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A Note on the Relationship Between Knots and Dichromatic Polynomial

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Abstract

Knot theory deals with how a circle or disjoint union of circles can be embedded in $\mathbb{R}^3$. Knots (or links) can be thought of topologically as circles embedded in space or geometrically as simple closed curves in space, however they can also be defined in a combinatorial sense. In other words, we can define the knots as equivalence classes of knot diagrams under an equivalence relation determined by certain diagrammatic movements. In this situation, it becomes much easier to manipulate (deform) the regular diagrams of the knots and their crossings. Based on these deformations, it becomes possible to define polynomials that are matched with specific coefficients. One of these special polynomials is the dichromatic polynomial. The definition of this polynomial has led to connections between knots and graph theory, as well as between knots and fields such as physics and biology. This study examines information regarding these relationships. The situations mentioned are examined in detail, both structurally and through calculations based on a specific example.

Keywords

Knot,
Link,
Graph,
Knot graph,
Dichromatic polynomial

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Düğümler ve Dikromatik Polinom Arasındaki İlişkiye Dair Bir Not

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Özet

Düğüm teorisi, bir çemberin veya çemberlerin ayrık birleşiminin $\mathbb{R}^3$ e nasıl gömülebileceğiyle ilgilenir. Düğümler (veya halkalar), topolojik olarak uzaya gömülü çemberler veya geometrik olarak uzayda basit kapalı eğriler olarak düşünülebilir, ancak kombinatoryal anlamda da tanımlanabilirler. Başka bir deyişle, düğümleri, belirli diyagramatik hareketlerle belirlenen bir denklik ilişkisi altında düğüm diyagramlarının denklik sınıfları olarak tanımlayabiliriz. Bu durumda, düğümlerin ve geçitlerinin düzenli diyagramlarını manipüle etmek (deforme etmek) çok daha kolay hale gelir. Bu deformasyonlara dayanarak, belirli katsayılarla eşleşen polinomlar tanımlamak mümkün hale gelir. Bu özel polinomlardan biri dikromatik polinomdur. Bu polinomun tanımı, düğümler ve graf teorisi arasında olduğu kadar, düğümler ve fizik ve biyoloji gibi alanlar arasında da bağlantılar kurulmasına yol açmıştır. Bu çalışma, bu ilişkilerle ilgili bilgileri incelemektedir. Bahsedilen durumlar, hem yapısal olarak hem de belirli bir örneğe dayalı hesaplamalar yoluyla ayrıntılı olarak incelenmektedir.

Anahtar kelimeler

Düğüm,
Halka,
Graf,
Düğüm grafı,
Dikromatik polinom

1. INTRODUCTION

Knot theory began as an attempt to comprehend the basis structure of the universe. However, over time, mathematicians came to view knot theory as a subfield of topology, the study of deformable shapes, because the springs of a knot could be made from a flexible material like string. Knot theory is a branch of topology concerned with the study of knotted loops. That is, a knot is a closed loop of string in $\mathbb{R}^3$. Of course, initially, this object can only acquire a mathematical value if it is a closed curve [1]. However, in recent years, open knots have also been mathematically defined and studied. There is a natural connection between knot theory and graph theory, and this connection began spontaneously [2]. This spontaneous connection has been developed by physicists and mathematicians, and the two theories have become almost inseparable. Definitions, invariants, and examples from both theories have been transferred to each other's fields [3]. With these transfers, the number of studies combining the two fields has rapidly increased. The study presented here is one such work. Knots (or links) are generally identified through knot diagrams; these are projections of the knot onto a plane, and certain cuts called crossings are made at the layered points to indicate which wire passes from above and which passes from below. When the information from the original knot is transferred to a diagram, this projection is called a regular diagram.

Knot theory essentially aims to define and classify knots and links, and to this end, a comprehensive knot table has been created [4]. This table is based on the number of crossings in each knot. While the number of distinct knots is small when the number of crossings is low, the number of distinct knots increases rapidly as the number of crossings increases. These distinct knots are placed in the knot table. In other words, the knots in the table are all distinct knots. In its final form, it has been determined that there are approximately 300 million prime knots with 19 crossings. Knot invariants have been developed to reveal this difference, and each has been an effective tool for making this classification. For this purpose, knot theorists have got help from graph theory. This assistance led to the emergence of the concept of knot graph. The relationship between knot and its planar graph was immediately defined and placed on a solid mathematical foundation.

