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SOFT STRUCTURES DERIVED FROM GROUPS

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Abstract. In this paper, we get soft structures what we call cyclicizer soft set, centralizer soft set, normalizer soft set, cosetial soft set, orbital soft set and stabilizer soft set using some group concepts such as cyclic, centralizer and normalizer of an element, coset and group action, in any given group. At the same time, we mentioned that they are a soft group. We discuss their soft set theoretic properties and give some theorems for groups. We proposed some necessary and sufficient conditions for two groups to be isomorphic using the soft set theory. We give relation between similarity of soft sets and groups.

1. Introduction

Group theory is at the center of the abstract algebra, and is almost as old as mathematics. As in mathematics, groups have applications in many other fields, such as physics and engineering. We know that a *group* is an algebraic structure that defines a binary operation on it and provides certain axioms, such as associativity, existence of unit and inverse elements. In addition to this, a *group isomorphism* is a bijective function which maintain group operations from one group to another. In group theory, the place of group isomorphism is very important. Because, isomorphic groups have the same properties and are indistinguishable from each other. For the foundation of group theory, we recommend Fraleigh's legendary book [4].

On the other hand, soft set theory is built by Molodtsov to model uncertainties in science and real life, mathematically, in 1999 [10]. Molodtsov described a soft set as a parametrization of subsets of any universal set. He also applied the soft set theory in many areas, such as analysis, game theory etc. The basic properties of the soft set theory are examined in [8, 3, 9, 5, 6, 7]. We briefly mention these informations in the preliminaries section. In [12], Pei and Miao showed that every soft set is an information system and vice versa. In this way, the applicability of soft sets to operations research and humanities is demonstrated. Aktaş and Çağman defined the concept of soft group over any given group in [1]. They defined a soft group over any given group as a parametrization of subgroups of a group sticking to Molodtsov's sense. They investigated fundamental properties of soft groups [1]. In [2], the concept of cyclic soft groups and their some applications on groups was given.

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In this paper, we give some specific soft sets derived from group concepts such as cyclic, centralizer, normalizer, coset, group action in any given group. We study basic soft set theoretic properties. We give some necessary and sufficient conditions for two groups to be isomorphic using the similarities of soft sets in soft set theory.

2. Preliminaries

As the preliminary information, which is necessary to study, give some definitions and properties.

2.1. **Soft Set Theory.** Let U be an initial universe, E be a set of parameters, $\mathcal{P}(U)$ be the power set of U, and $A \subseteq E$. Molodtsov [10] defined the soft set in the following manner:

Definition 2.1. [10] *A pair* (F, A) *is called a* soft set *over* U *where* F *is a mapping given by* $F : A \to \mathcal{P}(U)$.

Some set-theoretic operations defined by [8, 12]

Definition 2.2. [12] *For two soft sets* (F, A) *and* (G, B) *over a common universe* U*, we say that* (F, A) *is a* soft subset *of* (G, B) *and is denoted by* $(F, A)\widetilde{\subset}(G, B)$ *if*

(i) $A \subset B$ *and,*
(ii) $\forall a \in A,\ F(a) \subset G(a)$.

Definition 2.3. [12] *Two soft sets* (F, A) *and* (G, B) *over a common universe* U *are said* soft equal *if* (F, A) *is a soft subset of* (G, B)*, and* (G, B) *is a soft subset of* (F, A).

Definition 2.4. [12] *Let* (F, A) *and* (G, B) *be two soft sets over a common universe* U *such that* $A \cap B \neq \varnothing$. *The* soft intersection *of* (F, A) *and* (G, B) *is denoted by* $(F, A)\widetilde{\cap}(G, B)$*, and is defined as* $(F, A)\widetilde{\cap}(G, B) = (H, C)$*, where* $C = A \cap B$ *and for all* $c \in C$, $H(c) = F(c) \cap G(c)$.

