,n+1}^2 - 2C_{k,n}^2 = (r_1^{n+1} + r_2^{n+1})^2 - 2(r_1^n + r_2^n)^2$$

$$= r_1^{2n+2} + r_2^{2n+2} + 2(r_1r_2)^{n+1} - 2[r_1^{2n} + r_2^{2n} + 2(r_1r_2)^n]$$

$$= r_1^{2n}(r_1^2 - 2) + r_2^{2n}(r_2^2 - 2)\ -8(r_1r_2)^n$$

$$= r_1^{2n}(kr_1 + 2 - 2) + r_2^{2n}(kr_2 + 2 - 2) + (-2)^{n+3}.$$

$$= k\ C_{k,2n+1} + (-2)^{n+3}.$$

Theorem 3.5 Assume that $m \geq n$, then the following equality exists.

$$C_{k,2m}C_{k,2n} = C_{k,2m+2n} + 4^n C_{k,2m-2n}$$

Proof. By (1.2), we have

$$C_{k,2m}C_{k,2n} = (r_1^{2m} + r_2^{2m})(\ r_1^{2n} + r_2^{2n})$$

$$= r_1^{2m+2n} + r_1^{2m}r_2^{2n} + r_2^{2m}r_1^{2n} + r_2^{2m+2n}$$

$$= C_{k,2m+2n} + (r_1r_2)^{2n}(r_1^{2m-2n} + r_2^{2m-2n})$$

$$= C_{k,2m+2n} + 4^n C_{k,2m-2n}.$$

Theorem 3.6 The *k*-Jacobsthal-Lucas sequence verifies the following equality

$$C_{k,2n+2}^2 + C_{k,2n}^2 = C_{k,4n+4} + C_{k,4n} + 5.2^{2n+1}.$$

Proof. We use (1.2) for the proof:

$$C_{k,2n+2}^2 + C_{k,2n}^2 = (r_1^{2n+2} + r_2^{2n+2})^2 + (r_1^{2n} + r_2^{2n})^2$$

$$= r_1^{4n+4} + r_2^{4n+4} + r_1^{4n} + r_2^{4n} + 2(-2)^{2n+2} + 2(-2)^{2n}$$

$$= C_{k,4n+4} + C_{k,4n} + 2^{2n+1}5.$$

Theorem 3.7 Assume that $m \geq n$, then, the following equality holds:

$$C_{k,m+n}^2 + (2)^{2n}C_{k,m-n}^2 = C_{k,2m+2n} + 2^{2n}C_{k,2m-2n} + 4(-2)^{m+n}.$$

Proof. We employ (1.2) for the proof

$$C_{k,m+n}^2 + 2^{2n}C_{k,m-n}^2 = (r_1^{m+n} + r_2^{m+n})^2 + 2^{2n}(r_1^{m-n} + r_2^{m-n})^2$$

$$= r_1^{2m+2n} + 2r_1^{m+n}r_2^{m+n} + r_2^{2m+2n} + 2^{2n}(r_1^{2m-2n} + 2r_1^{m-n}r_2^{m-n} + r_2^{2m-2n})$$

$$= C_{k,2m+2n} + 2(-1)^{m+n}2^{m+n} + 2^{2n}[C_{k,2m-2n} + 2(-1)^{m-n}2^{m-n}]$$

$$= C_{k,2m+2n} + 2^{2n}C_{k,2m-2n} + 2(-2)^{n+m} + 2(-1)^{m+n}2^{m+n}$$

$$= C_{k,2n+2m} + 2^{2n}C_{k,2m-2n} + 4(-2)^{m+n}.$$

4. Some Features Between *k*-Jacobsthal and *k*-Jacobsthal-Lucas Sequence

In the section, we present novel features between the *k*-Jacobsthal and the *k*-Jacobsthal-Lucas sequences.

Theorem 4.1 Let $k \in \mathbb{R}^+$ and n be a non-negative integer. The following recurrence relations hold

$$J_{k,n+1} = r_1 J_{k,n} + r_2^n$$

$$C_{k,n+1} = r_1 C_{k,n} - (r_1 - r_2)r_2^n$$

where $r_1 = \dfrac{k + \sqrt{k^2+8}}{2}$, $r_2 = \dfrac{k - \sqrt{k^2+8}}{2}$.