A knot is any simple closed curve in the 3-dimensional space $\mathbb{R}^3$ without its points of intersection, and a link is any union of simple closed curves that do not intersect. A graph can be thought of as a shape consisting of points and line segments (topologically it is called a 1-complex) [5]. A graph corresponding to each knot (link) can be assigned, and such a graph is called a knot (link) graph [6]. Any knot diagram: it defines a plane graph whose vertices are intersection points and whose edges are paths between consecutive intersection points. Exactly one face of this planar graph is unbounded, each of the others is homeomorphic to a 2-dimensional disk. It can be introduced a planar graph that is naturally related to any knot or link diagram, and shown how these graphs can be used to examine the algebraic invariants of knots. This study explains the relationship between a type of polynomial, primarily related to graphs, and a knot, and how this polynomial is calculated for the knot. The polynomial in question is a dichromatic polynomial.

2. PRELIMINARIES

Mathematically, knots are modeled as closed loops, that is, loops with no beginning or end (see Figure 1). Additionally, a knot must not intersect with itself.

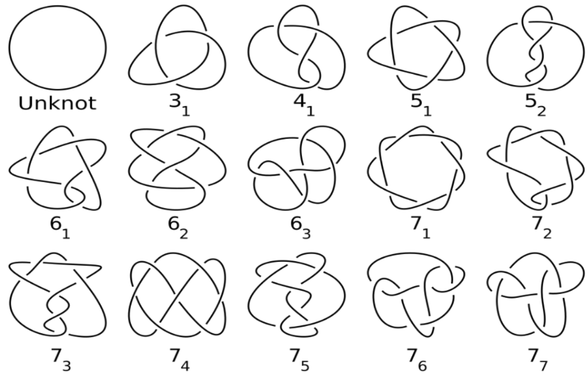

Figure 1. Some examples of mathematical knots [7]

Let $f: X \to Y$ be a function, where X and Y are two topological spaces. If there is a homeomorphism from the topological space X to the subspace $f(X)$ in Y, the space X is said to be embedded in the space Y. A knot K is a continuous embedding of 1-sphere S^1 into 3-sphere S^3 [4]. In other words, a knot is a one-to-one continuous transformation $K: S^1 \to S^3$. Note that $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

A sequential finite combination of knots that do not intersect with each other is called a link. Each knot that makes up the link is called a component of the link (see Figure 2).

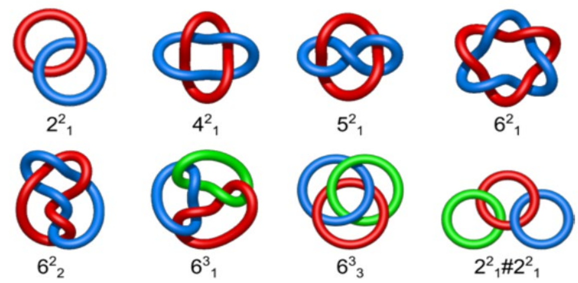

Figure 2. Some examples of links [8]

Unless otherwise stated, it will be assumed that the situations that are valid for knots are also valid for links. In fact, the shapes given as knots so far are regular diagrams in two-dimensional space of the knots embedded in S^3. Almost all of the work on knots is carried out on these regular diagrams, which are considered to be a complete representation of the knot. The narrow region on a regular diagram that shows that the arc of a knot passes from above or below is called the crossing point of the knot (simply crossing). Additionally, knots are named according to their crossing numbers. If the crossings in the regular diagram of a knot progress in an under-upper-under-upper-... (or upper-under-upper-under-...) pattern, this knot is called an alternating knot [9]. Otherwise, this knot is called a nonalternating knot.

Let V be a finite, non-empty set of the form $V = \{v_1, v_2, v_3, \dots, v_n\}$, whose elements are called vertices. The binary structure (V, E) consisting of a finite set of edges in the form $E = \{e = v_i v_j : v_i, v_j \in V\}$ connecting these points or the point itself (which does not give any geometric or positional information but only shows the relationship between the points) is called a graph [10]. A graph is denoted by $G = (V, E)$ or simply G. If there is at least one edge between any vertices v_1 and v_2 of G, it is said that they are adjacent or neighbor. Similarly, edges in any graph that have a common vertex are called neighbor edges. The number of elements of the set V which is the set of vertices in a graph G is called the order of the graph G and is denoted by $|V(G)|$. A sequence with a finite number of elements consisting of neighboring vertices and edges in a graph is called a walk, and the symbol W is used to express the walk, and the length of this walk is determined by the number of edges in the sequence. If each edge and each vertex in a walk occurs once, this walk is called a path. If a path starts from the same vertex and ends at the same vertex again, it is called a cycle. Let v be any vertex in a graph G. Accordingly, the degree of v is the number of edges connected to v. In a graph, the vertex with the smallest degree is called the vertex with the minimum degree while the vertex with the largest degree is called the vertex with the maximum degree. The degree of the vertex of the graph containing a loop is taken as $+2$. If the degree of the vertex v is equal to zero, v is called isolated vertex, and if the degree of the vertex v is equal to one, v is called pendant (end vertex). If there is always a path between any two arbitrary vertices of a graph, this graph is called a connected graph. Otherwise, this graph is called a disconnected graph. Graphs can be considered to model a network, and thus graph-theoretical parameters can be used for this network [11].