Definition 2.5. [8] *The* soft union *of two soft sets* (F, A) *and* (G, B) *over a common universe* U *is the soft set* (H, C) *, denoted by* $(F, A)\widetilde{\cup}(G, B) = (H, C)$*, where* $C = A \cup B$*, and* $\forall c \in C$,

$$H(c) = \begin{cases} F(c) & , \text{if } c \in A - B \\ G(c) & , \text{if } c \in B - A \\ F(c) \cup G(c) & , \text{if } c \in A \cap B \end{cases}$$

Definition 2.6. [3] *Let* U *be an initial universe set,* E *be the universe set of parameters, and* $A \subset E$.

(i) (F, A) *is called a* relative null soft set *(with respect to the parameter set* A*), denoted by* Φ_A*, if* $F(a) = \varnothing$ *for all* $a \in A$.

(ii) (F, A) *called a* relative whole soft set *(with respect to the parameter set* A*), denoted by* $\mathcal{U}_A$*, if* $F(a) = U$ *for all* $a \in A$.

The relative whole soft set $\mathcal{U}_E$ *with respect to the universe set of parameters* E *is called the* absolute soft set *over* U.

Definition 2.7. [8] *Let* (F, A) *and* (G, B) *be two soft sets over the common universe* U*. Then the operation* **AND** *between* (F, A) *and* (G, B) *denoted by* $(F, A) \wedge (G, B)$ *and is defined by* $(F, A) \wedge (G, B) = (H, A \times B)$ *where* $H((a, b)) = F(a) \cap G(b)$*, for all* $(a, b) \in A \times B$.

Definition 2.8. [8] *Let (F, A) and (G, B) be two soft sets over the common universe U. Then the operation* **OR** *between (F, A) and (G, B) denoted by $(F, A) \vee (G, B)$ and is defined by $(F, A) \vee (G, B) = (H, A \times B)$ where $H((a, b)) = F(a) \cup G(b)$, for all $(a, b) \in A \times B$.*

Definition 2.9. [12] *The* soft complement *of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(a) = U - F(a)$ for all $a \in A$.*

Definition 2.10. [12] *Let (F, A) be a soft set over U. We say that (F, A) is a* partition-type soft set *over U if*

(i) *For all $a \in A$, $F(a) \neq \varnothing$.*
(ii) $\bigcup_{a \in A} F(a) = U$.
(iii) *For any two parameters $a_i, a_j \in A$, $F(a_i) \neq F(a_j) \Rightarrow F(a_i) \cap F(a_j) = \varnothing$.*

Moreover, we say that (F, A) is a covering-type soft set *over U if*

(i) *For all $a \in A$, $F(a) \neq \varnothing$.*
(ii) $\bigcup_{a \in A} F(a) = U$.

In [7], Kim and Min defined the concept of a full soft set as is follows,

Definition 2.11. [7] *Let (F, A) be a soft set over U. We say that (F, A) is a* full soft set *if $\bigcup_{a \in A} F(a) = U$.*

Note that, every partition-type soft set is a covering-type soft set and every covering-type soft set is a full soft set.

In [9], Min has introduced the concept of similarity between soft sets and investigated some properties. He defined the concept of similarity between soft sets as follows:

Definition 2.12. [9] *Let (F, A) and (G, B) be soft sets over a common universe set U. Then (F, A) is* similar *to (G, B) (simply $(F, A) \cong (G, B)$) if there exists a bijective function $\phi : A \to B$ such that $F(x) = (G \circ \phi)(x)$ for every $x \in A$, where $(G \circ \phi)(x) = G(\phi(x))$.*

In [6], we have generalized form of similarity relation on soft sets as follows.

Definition 2.13. [6] *Let E be a set of parameters, U and V be two universes and (F, A) and (G, B) be soft sets over U and V respectively, where $A, B \subseteq E$. We called that (F, A) is* similar *to (G, B) if there exist bijective functions $f : U \to V$ and $\phi : A \to B$ such that $(f \circ F)(\alpha) = (G \circ \phi)(\alpha)$ for every $\alpha \in A$.*

2.2. **Soft Groups.** In [1], Aktaş and Çağman defined the concept of soft group as follows;

Definition 2.14. [1] *Let G be a group, (F, A) be a soft set over G. Then (F, A) is said to be a* soft group *over G if and only if $F(a)$ is a subgroup of G for all $a \in A$.*

Example 2.15. *Let $A = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ and $G = (\mathbb{R}, +)$ be a group of real numbers with addition.*

$$(F, A) = \{\clubsuit = \mathbb{Z}, \diamondsuit = \mathbb{Q}, \heartsuit = \mathbb{R}, \spadesuit = \{0\}\}$$

is a soft set over $\mathbb{R}$, and so (F, A) is a soft group over $\mathbb{R}$.