Proof. If we use (1.1), we see the proof easily.

Theorem 4.2 The *k*-Jacobsthal and the *k*-Jacobsthal-Lucas sequences verify the following equality

$$J_{k,2n}C_{k,2n+1} = J_{k,4n+1} - 4^n.$$

Proof. For the proof, the Binet formulas of the k-Jacobsthal and the k-Jacobsthal-Lucas sequences are used.

$$J_{k,2n}C_{k,2n+1} = \frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}(r_1^{2n+1} + r_2^{2n+1})$$

$$= \frac{r_1^{4n+1} - r_2^{4n+1}}{r_1 - r_2} + \frac{(r_1 r_2)^{2n}[r_2 - r_1]}{r_1 - r_2}$$

$$= J_{k,4n+1} - 4^n.$$

Theorem 4.3 The following relation is obtained

$$J_{k,2n+3}C_{k,2n+1} = J_{k,4n+4} + k(-2)^{2n+1}.$$

Proof. According to (1.1) and (1.2), we establish

$$J_{k,2n+3}C_{k,2n+1} = \frac{r_1^{2n+3} - r_2^{2n+3}}{r_1 - r_2}(r_1^{2n+1} + r_2^{2n+1})$$

$$= \frac{r_1^{2n+3}r_1^{2n+1} + r_1^{2n+3}r_2^{2n+1} - r_2^{2n+3}r_1^{2n+1} - r_2^{2n+3}r_2^{2n+1}}{r_1 - r_2}$$

$$= \frac{r_1^{4n+4} - r_2^{4n+4}}{r_1 - r_2} + \frac{r_1^{2n+3}r_2^{2n+1} - r_2^{2n+3}r_1^{2n+1}}{r_1 - r_2}$$

$$= J_{k,4n+4} + k(-2)^{2n+1}.$$

Theorem 4.4 The following relation is provided

$$J_{k,n}C_{k,2n} = J_{k,3n} - (-2)^n J_{k,n}.$$

Proof. According to (1.1) and (1.2), we get

$$J_{k,n}C_{k,2n} = \frac{r_1^n - r_2^n}{r_1 - r_2}\left(r_1^{2n} + r_2^{2n}\right)$$

$$= \frac{r_1^{3n} - r_2^{3n}}{r_1 - r_2} + (r_1 r_2)^n \frac{(r_2^n - r_1^n)}{r_1 - r_2}$$

$$= J_{k,3n} - (-2)^n J_{k,n}.$$

Theorem 4.5 The following equations are computed:

$$J_{k,n}J_{k,m+1} + J_{k,n-1}J_{k,m} = \frac{C_{k,m+n+1} + C_{k,m+n-1} + (-2)^{n-1}C_{k,m-n+1}}{k^2+8}$$

$$C_{k,n}C_{k,m+1} + C_{k,n-1}C_{k,m} = C_{k,m+n+1}C_{k,m+n-1} - (-2)^{n-1}C_{k,m-n+1}.$$

Proof . According to (1.1) and (1.2), we denote

$$J_{k,n}J_{k,m+1} + J_{k,n-1}J_{k,m} = \frac{(r_1^n - r_2^n)}{r_1 - r_2}\frac{(r_1^{m+1} - r_2^{m+1})}{r_1 - r_2} + \frac{(r_1^{n-1} - r_2^{n-1})}{r_1 - r_2}\frac{(r_1^m - r_2^m)}{r_1 - r_2}$$

$$= \frac{r_1^{m+n+1} - r_1^n r_2^{m+1} - r_1^{m+1} r_2^n + r_2^{m+n+1} + r_1^{m+n-1} - r_1^{n-1} r_2^m - r_1^m r_2^{n-1} + r_2^{m+n-1}}{(r_1 - r_2)^2}$$

$$= \frac{C_{k,m+n+1} + C_{k,m+n-1} - r_1^n r_2^m (r_2 + r_1^{-1}) - r_1^m r_2^n (r_1 + r_2^{-1})}{(r_1 - r_2)^2}$$