3. RESULTS AND DISCUSSION

Polynomials are very useful tools in studies involving knots and rings. First, let's look at how to calculate the polynomial for graphs. A dichromatic polynomial that has two variables where q and v are variables is denoted by $Z_G(q, v)$. This is a polynomial defined for planar graphs. That is, it does not bound up with how the polynomial graph is embedded in three-dimensional space, however it bounds up with the type of isomorphism of the graph. So, it is different significantly according to the polynomials of knots and links. It should be noted here that multiple edges on the graph are allowed to share identical endpoints and to have edges which start and end at identical vertex.

Dichromatic polynomials are described by three formulas below [12]:
Formula 1: $Z(\bullet) = q$,
this rule states that the polynomial of a graph with a single vertex is equal to q only.
Formula 2: $Z(\bullet G) = qZ(G)$,
the rule states that attaching a novel vertex to a graph which is not connected by an edge produces the polynomial of graph to be multiplied by q.

Formula 3: ,
this rule states that when a certain edge of the graph G is chosen, the polynomial for G is gotten by attaching the graph's polynomial by this edge removed to the polynomial of the graph with that edge contracted to a single vertex, multiplied by "v". Provided that the rule is applied to an edge that starts and ends at identical vertex it is yielded the following result:

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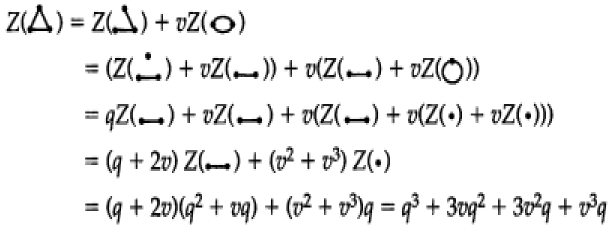
Let us give some computational examples. Firstly, for $\bullet\!\!-\!\!\bullet$:

$$Z(\bullet\!\!-\!\!\bullet) = Z(\bullet\ \ \bullet) + vZ(\bullet) = q^2 + vq.$$

Let us give a more detailed example [12]:



How can it be said that we will get the same answer when we calculate the polynomial of a graph in two distinct ways (by selecting and removing various edges at diversified phases)? In other words, is the polynomial well-defined? Proof will not be given here, but it should be known that the answer to this question is always yes. Furthermore, unlike knot polynomials, which are entirely dependent on a specific knot, this graph polynomial is not bounds up with how the graph is situated in space. Which vertices the edges connect to other vertices is the only thing that matters.

This polynomial has various applications. An example of its use is as follows [12]: Suppose that we have a given graph and we want to color each vertex of this graph with one of q possible colors, so, two vertices connected by a single edge should not have the same color. Coloring a graph in this way is called vertex coloring. In fact, vertex coloring applications are also encountered in the real world. For example, let us assume that there are a number of VHF television stations in a certain areas of the country, and the signals of some of these stations interfere with each other. There exist 17 channels, however there are many more stations than channels. Thus, a graph is constructed where each station is a vertex and an edge between two stations purports that these stations are near sufficient to each other that their signals will interfere by each other unless they are served different channels. The aim is to accurately color the graph such that every station is assigned a channel, but no two stations sharing an edge receive the same channel (channel numbers from 1 to 17 are used instead of colors). Surprisingly, if we take the value of $v = -1$ in the dichromatic polynomial of the

graph here, it is obtained completely the number of different vertex colorings of the graph. For instance, on account of the dichromatic polynomial of a triangular graph is $q^3 + 3vq^2 + 3v^2q + v^3q$, the number of vertex colorings of the triangular graph is expressed as:
$q^3 + 3(-1)q^2 + 3(-1)^2q + (-1)^3q = q^3 - 3q^2 + 2q$.

Let us see whether this is true or not. Given a triangular graph and q possible colors for its vertices, we can color the first vertex with any of those q colors. Since the second vertex is connected to the first vertex by an edge, we can only paint it with one of the remaining $q-1$ colors. Since the third vertex is connected to the first two, it can be painted with one of the unused $q-2$ colors. Therefore, the total number of ways to color the vertices in the graph such that no two connected vertices have the same color is $q(q-1)(q-2) = q^3 - 3q^2 + 2q$. That is completely identical conclusion obtained by placing the value $v = -1$ into the dichromatic polynomial.