Definition 2.16. [1] *Let (F, A) be a soft group over G. Then,*

(1) (F, A) *is said to be an* identity soft group *over* G *if* $F(a) = \{e_G\}$ *for all* $a \in A$*, where* e_G *is the identity element of* G*.*
(2) (F, A) *is said to be an* absolute soft group *over* G *if* $F(a) = G$ *for all* $a \in A$*.*

Definition 2.17. [1] *Let* (F, A) *and* (H, B) *be to soft set over* G*. Then* (H, B) *is a* soft subgroup *of* (F, A)*, written* $(H, B)\widetilde{<}(F, A)$*, if*

(1) $B \subseteq A$,
(2) $H(b) < F(b)$ *for all* $b \in B$.

Definition 2.18. [1] *Let* (F, A) *be a soft group over* G *and* (H, B) *be a soft subgroup of* (F, A)*. Then we say that* (H, B) *is a* normal soft subgroup *of* (F, A)*, written* $(H, B)\widetilde{\triangleleft}(F, A)$*, if* $H(b)$ *is a normal subgroup of* $F(b)$ *for all* $b \in B$*.*

Definition 2.19. [1] *Let* (F, A) *and* (H, B) *be two soft groups over* G_1 *and* G_2*, respectively. The* product *of soft groups* (F, A) *and* (H, B) *is defined as* $(F, A) \times (H, B) = (K, A \times B)$*, where* $K(x, y) = F(a) \times H(b)$ *for all* $(a, b) \in A \times B$*.*

2.3. A Brief Overview Some Concepts in Group Theory.

Definition 2.20. *Let* G *be a group and* S *be a subgroup of* G*.*

Cyclic Subgroup: [4] *The* cyclic subgroup *of* G *generated by an element* $x \in G$ *is defined to be*

$$\langle x \rangle = \{x^n \mid n \in \mathbb{Z}\},$$

where $\mathbb{Z}$ *is the set of integers.*

Centralizer: [11] *The* centralizer *of an element* $x \in G$ *is defined to be*

$$C(x) = \{y \in G \mid xy = yx\}.$$

Normalizer: [11] *The* normalizer *of an element* $x \in G$ *is defined to be*

$$N(x) = \{y \in G \mid y^{-1}xy \in \langle x \rangle\}.$$

Left - Right Coset: [4] *The* left coset *of* S *in* G *is defined to be*

$$xS = \{xy \mid y \in S\},$$

and the right coset *of* S *in* G *is defined to be*

$$Sx = \{yx \mid y \in S\},$$

where $x \in G$*.*

Definition 2.21. [4] *Let* G *be a group and* X *be a non-empty set. It is called that* $* : G \times X \to X$ *is a* group action *of* G *on* X *that satisfies the following two axioms;*

(a) $e * x = x$*, for all* $x \in X$ *and* $e \in G$ *is an identity element of* G*.*
(b) $(gh) * x = g * (h * x)$*, for all* $g, h \in G$ *and for all* $x \in X$*.*

It is called that X *is* G-set *if* G *is acting on* X*.*

Definition 2.22. [4] *Let* G *be a group acting on a set* X*. The* orbit *of an element* $x \in X$ *is defined as*

$$\mathbf{Orb}(x) = \{y \in X \mid \exists g \in G, y = g * x\}.$$

That is $\mathbf{Orb}(x) = G * x$*.*

At the same time, for given $x \in X$*, its* stabilizer *is defined as*

$$\mathbf{Stab}(x) = \{g \in G \mid g * x = x\}$$

which is subset of G*.*

Note that, for each $x \in X$, $\mathbf{Stab}(x)$ is a subgroup of G. It is called a *stabilizer subgroup* of G or *isotropy subgroup* of x.