$$= \frac{C_{k,m+n+1} + C_{k,m+n-1} + r_1^n r_2^m \left(\frac{r_1 r_2 + 1}{r_1}\right) + r_1^m r_2^n \left(\frac{r_1 r_2 + 1}{r_2}\right)}{(r_1 - r_2)^2}$$

$$= \frac{C_{k,m+n+1} + C_{k,m+n-1} - r_1^{n-1} r_2^m - r_1^m r_2^{n-1}}{(r_1 - r_2)^2}$$

$$= \frac{C_{k,m+n+1} + C_{k,m+n-1} + (-2)^{n-1}C_{k,m-n+1}}{k^2+8}.$$

Theorem 4. 6 The following relations are evaluated:

$$J_{k,m+6} = k(k^2+6)J_{k,m+3} + 8J_{k,m}$$

$$C_{k,m+6} = k(k^2+6)C_{k,m+3} + 8C_{k,m}.$$

Proof. The proof will be conducted by employing the recurrence relation of the k-Jacobsthal sequence and some algebraic operations

$$J_{k,m+6} = (k^2+4)J_{k,m+4} - 4J_{k,m+2}$$

$$=(k^2+4)kJ_{k,m+3} + 2k^2 J_{k,m+2} + 8J_{k,m+2} - 4J_{k,m+2}$$

$$=(k^2+4)kJ_{k,m+3}+2kJ_{k,m+3}+(2k^2+4)J_{k,m+2}-2kJ_{k,m+3}$$

$$=(k^3+6k)J_{k,m+3}+(2k^2+4)J_{k,m+2}-2k(kJ_{k,m+2}+2J_{k,m+1})$$

$$=(k^3+6k)J_{k,m+3}+2k^2J_{k,m+2}-2k^2J_{k,m+2}+4J_{k,m+2}-4kJ_{k,m+1}$$

$$=(k^3+6k)J_{k,m+3}+4\left(kJ_{k,m+1}+2J_{k,m}\right)-4kJ_{k,m+1}$$

$$=k(k^2+6)J_{k,m+3}+8J_{k,m}$$

The other proof is made by a similar way.

5. Sum Formulas on *k*-Jacobsthal and *k*-Jacobsthal-Lucas Sequences

In the section, we present novel sum formulas for the *k*-Jacobsthal and the *k*-Jacobsthal-Lucas sequences.

Theorem 5.1 The following binomial sum is evaluated:

$$\sum_{i=0}^{n}\binom{n}{i}\frac{k^iJ_{k,i}}{2^i}=\frac{J_{k,2n}}{2^n}.$$

Proof. The proof is denoted by (1.1), the binomial expansion, and the difference equation for the recurrence relation as $r_1^2=kr_1+2,\quad r_2^2=kr_2+2$.

$$\sum_{i=0}^{n}\binom{n}{i}\frac{k^iJ_{k,i}}{2^i}=\frac{2^n}{2^n}\sum_{i=0}^{n}\binom{n}{i}\frac{k^i}{2^i}\left(\frac{r_1^i-r_2^i}{r_1-r_2}\right)=\frac{1}{2^n(r_1-r_2)}\left[\sum_{i=0}^{n}\binom{n}{i}(kr_1)^i2^{n-i}-\sum_{i=0}^{n}\binom{n}{i}(kr_2)^i2^{n-i}\right]$$

$$=\frac{1}{2^n(r_1-r_2)}[(kr_1+2)^n-(kr_2+2)^n]=\frac{1}{2^n}\left(\frac{r_1^{2n}-r_2^{2n}}{r_1-r_2}\right)=\frac{J_{k,2n}}{2^n}.$$

Theorem 5.2 The following binomial sums are obtained:

$$\sum_{i=0}^{2m+1}\binom{2m+1}{i}J_{k,2i}(-2)^{2m+1-i}=\frac{k^{2m+1}C_{k,2m+1}}{r_1-r_2}$$

$$\sum_{i=0}^{2m}\binom{2m}{i}(-2)^{2m-i}C_{k,2i}=k^{2m}C_{k,2m}.$$

Proof. The proof will be carried out using the Binet formulas and recurrence relations for the *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers and the properties of the summation formula:

$$\sum_{i=0}^{2m+1}\binom{2m+1}{i}J_{k,2i}(-2)^{2m+1-i}=\frac{1}{r_1-r_2}\sum_{i=0}^{2m+1}\binom{2m+1}{i}(-2)^{2m+1-i}[r_1^{2i}+r_2^{2i}]$$

$$=\frac{1}{r_1-r_2}(r_1^2-2)^{2m+1}+(r_2^2-2)^{2m+1}$$

$$=\frac{(kr_1)^{2m+1}+(kr_2)^{2m+1}}{r_1-r_2}$$

and, we get

$$\sum_{i=0}^{2m}\binom{2m}{i}(-2)^{2m-i}C_{k,2i} = \sum_{i=0}^{2m}\binom{2m}{i}(-2)^{2m-i}(r_1^2)^i + \sum_{i=0}^{2m}\binom{2m}{i}(-2)^{2m-i}(r_2^2)^i$$

$$= (r_1^2 - 2)^{2m} + (r_2^2 - 2)^{2m}$$

$$= (kr_1)^{2m} + (kr_2)^{2m} = k^{2m}C_{k,2m}.$$

Theorem 5.3 The following binomial sum is deduced:

$$\sum_{i=0}^{2n}\binom{2n}{i}(-2)^{2n-i}J_{k,i}C_{k,i} = k^{2n}J_{k,2n}.$$

Proof. We shall demonstrate this by employing the binomial expansion (1.1) and (1.2) for the *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers:

$$\sum_{i=0}^{2n}\binom{2n}{i}(-2)^{2n-i}J_{k,i}C_{k,i} = \sum_{i=0}^{2n}\binom{2n}{i}(-2)^{2n-i}\left(\frac{r_1^i - r_2^i}{r_1 - r_2}\right)(r_1^i + r_2^i)$$

$$= \frac{1}{r_1 - r_2}\sum_{i=0}^{2n}\binom{2n}{i}(r_1^2)^i(-2)^{2n-i} - \frac{1}{r_1 - r_2}\sum_{i=0}^{2n}\binom{2n}{i}(r_2^2)^i(-2)^{2n-i}$$

$$= \frac{1}{r_1 - r_2}[(r_1^2 - 2)^{2n} - (r_2^2 - 2)^{2n}]$$

$$= \frac{k^{2n}}{r_1 - r_2}[r_1^{2n} - r_2^{2n}] = k^{2n}J_{k,2n}.$$

Theorem 5.4 The following sum is obtained:

$$\sum_{i=0}^{n-1}\frac{C_{k,i}J_{k,i}}{2^k} = \frac{-4^n J_{k,2-2n} - J_{k,2n} + J_{k,2}}{2^k(1-k^2)}.$$

Proof. The proof will be carried out by employing (1.1 and (1.2) and geometric series expansions:

$$\sum_{i=0}^{n-1}\frac{C_{k,i}J_{k,i}}{2^k} = \frac{1}{2^k}[(r_1^0 + r_2^0)\left(\frac{r_1^0 - r_2^0}{r_1 - r_2}\right) + (r_1^1 + r_2^1)\left(\frac{r_1^1 - r_2^1}{r_1 - r_2}\right) + (r_1^2 + r_2^2)\left(\frac{r_1^2 - r_2^2}{r_1 - r_2}\right) + \cdots$$

$$+ (r_1^{n-1} + r_2^{n-1})\left(\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\right)]$$

$$= \frac{1}{2^k(r_1 - r_2)}(r_1^2 - r_2^2 + r_1^4 - r_2^4 + \cdots + r_1^{2n-2} - r_2^{2n-2})$$

$$= \frac{1}{2^k(r_1 - r_2)}[(r_1^2 + r_1^4 + \cdots + r_1^{2n-2}) - (r_2^2 + r_2^4 + \cdots + r_2^{2n-2})$$