On the given television stations, it is taken place the values $v = -1$ and $q = 17$ into the dichromatic polynomial of graph corresponding to station interference, also provided that the conclusion is bigger than zero, it is known that the graph has at least one vertex coloring and therefore at least one channel assignment option to prevent interference. Let us prove that the dichromatic polynomial of a graph gives the number of vertex colorings as appraised for $v = -1$. The number of vertex colorings for any graph G that has no edges, i.e., graphs consisting only of vertices, is given by $Z_G(q, -1)$. Now let us prove this for graphs with edges. Assume that it has been proven this for a graph that has m edges. Whether it can be demonstrated that it is also true for any graph that has $m+1$ edges, then the principle of induction shows that it is true for a graph. Assume that G is a graph that has $m+1$ sides and E is a side connecting two distinct vertices A and B of G. Formula 3 for the dichromatic polynomial when $v = -1$ states that , where G' and G'' are the two novel graphs seeming for that equality. Both G' and G'' own m vertices. So,  represents the number of vertex colorings of G', and  represents the number of vertex colorings of G''. However, the number of vertex colorings of G is obtained by subtracting the number of colorings that A and B own identical coloring from the number of vertex colorings of G'. The number of vertex colorings in G' that A and B own identical color is completely identical as the number of vertex colorings in G''. Therefore, the expression $Z(G') - Z(G'')$ represents the number of vertex colorings of G. However, $Z(G)$ is stated by the equality. Therefore, $Z(G)$ is the number of vertex colorings of the graph G.

That is a surprising reality. For example, one of very hard theorems proven recently is the theorem known as the Four-Colors Theorem. This theorem states that any map of countries in a plane can be colored by 4 colors, and that two countries of the same color won't share borders on the resulting map. This theorem was proven by Wolfgang Haken and Kenneth Appel [12]. Provided that we consider the binary graph of the country map as in Figure 3, we ask whether the vertices of this planar graph can be colored by 4 or fewer colors such that no two vertices sharing an edge have the same color [12]. However, the planar graph G is colored by 4 colors iff $q = 4$ and $v = -1$, $Z(G) \neq 0$. Therefore, the Four Colors Theorem is equivalent to proving that for every planar graph G, $Z(G) \neq 0$ when $q = 4$ and $v = -1$.

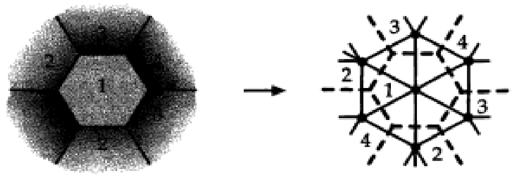


Figure 3. Coloring a map is the same as coloring a planar graph [12]

Now let us show that the dichromatic polynomial is related to polynomials from knots and links. A planar graph is every time associated with a knot. Each edge of the graph forms an intersection point in the link, and we have two options for placing this intersection point. We can shade the planar regions formed by the link in a checkerboard fashion, selecting all intersections such that the region outside the link remains unshaded, with the shaded regions located to the north and south, so that the thread passing through the top moves from southwest to northeast, as shown in Figure 4. The planar graph then transforms into an alternating knot. Let G be the planar graph, and the ensuing alternating knot is denoted by $L(G)$ (see Figure 4 as an example).

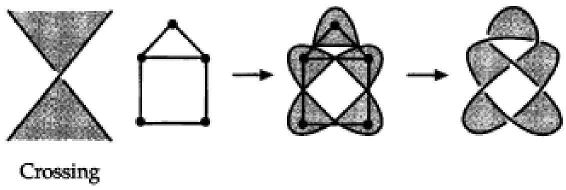
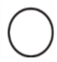


Figure 4. A planar graph associated with an alternating knot [12]

It will be indicated that the dichromatic polynomial of a planar graph G can be obtained as the "brackets" polynomial of its corresponding alternating knot $L(G)$. For knot and link regular diagrams, the brackets polynomial is defined with three laws below:

Law 1: <  > = 1,

Law 2: < L ∪  > = $-(A^{-2} + A^2)$ < L >,

Law 3: <  >= A <) (> + A^{-1} <  >.

Now let us define the square brackets polynomial for a knot or link diagram. This polynomial owns two variables which are q and v, it is described with identical 3 rules, but by dissimilar coefficients:

Rule 1: [◯] $= q^{1/2}$,

Rule 2: [L ∪ ◯] $= q^{1/2}[L]$,

Rule 3: < ╳ > $= q^{-1/2}v$ [ ) ( ] + [⌣⌢].

The square brackets polynomial which is emerged is not an absolute constant for knots and links. However, for the projection of a knot or link, it can be reckoned the square brackets taking care not to isotopically eliminate intersections in the projections of the connections, as this could alter the result. For example, although the knot indicated in Figure 5 is a projection of the trivial knot, its square brackets polynomial will not $q^{1/2}$.