Definition 2.23. [4] *Let X and Y be two G-sets, and $f : X \to Y$ be a function. It is called that f is a* morphism of G-sets *or* G-function, *if $f(g * x) = g * f(x)$ for all $g \in G$ and $x \in X$. If f is a bijective G-function then we call that f is an* isomorphism.

Definition 2.24. [4] *Let G be a group. For any $g \in G$, the element aga^{-1} is called* conjugate of g *with respect to a. The automorphism $f : G \to G$ such that $f(x) = axa^{-1}$ is called* conjugation.

Let G be a group. If we define the operation $\cdot : G \times G \to G$ such that $g \cdot x = gxg^{-1}$, then we have an action $\cdot$, and G acts on itself. We call that $\cdot$ is *conjugation action*.

3. Soft Structures in Groups

In this section, we will give some specific soft set definitions on a group using the group theoretic concepts mentioned in the previous section.

Definition 3.1. *Let G be a group. If we define the mapping $\mathbf{c} : G \to \mathcal{P}(G)$ such that $\mathbf{c}(x) = C(x)$ for all $x \in G$, then we called that $(\mathbf{c}, G)$ is a* centralizer soft set *over G.*

Definition 3.2. *Let G be a group. We define the mapping $\mathbf{n} : G \to \mathcal{P}(G)$ such that $\mathbf{n}(x) = N(x)$ for all $x \in G$, then we called that $(\mathbf{n}, G)$ is a* normalizer soft set *over G.*

Definition 3.3. *Let G be a group and S be a subgroup of G. We define the mapping $\mathbf{k}_L : G \to \mathcal{P}(G)$ such that $\mathbf{k}_L(x) = xS$ for all $x \in G$, then we called that $(\mathbf{k}_L, G)$ is a* left cosetial soft set *over G with respect to S.*

Dually, if $\mathbf{k}_R : G \to \mathcal{P}(G)$ such that $\mathbf{k}_R(x) = Sx$ for all $x \in G$, then $(\mathbf{k}_R, G)$ is a right cosetial soft set *over G with respect to S.*

Definition 3.4. *Let G be a group. We define the mapping $\mathbf{cy} : G \to \mathcal{P}(G)$ such that $\mathbf{cy}(x) = \langle x \rangle$ for all $x \in G$, then we called that $(\mathbf{cy}, G)$ is a* cyclicizer soft set *over G.*

Example 3.5. *Let $X = \mathbb{R} - \{0, 1\}$. Suppose that $G = \{f, g, h, i, j, k\}$ such that $f, g, h, i, j, k : X \to X$ and defined by $f(x) = \dfrac{1}{1-x}$, $g(x) = \dfrac{x-1}{x}$, $h(x) = \dfrac{1}{x}$, $i(x) = x$, $j(x) = 1 - x$, $k(x) = \dfrac{x}{x-1}$. G is a group with respect to composition of functions and its Cayley Table is as follows:*

From Definition 3.1 and Table 1, we have that $\mathbf{c}(f) = \{f, g, i\}$, $\mathbf{c}(g) = \{f, g, i\}$, $\mathbf{c}(h) = \{h, i\}$, $\mathbf{c}(i) = G$, $\mathbf{c}(j) = \{i, j\}$ and $\mathbf{c}(k) = \{i, k\}$. Then the centralizer soft set over G is

$$(\mathbf{c}, G) = \{f = \{f, g, i\}, g = \{f, g, i\}, h = \{h, i\}, i = G, j = \{i, j\}, k = \{i, k\}\}.$$

From Definition 3.4 and Table 1, we have that $\mathbf{cy}(f) = \{f, g, i\}$, $\mathbf{cy}(g) = \{f, g, i\}$, $\mathbf{cy}(h) = \{h, i\}$, $\mathbf{cy}(i) = \{i\}$, $\mathbf{cy}(j) = \{i, j\}$, $\mathbf{cy}(k) = \{i, k\}$. Then the cyclicizer soft set over G is

$$(\mathbf{cy}, G) = \{f = \{f, g, i\}, g = \{f, g, i\}, h = \{h, i\}, i = \{i\}, j = \{i, j\}, k = \{i, k\}\}.$$