After we use geometric sum formula, we get

$$= \frac{1}{2^k(r_1 - r_2)}\left(\frac{r_1^{2n} - r_1^2}{r_1^2 - 1} - \frac{r_2^{2n} - r_2^2}{r_2^2 - 1}\right)$$

$$= \frac{1}{2^k(r_1 - r_2)} \frac{r_1^{2n}r_2^2 - r_1^2r_2^2 - r_1^{2n} + r_1^2 - r_2^{2n}r_1^2 + r_1^2r_2^2 + r_2^{2n} - r_2^2}{(r_1^2 - 1)(r_2^2 - 1)}$$

$$= \frac{1}{2^k(r_1 - r_2)} \frac{(r_1r_2)^{2n}(r_2^{2-2n} - r_1^{-2n}) + r_2^{2n} - r_1^{2n} + r_1^2 - r_2^2}{(r_1^2 - 1)(r_2^2 - 1)}$$

By the property $(r_1^2 - 1)(r_2^2 - 1) = 5 - (r_1^2 + r_2^2)$=5-($(r_1 + r_2)^2 - 2(r_1r_2)$ =1$-k^2$, we have

$$= \frac{1}{2^k(1 - k^2)}[-(r_1r_2)^{2n}\left(\frac{r_1^{2-2n} - r_2^{2-2n}}{r_1 - r_2}\right) - \left(\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right) + \left(\frac{r_1^2 - r_2^2}{r_1 - r_2}\right)]$$

$$= \frac{1}{2^k(1 - k^2)}[\frac{-(r_1r_2)^{2n}}{r_1 - r_2}\left(\frac{r_2^{2n-2} - r_1^{2n-2}}{(r_1r_2)^{2n-2}}\right) - (r_1^{2n} - r_2^{2n}) + (r_1^2 - r_2^2)]$$

$$\frac{-4J_{k,2n-2} - J_{k,2n} + J_{k,2}}{2^k(1 - k^2)}.$$

Conclusion

Some interesting features are obtained for *k*-Jacobsthal and *k*-Jacobsthal-Lucas sequence in the study. We think that it can be found many more identities for these numbers. The Jacobsthal and Jacobsthal-Lucas sequences may attract attention as Fibonacci and Lucas numbers by virtue of charming identities in scientific area.

References

1) Horadam, A. F. (1996). Jacobsthal Representation Numbers. The Fibonacci Quarterly, 34, 40–54.
2) Falcon, S. Plaza, A. (2007). On the Fibonacci k-Numbers. Chaos, Solitons&Fractals 32, 1615-1624.
3) Falcon, S. (2011). On the k-Lucas Numbers. International Cournal contemp. Math. Sciences, 6(21), 1039-1050.
4) Cerin, Z. (2007). Sums of Squares and Products of Jacobsthal Numbers. Journal of Integer Sequences, 10, Article 07.2.5.
5) Cerin, Z. (2007). Formulae for Sums of Jacobsthal Lucas Numbers. International Mathematical Forum, 2, 1969-1984.
6) Bolat, C. (2008). Properties and Applications of k-Fibonacci. k-Lucas Numbers. M.S thesis. Selcuk Univercity, Konya, Turkey.
7) Godase, A. D., Dhakne, M. B. (2014). On the properties of k-Fibonacci and k-Lucas numbers, International Journal of Advancesin Applied Mathematics and Mechanics, 2, 1, 2014, 100-106.
8) Catarino, P. (2014). On Some Identities for k-Fibonacci Sequence. International Journal contemp. Math. Sciences, 9(1**)**, 37-42.
9) Jhala, D., K. Sisodiya and Rathore, G.P.S. (2013). On some identities for k-Jacobsthal numbers, Int. Journal of Math. Analysis, 7(12) 551-556.
10) Srisawat, S., Sriprad, W., Sthityanak, O. (2015). On the k-Jacobsthal Numbers by Matrix Methods. Progress in Applied Science and Technology, 5(1), 70–76.
11) Uygun S., Eldoğan H. (2016). k-Jacobsthal and k-Jacobsthal Lucas Matrix Sequences, International Mathematical Forum, 11(3), 145-154.
12) Uygun S., Owusu, E. (2016). A New Generalization of Jacobsthal Numbers (Bi-Periodic Jacobsthal Sequences). Journal of Mathematical Analysis, 5, 728-39.
13) N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, 2006.
14) Zhang, Z. (1997). Some Identities Involving Generalized Second-order Integer Sequences, The Fibonacci Quarterly, 35(3), 265–267.

Acknowledgment

Thank you very much to the editor and the referee for their valuable comments.