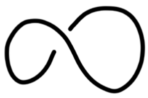


Figure 5. A projection of the trivial knot

Theorem 3.1: The equality $Z_G(q, v) = q^{N/2}[L(G)]$ holds, where $L(G)$ is the knot associated with the graph G and N is the number of vertices of the planar graph G [12].

Proof: First of all, let us show this for graphs that own no edges. Let us consider a graph with an unique vertex (see Figure 6). Thus, the knot $L(G)$ associated with the graph G is simply a regular diagram of an trivial knot. Therefore, the square brackets polynomial for $L(G)$ will be $q^{1/2}$ according to the initial law of calculating the square brackets polynomial. By multiplying it with $q^{N/2}$ when $N = 1$, we obtain the value of q. This is the dichromatic polynomial of a graph with an unique vertex. So, the theorem is proven true for the case where the graph G has only one vertex.

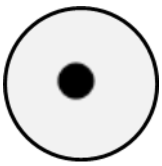


Figure 6. The state of a single vertex graph and its associated knot

The theorem also holds about a graph which owns only vertices and does not own edges, because whether G is a graph that owns only vertices, thus $L(G)$ is an trivial link obtained by having some trivial link components surrounding each vertex.

Now we will evaluate the status of the third law for a graph G. It will be used induction method with respect to the number of edges in the graph. The theorem proven for graphs which do not have any edge. Assume that the theorem is true for all graphs which are less edges than graph G. Then. it can be shown that it is also true for the graph.

Assume that N represent the number of vertices in graph G. Let G' and G'' represent the graphs shown in Figure 7. Because G' and G'' have less edges than G, it is obtained that $Z(G') = q^{N/2}[L(G')]$ and $Z(G'') = q^{(N-1)/2}[L(G'')]$.

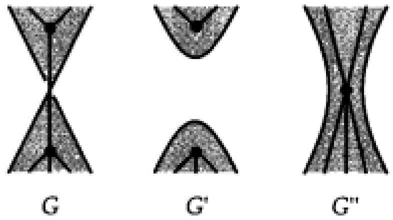


Figure 7. Dual graphs [12]

Our aim is to demonstrate that $Z(G) = q^{N/2}[L(G)]$. According to the third law, the dichromatic polynomial is calculated as

$$Z(G) = Z(G') + vZ(G'')$$
$$Z(G) = q^{N/2}[L(G')] + vq^{(N-1)/2}[L(G'')]$$
$$Z(G) = q^{N/2}\left([L(G')] + vq^{-1/2}[L(G'')]\right)$$
$$Z(G) = q^{N/2}[L(G)].$$

Thus, the proof of the theorem is complete.

Depending on the coefficients selected for the “brackets” laws, it can be calculated polynomials for knots or graphs. That also owns implications for statistical mechanics [13]. In statistical mechanics, there is a close relationship between the Ising model and this dichromatic polynomial. In this model, every vertex of a graph is permitted to have one of two states. The type of graph here is a lattice, where the vertices and edges form a regular pattern in space. Metals are a prime example. Metals occur molecules located at the vertices of a lattice pattern in three-dimensional space. Furthermore, the two states imply that a value of $+1$ or -1 is assigned to the suitable vertex. For that magnetization example, the $+1$ state fits to the magnetic axis (spin vector) pointing upwards, and the -1 state fits to the magnetic axis pointing downwards. It is possible to generalize this to a novel model named by the Potts model (see Figure 8). It is used a planar graph here its every vertex can be considered a particle, but at the moment, in place of just two states, it is accepted that every vertex can own one of q states such that q is a positive integer. That is very useful to imagine the q states that a particle can own as q potential colors that the particle can own [12].

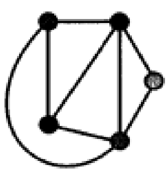


Figure 8. A specific case for Potts model [12]

When it is calculated the dichromatic polynomial of the graph and substitute the correct substitution for the variable v, the dichromatic polynomial of the graph becomes fully the quotient function of the Potts model. The quotient function of the planar graph is defined as:

$$P = \sum e^{-E(S)/kT}.$$

Here we sum up all potential states of the system. In the definition, k is the Boltzmann constant and T is the temperature of the system. The energy of the system in a specific state S is indicated by $E(S)$. That energy is the sum of the interaction energies of the edges, defined as:

$$E(S) = \sum E(S_i, S_j),$$

and the sum is calculated on all pairs of vertices connected by edges in the graph. The state S is indicated with personal states of the vertices, therefore $S = (s_1, s_2, \ldots, s_n)$. In that special model, the interaction energy round an edge is chosen as $E(s_i, s_j) = \delta(s_i, s_j)$ such that δ is called the Kronecker delta function and is defined as:

$$\delta(a, b) =) \begin{cases} 1, & if\ a = b \\ 0, & if\ a \neq b \end{cases}.$$

So, an edge in the graph conduces to the energy of the state of interest just, while the vertices at its two endpoints are in identical state (i.e., they have the same assigned colors.).