$\circ$	f	g	h	i	j	k
f	g	i	k	f	h	j
g	i	f	j	g	k	h
h	j	k	i	h	f	g
i	f	g	h	i	j	k
j	k	h	g	j	i	f
k	h	j	f	k	g	i

TABLE 1. Cayley Table of G

From Definition 3.2 and Table 1, we have that

$$\mathbf{n}(f) = \{y \mid y^{-1} \circ f \circ y \in \langle f \rangle\} = \{y \mid y^{-1} \circ f \circ y \in \{f, g, i\}\} = G,$$

$\mathbf{n}(g) = G$, $\mathbf{n}(h) = \{h, i\}$, $\mathbf{n}(i) = G$, $\mathbf{n}(j) = \{i, j\}$, $\mathbf{n}(k) = \{i, k\}$. *Then the normalizer soft set over* G *is*

$$(\mathbf{n}, G) = \{f = G, g = G, h = \{h, i\}, i = G, j = \{i, j\}, k = \{i, k\}\}.$$

Let take the subgroup $S = \{i, j\}$ *of* G. *Then we have that* $\mathbf{k}_L(f) = \{f, h\}$, $\mathbf{k}_L(g) = \{g, k\}$, $\mathbf{k}_L(h) = \{f, h\}$, $\mathbf{k}_L(i) = \{i, j\}$, $\mathbf{k}_L(j) = \{i, j\}$, $\mathbf{k}_L(k) = \{g, k\}$. *Hence, the left cosetial soft set over* G *with respect to* $S = \{i, j\}$ *is*

$$(\mathbf{k}_L, G) = \{f = \{f, h\}, g = \{g, k\}, h = \{f, h\}, i = \{i, j\}, j = \{i, j\}, k = \{g, k\}\}.$$

We can calculate the right cosetial soft set $(\mathbf{k}_R, G)$ *over* G *with respect to* S *in the same way.*

Now, let's give some of the results we obtained from the definitions given above.

Theorem 3.6. *Let* G *be a group and* $(\mathbf{cy}, G)$ $(\mathbf{c}, G)$ *and* $(\mathbf{n}, G)$ *be cyclicizer, centralizer and normalizer soft sets over* G*, respectively. Then*

$$(\mathbf{cy}, G)\widetilde{\subset}(\mathbf{c}, G)\widetilde{\subset}(\mathbf{n}, G).$$

Proof. Since $\langle x \rangle \subset C(x) \subset N(x)$ for all $x \in G$, then $(\mathbf{cy}, G)\widetilde{\subset}(\mathbf{c}, G)\widetilde{\subset}(\mathbf{n}, G)$. □

Theorem 3.7. *Let* G *be a group. The centralizer soft set* $(\mathbf{c}, G)$*, the normalizer soft set* $(\mathbf{n}, G)$ *and the cyclicizer soft set* $(\mathbf{cy}, G)$ *over* G *are soft groups over* G.

Proof. Since $\mathbf{c}(x) = C(x)$, $\mathbf{n}(x) = N(x)$ and $\mathbf{cy}(x) = \langle x \rangle$ are subgroups of G, for all $x \in G$, then $(\mathbf{c}, G)$, $(\mathbf{n}, G)$ and $(\mathbf{cy}, G)$ are soft group over G from Definition 2.14. □

Theorem 3.8. G *is an abelian group if and only if* $(\mathbf{c}, G)$ *is the absolute soft set over* G.

Proof. From Definition 3.1, we have the centralizer soft set $(\mathbf{c}, G)$ such that $\mathbf{c}(x) = \{y \mid xy = yx\}$ where $\mathbf{c} : G \to \mathcal{P}(G)$ is a function. Since G is an abelian group, i.e. $xy = yx$ for all $x, y \in G$, then we obtain that $\mathbf{c}(x) = G$ for all $x \in G$. Hence $(\mathbf{c}, G) = \mathcal{G}$.

On the other hand, suppose that $(\mathbf{c}, G) = \mathcal{G}$, then $\mathbf{c}(x) = \{y \mid xy = yx\} = G$ for all $x \in G$. Obviously, $xy = yx$ for all $x, y \in G$, then G is an abelian group. □

Theorem 3.9. *If* G *is cyclic group then* $(\mathbf{c}, G)$ *is the absolute soft set over* G.

Proof. If G is cyclic then G is abelian. From Theorem 3.8, we have $(\mathbf{c}, G)$ is the absolute soft set. □

Theorem 3.10. *$(\mathbf{c}, G)$ and $(\mathbf{n}, G)$ are full soft sets over G.*

Proof. Let G be a group. For the identity element $e_G \in G$, we have obviously that $xe_G = e_G x$ for all $x \in G$, i.e. each element of G is commutative with the identity element. Hence, $\mathbf{c}(e_G) = G$ for $e_G \in G$. Consequently, for all $x \in G$, we obtain that $\bigcup_{x \in G} \mathbf{c}(x) = G$, i.e. $(\mathbf{c}, G)$ is full.

In addition, we have $(\mathbf{c}, G)\widetilde{\subset}(\mathbf{n}, G)$ from Theorem 3.6. So, $(\mathbf{n}, G)$ is full also. □

Theorem 3.11. *Let G be a group. If $(\mathbf{cy}, G)$ is an identity soft group, then G is the trivial group.*

Proof. For any $x \in G$, we have $\mathbf{cy}(x) = \langle x \rangle = \{e_G\}$. Then $x = e_G$. Thus G is trivial. □

Theorem 3.12. *Let G be a group. $(\mathbf{cy}, G)$ is the normal soft subgroup of $(\mathbf{n}, G)$.*

Proof. It is obvious. □

Theorem 3.13. *Let G be a group and S be a subgroup of G. Then $(\mathbf{k}_L, G)$ and $(\mathbf{k}_R, G)$ are partition-type soft sets and covering-type soft sets over G.*

Proof. For any $x \in G$, we have $x \in \mathbf{k}_L(x) = xS \neq \varnothing$. Then we obtain that $\bigcup_{x \in G} \mathbf{k}_L(x) = G$. We know that either $\mathbf{k}_L(x) = \mathbf{k}_L(y)$ or $\mathbf{k}_L(x) \cap \mathbf{k}_L(y) = \varnothing$ for each $x, y \in G$. So, if we take $\mathbf{k}_L(x) \neq \mathbf{k}_L(y)$ then we have $\mathbf{k}_L(x) \cap \mathbf{k}_L(y) = \varnothing$ for each $x, y \in G$. From Definition 2.10, we obtain that $(\mathbf{k}_L, G)$ is a partition-type soft set and so covering-type soft set over G.

The same arguments apply to $(\mathbf{k}_R, G)$. □

Obviously, we have following theorems for cosetial soft sets over a group.

Theorem 3.14. *Let G be a group and S be a subgroup of G. If S is normal, then $(\mathbf{k}_L, G) = (\mathbf{k}_R, G)$.*

Theorem 3.15. *If G is an abelian group, then $(\mathbf{k}_L, G) = (\mathbf{k}_R, G)$.*

In [2], Aktaş and Özlü defined the concept of cyclic soft group as follows:

Definition 3.16. [2] *Let (F, A) be a soft group over G and X an element of $\mathcal{P}(G)$. The set $\{(a, \langle x \rangle) \mid a \in A, x \in G\}$ is called a soft subset of (F, A) generated by the set X and denoted by $\langle X \rangle$. If $(F, A) = \langle X \rangle$, then the soft group (F, A) is called the* cyclic soft group *generated by X.*

From this definition, we can give following theorem.