Theorem 3.2: Let $v = e^{-(1/kT)} - 1$. Then, for the planar graph G, the dichromatic polynomial $Z_G(q, v)$ happens the quotient function for Potts model over the graph G, and the variable q in the polynomial gives the number of potential states at each vertex [12].

Proof: Let us rewrite it, starting from the quotient function, to demonstrate that it is a dichromatic polynomial.

$$P = \sum e^{-E(S)/kT} = \sum e^{-\sum \delta(s_i, s_j)/kT} = \sum \prod \left(e^{-1/kT}\right)^{\delta(s_i, s_j)}$$

$$P = \sum \prod \left(1 + v\delta(s_i, s_j)\right).$$

That result comes out in the preceding line by substituding the two possibilities for the amounts in the function ($\delta = 0\ or\ \delta = 1$) and seeing that these two expressions are equal in both cases. We assert that the expression $\sum \prod \left(1 + v\delta(s_i, s_j)\right)$ is completely a dichromatic polynomial with variables q, v. In order to make that happen, it will be indicated that this polynomial fulfills the three laws required to calculate the dichromatic polynomial. Because we previously showed that the dichromatic polynomial is well-defined, A polynomial fulfilling the identical describing laws should itself be a dichromatic polynomial.

At the outset, provided that there is a graph G which occurs an unique vertex and no edges, the quotient function P is presented by adding on entire states s as follows:

$$\sum \prod \left(1 + v\delta(s_i, s_j)\right) = \sum 1.$$

Since the vertex can take the state q, $P(\cdot) = \sum 1 = q$. Therefore, the quotient function satisfies the initial law of dichromatic polynomials. The other law is shown by discovering the quotient function of a graph that has an extra vertex not connected to the remainder of the graph via any edge, that is, if the graph is reduced using this logic, $P(\cdot\, G) = qP(G)$ is obtained.

Finally, let us show that the final law holds for the dichromatic polynomial. It should be indicated that the quotient function fulfills the following:

Let us assume that we label all the vertices in graph G with integers $\{1, 2, \ldots, n\}$. Assume that a and b is two vertices labelled by integers in G and the edge e connects them. Let us assume that, as shown in Figure 9, G' is the reduced graph where the edge e is deleted from G, and G'' is also the reduced graph where the edge e is contracted in G, resulting in the identification of the vertices a, b. The aim is to demonstrate that $P(G) = P(G') + vP(G'')$. Since the quotient function $P(G) = \sum \prod \left(1 + v\delta(s_i, s_j)\right)$ for the planar graph G is defined as follows, for a specific state s of the whole graph, the term corresponding to s in that addition is:

$$\prod \left(1 + v\delta(s_i, s_j)\right) = \left(1 + v\delta(s_a, s_b)\right) \prod_{(i,j) \neq (a,b)} \left(1 + v\delta(s_i, s_j)\right)$$

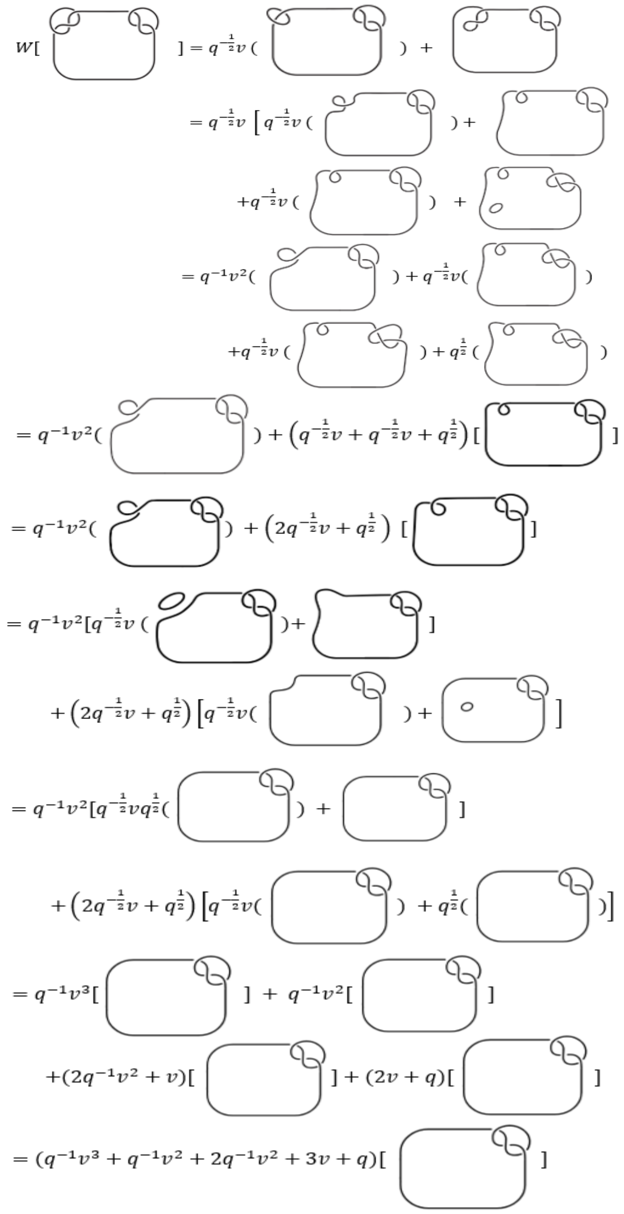
$$\prod \left(1 + v\delta(s_i, s_j)\right) = \prod_{(i,j) \neq (a,b)} \left(1 + v\delta(s_i, s_j)\right) + v\delta(s_a, s_b) \prod_{(i,j) \neq (a,b)} \left(1 + v\delta(s_i, s_j)\right),$$