Theorem 3.17. *The cyclicizer soft set over any group G is a cyclic soft group over G.*

Proof. From Theorem 3.7, $(\mathbf{cy}, G)$ is a soft group over G. Since $\mathbf{cy}(x) = \langle x \rangle$ for any $x \in G$, $(\mathbf{cy}, G)$ is generated by G, i.e. $(\mathbf{cy}, G) = \langle G \rangle$. Hence, we obtain that $(\mathbf{cy}, G)$ is a cyclic soft group over G from Definition 3.16. □

Theorem 3.18. *Let G and G' be groups. $(\mathbf{c}, G) \cong (\mathbf{c}', G')$ if and only if G is isomorphic to G'.*

Proof. Suppose that $(\mathbf{c}, G) \cong (\mathbf{c}', G')$. In this case, there exists a bijection $f : G \to G'$ such that $\mathbf{c}' \circ f = f^* \circ \mathbf{c}$. For any $x \in G$, we have

$$(\mathbf{c}' \circ f)(x) = \mathbf{c}'(f(x)) = \{y \mid yf(x) = f(x)y\}$$

and

$$(f^* \circ \mathbf{c})(x) = f^*(c(x)) = f^*(\{t \mid xt = tx\}) = \{f(t) \mid f(xt) = f(tx)\}.$$

Since f is a bijection and $\mathbf{c}' \circ f = f^* \circ \mathbf{c}$, we obtain that $f(t) = y$ and $yf(x) = f(tx)$. Hence, we have $f(xt) = f(x)f(t)$, i.e. f is a homomorphism. Thus f is an isomorphism from G to G'.

Other side of the proof of theorem is obvious. □

The definitions of the soft sets obtained using the group action concept are as follows.

Definition 3.19. *Let G be a group, X be a non-empty set and $* : G \times X \to X$ be a group action on X. It is called that $(\mathbf{Orb}, X)$ is an* orbital soft set *over X, such that $\mathbf{Orb} : X \to \mathcal{P}(X)$ is a set valued mapping where $\mathbf{Orb}(x)$ is an orbit of x in X.*

Besides,

Definition 3.20. *Let G be a group, X be a non-empty set and $* : G \times X \to X$ be a group action on X. It is called that $(\mathbf{Stab}, X)$ is a* stabilizer soft set *over G, such that $\mathbf{Stab} : X \to \mathcal{P}(G)$ is a set valued mapping where $\mathbf{Stab}(x)$ is a stabilizer of x in X.*

Example 3.21. *Let $G = \{1, -1\}$ be group and $X = \mathbb{R}$ be the set of real numbers. Define $* : \{1, -1\} \times \mathbb{R} \to \mathbb{R} : (x, y) \mapsto xy$ multiplication of real numbers. So, $*$ is a group action and $\mathbb{R}$ is a G-set. For arbitrary $x \in \mathbb{R}$, we have that $\mathbf{Orb}(x) = \{x, -x\}$. Hence,*

$$(\mathbf{Orb}, \mathbb{R}) = \{x = \{x, -x\} \mid x \in \mathbb{R}\}.$$

From Definition 2.22, since $\mathbf{Stab}(x)$ is a subgroup of G for each $x \in X$. Then we obtain following theorem, obviously.

Theorem 3.22. *Let G be a group and X be a G-set. Then $(\mathbf{Stab}, X)$ is a soft group over G.*

Theorem 3.23. *Let G be a group and X be a G-set. $(\mathbf{Orb}, X)$ is a partition-type soft set over X.*

Proof. From definition of orbit (Definition 2.22), we have an equivalence relation on X and defined by $x_1 \sim x_2 \Leftrightarrow gx_1 = x_2$ for some $g \in G$. Therefore, $\mathbf{Orb}(x)$ is an equivalence class for each $x \in X$, so the collection of orbits is a partition for X. Hence, $(\mathbf{Orb}, G)$ is a partition-type soft set over X from Definition 2.10. □

Theorem 3.24. *$(\mathbf{Orb}, X)$ is similar to $(\mathbf{Orb}, Y)$ if and only if X is isomorphic to Y.*

Proof. Suppose that X is isomorphic to Y. Then we have an isomorphism $f : X \to Y$ such that f is bijective and $f(g * x) = g * f(x)$ for all $g \in G$ and for all $x \in X$.

For any $x \in X$, we have

$$\begin{aligned}(f^* \circ \mathbf{Orb})(x) &= f^*[\{y \in X \mid \exists g \in G, y = g * x\}] \\ &= \{f(x) \in f[X] \mid \exists g \in G, f(y) = f(g * x)\} \\ &= \{f(x) \in f[X] \mid \exists g \in G, f(y) = g * f(x)\} \\ &= \{f(x) \in Y \mid \exists g \in G, f(y) = g * f(x)\} \\ &= \mathbf{Orb}(f(x)) \\ &= (\mathbf{Orb} \circ f)(x)\end{aligned}$$

Hence $(\mathbf{Orb}, X)$ is similar to $(\mathbf{Orb}, Y)$ from Definition 2.13.

On the other hand, suppose that $(\mathbf{Orb}, X)$ is similar to $(\mathbf{Orb}, Y)$. We say that $y = g * x$ for any $y \in X$. Then $y \in \mathbf{Orb}(x)$. In this way, we obtain that

$$f(y) \in f[\mathbf{Orb}(x)] = f^*(\mathbf{Orb}(x)) = \mathbf{Orb}(f(x)),$$

i.e. $f(g * x) \in \mathbf{Orb}(f(x))$. From here, we get $f(g * x) = g * f(x)$ as a result since f is a bijective function. Hence f is an isomorphism from X to Y. □

Theorem 3.25. *$(\mathbf{Stab}_X, X)$ is similar to $(\mathbf{Stab}_Y, Y)$ if and only if X is isomorphic to Y.*

Proof. Suppose that $(\mathbf{Stab}_X, X) \cong (\mathbf{Stab}_Y, Y)$. Then we have a bijection $f : X \to Y$ such that $\mathbf{Stab}_X = \mathbf{Stab}_Y \circ f$. Since f is a bijection, then we have that there is one and only one $y \in Y$ such that $f(x) = y$ for each $x \in X$. Let us take $g \in \mathbf{Stab}_Y(y) = \mathbf{Stab}_Y(f(x))$, so we have $g * y = y$ such that $f(x) = y$. Then we obtain that $g * f(x) = f(x)$. Since $\mathbf{Stab}_X = \mathbf{Stab}_Y \circ f$, we have $g \in \mathbf{Stab}_X(x)$, so $g * x = x$. Thus we obtain that $f(g * x) = f(x)$ for each $x \in X$, since f is a bijection. Hence, we have $f(g * x) = f(x) = g * f(x)$. Thus f is an isomorphism from X to Y.

If we take f is an isomorphism, then it is obvious that $\mathbf{Stab}_X = \mathbf{Stab}_Y \circ f$ for each $x \in X$. Thus $(\mathbf{Stab}_X, X) \cong (\mathbf{Stab}_Y, Y)$. □

Theorem 3.26. *Let G be a group and G acts on itself by left multiplication. Then $(\mathbf{Stab}, G)$ is an identity soft group.*

Proof. For any $x \in G$, we have $\mathbf{Stab}(x) = \{g \in G \mid gx = x\} = \{e_G\}$. From Definition 2.16, we get that $(\mathbf{Stab}, G)$ is an identity soft groups. □

Theorem 3.27. *Let G be a group and G acts on itself by conjugation. Then $(\mathbf{Stab}, G) = (\mathbf{c}, G)$.*

Proof. For any $x \in G$,

$$\mathbf{Stab}(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \mid gx = xg\} = \mathbf{c}(x).$$

Thus we have that $(\mathbf{Stab}, G) = (\mathbf{c}, G)$ from Definition 2.24 and Definition 3.1. □

4. Conclusion

As we have already stated, group theory is a major area in abstract algebra and very important tool in mathematics. Besides, the soft set theory, which has a wide application in many areas, is one of the most popular topics of recent times in mathematics. In this article, we derived some specific soft sets from a given group and discussed basic properties. We give some necessary and sufficient conditions for isomorphic groups using soft set theoretic concepts. In [12], Pei and Miao showed that every soft set over any given universal set is an information system. By using this way, soft sets derived from groups also naturally become an information system. Therefore, as a future study, the effect of groups on information systems can be investigated. Of course, it also can be investigated in future that effect of soft sets in group theory.

The author hopes that this article sheds light on a way of working scientists in this field.

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