the law of distribution was applied here and let us classify the term before the plus sign on the right side in the final state of equality as the first term and the term after the plus sign as the second term. The first term of the equation above is exactly the same as the term in the quotient function corresponding to the state s of G'. For the second term, it should be noted that $\delta(s_a, s_b) = 1$ when the colors of vertices a and b are the same in the state s. Then, the second term is simply the product of v and the term in the quotient function fitting to the state s of G''. As a and b own dissimilar colors in state s, the value $\delta(s_a, s_b)$ is equal to zero so, the second term is lost. Therefore, that is normal for it to disappear because of no situation fitting to G'', the two vertices are reduced to a single vertex and cannot have different colors. Thus, if we sum up all possible cases we obtain the equality $P(G) = P(G') + vP(G'')$, which is the desired thing. So, since the quotient function P fulfills the laws about calculating dichromatic polynomials that will be a dichromatic polynomial. Hence, the proof is complete.

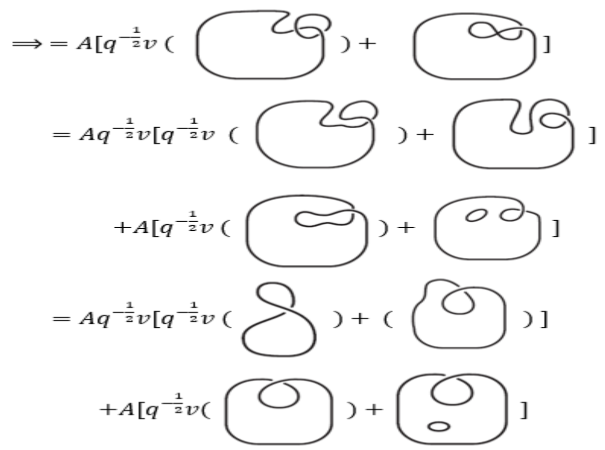
Here, it is seen that the dichromatic polynomial of this graph, which can be calculated using the skein relation on the alternating knot corresponding to the graph, is actually equal to the partition function of the statistical mechanics model known as the Potts model.

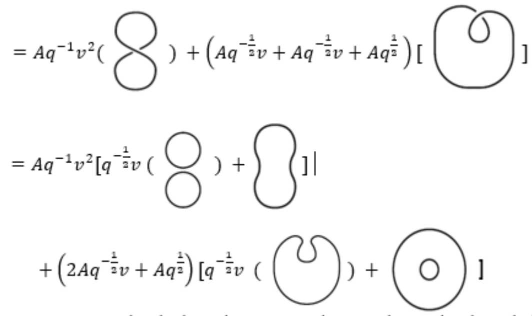
4. APPLICATION

As an application, let us demonstrate how to perform a step-by-step dichromatic polynomial calculation using the regular diagram of a 6-crossing alternating knot. Here, the symbol $W(K)$ (instead of $[L(K)]$) is used for the brackets polynomial of any knot K.



(Let $q^{-1}v^3 + 3q^{-1}v^2 + 3v + q = A$)





$$= Aq^{-1}v^2\left[q^{-\frac{1}{2}}vq^{\frac{1}{2}}q^{\frac{1}{2}} + q^{\frac{1}{2}}\right] + \left(2Aq^{-\frac{1}{2}}v + Aq^{\frac{1}{2}}\right)\left[q^{-\frac{1}{2}}vq^{\frac{1}{2}} + q^{\frac{1}{2}}q^{\frac{1}{2}}\right]$$

$$= Aq^{-1}v^2\left[q^{\frac{1}{2}}v + q^{\frac{1}{2}}\right] + \left(2Aq^{-\frac{1}{2}}v + Aq^{\frac{1}{2}}\right)[v+q]$$

$$= Aq^{-1}v^2q^{\frac{1}{2}}v + Aq^{-1}v^2q^{\frac{1}{2}} + 2Aq^{-\frac{1}{2}}vv + Aq^{\frac{1}{2}}v + 2Aq^{-\frac{1}{2}}vq + Aq^{\frac{1}{2}}q$$

$$= Aq^{-\frac{1}{2}}v^3 + Av^2q^{-\frac{1}{2}} + 2Aq^{-\frac{1}{2}}v^2 + Aq^{\frac{1}{2}}v + 2Aq^{\frac{1}{2}}v + Aq^{\frac{3}{2}}$$

$$= Aq^{-\frac{1}{2}}v^3 + 3Av^2q^{-\frac{1}{2}} + 3Aq^{\frac{1}{2}}v + Aq^{\frac{3}{2}}$$

(If we substitute the equality $A = q^{-1}v^3 + 3q^{-1}v^2 + 3v + q$)

$$\Longrightarrow = [(q^{-1}v^3 + 3q^{-1}v^2 + 3v + q)(q^{-\frac{1}{2}}v^3)]$$
$$+[3(q^{-1}v^3 + 3q^{-1}v^2 + 3v + q)(q^{-\frac{1}{2}}v^2)]$$
$$+[3(q^{-1}v^3 + 3q^{-1}v^2 + 3v + q)(q^{\frac{1}{2}}v)]$$
$$+[(q^{-1}v^3 + 3q^{-1}v^2 + 3v + q)(q^{\frac{3}{2}})].$$

Thus, it is obtained that

$W($  $)$

$$= v^6q^{-\frac{3}{2}} + 3q^{-\frac{3}{2}}v^5 + 3v^4q^{-\frac{1}{2}} + q^{\frac{1}{2}}v^3 + 3v^5q^{-\frac{3}{2}} + 9v^4q^{-\frac{3}{2}} + 9q^{-\frac{1}{2}}v^3 + 3v^2q^{\frac{1}{2}} + 3v^4q^{-\frac{1}{2}} + 9v^3q^{-\frac{1}{2}} + 9q^{\frac{1}{2}}v^2 + 3vq^{\frac{3}{2}} + v^3q^{\frac{1}{2}} + 3v^2q^{\frac{1}{2}} + 3v\,q^{\frac{3}{2}} + q^{\frac{5}{2}} \quad (1)$$

Equality (1) is the square brackets polynomial of the knot. The shaded regular projection and corresponding graph of the 6-crossings alternating knot under consideration are given in Figure 9.

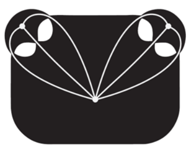
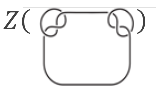


Figure 9. The shaded planar projection of a 6-crossings knot

$Z($  $) = q^{\frac{N}{2}}.W($  $) \Longrightarrow$

$Z($  $)$

$$= q^{\frac{3}{2}}.[v^6q^{-\frac{3}{2}} + 3q^{-\frac{3}{2}}v^5 + 3v^4q^{-\frac{1}{2}} + q^{\frac{1}{2}}v^3 + 3v^5q^{-\frac{3}{2}} + 9v^4q^{-\frac{3}{2}} + 9q^{-\frac{1}{2}}v^3 + 3v^2q^{\frac{1}{2}} + 3v^4q^{-\frac{1}{2}} + 9v^3q^{-\frac{1}{2}} + 9q^{\frac{1}{2}}v^2 + 3vq^{\frac{3}{2}} + v^3q^{\frac{1}{2}} + 3v^2q^{\frac{1}{2}} + 3v\,q^{\frac{3}{2}} + q^{\frac{5}{2}}]$$

$$= v^6 + 3v^5 + 3v^4q + q^2v^3 + 3v^5 + 9v^4 + 9qv^3$$
$$+3v^2q^2 + 3v^4q + 9v^3q + 9v^2q^2 + 3vq^3 + v^3q^2$$
$$+ 3v^2q^2 + 3v\, q^3 + q^4$$
$$= v^6 + 6v^5 + 6v^4q + 9v^4 + 18v^3q + 2v^3q^2 + 15v^2q^2 + 6vq^3 + q^4 \quad (2)$$

Equality (2) is the dichromatic polynomial of the knot.

5. CONCLUSION

Mathematicians and physicists continue their research into the relationships between knots and dichromatic polynomial. Clearly, much more can be added to this relationship. Indeed, it is possible to extend this relationship to other graph polynomials.

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