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EXISTENCE RESULTS FOR AN IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH A NON-COMPACT SEMIGROUP

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Abstract. In this paper we study a fractional differential equations problem with not instantaneous impulses involving a non-compact semigroup. We present some concepts and facts about the strongly continuous semigroup and the measure of noncompactness. After that we give an existence theorem of our problem using a condensing operator and the measure of noncompactness.

1. Introduction

The concept of fractional differential equations has become more popular among mathematicians, and is studied extensively in the recent years for its many applications, for more details about fractional differential equations we refer the readers to [13, 14, 9, 16, 17, 22, 11].

Furthermore, impulsive differential equations have known rapid growth because they play a main role in describing modern problems in fields such as physics, biology, economics and population dynamics; for more details the reader can see [15, 1, 7].

A lot of models about fractional impulsive differential equations were studied recently, for more details we give the references [19, 12, 21] and the references therein.

In [3] P. Chen, X. Zhang and Y. Li studied the existence of mild solutions for the initial value problem

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)), \ t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., m, \\ x(t) = g_i(t, x(t)), \ t \in (t_i, s_i], i = 1, 2, ..., m, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Where $A : D(A) \subset E \longrightarrow E$ is a closed linear operator, $-A$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ in E, here the semigroup $(T(t))_{t\geq 0}$ is non-compact.

Motivated by this work, in this article we study the impulsive fractional evolution equation

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$$
(1.2) \qquad \begin{cases} {}^cD^{\alpha}x(t) = Ax(t) + f(t, x(t), Bx(t)) + C(t)u(t), \ t \in (s_i, t_{i+1}], \\ \qquad\qquad\qquad\qquad\qquad i = 0, 1, 2, ..., m, \ u \in U_{ad}, \\ {}^cD^{\beta}x(t) = g_i(t, x(t)), \ t \in (t_i, s_i], i = 1, 2, ..., m, \\ x(0) = x_0, \end{cases}
$$

involving ${}^cD^{\alpha}$ and ${}^cD^{\beta}$ which are the Caputo fractional derivatives of order $\alpha \in (0,1)$ and $\beta \in (0,1)$ respectively with the lower limit zero, $A : D(A) \subset X \longrightarrow X$ is the generator of a non-compact C_0-semigroup of bounded operators $(T(t))_{t \geq 0}$ on a Banach space $(X, \| \ . \ \|)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C(J \times X, X)$, and $Bx(t) = \int_0^t B(t,s)x(s)ds$, $B \in C(D_1, \mathbb{R}^+)$, with $D_1 = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq \omega\}$, $K_0 = \max_{(t,s) \in D_1} B(t,s)$ and U_{ad} is a set that will be defined later.

The rest of the paper is organized as follows. In section 2 we present the notations, definitions and preliminary results needed in the following sections. In section 3, a suitable concept of PC-mild solutions for our problems is introduced. Section 4 is concerned with the existence results of our problems.

2. Preliminaries and notations

Let us set $J = [0,T], J_0 = [0,t_1], J_1 = (t_1,t_2], ..., J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, t_{m+1}]$ and introduce the space $PC(J,X) := \{x : J \to X : x \in C((t_i, t_{i+1}], X), i = 0, 1, ..., m \textit{ and there exist } x(t_i^-) \textit{ and } x(t_i^+), \ i = 1, ..., m \textit{ with } x(t_i^-) = x(t_i^+)\}$. It is clear that $PC(J,X)$ is a Banach space with the norm $\|u\|_{PC} = sup\{\|u(t)\| : t \in J\}$. Let Y be a separable reflexive Banach space where controls u takes values, and $P_f(Y)$ is a class of nonempty closed and convex subsets of Y. We suppose that the multivalued map $w : [0,a] \longrightarrow P_f(Y)$ is measurable, $w(.) \subset E$, where E is bounded set of Y, and the admissible control set $U_{ad} = \Big\{u \in L^p(E) : u(t) \in w(t), a.e\Big\}$, $p > \frac{1}{\tau}$, ($\tau \in (0, \alpha)$), for more details about admissible control set, we refer the readers to [6].

Let us recall the following well-known definitions.

Definition 2.1. [17] The Riemann-Liouville fractional integral of order q with lower limit zero for a function f is defined as

$$
I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \ q > 0,
$$

provided the integral exists, where $\Gamma(.)$ is the gamma function.

Definition 2.2. [17] The Riemann-Liouville derivative of order q with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$
{}^LD^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-q-1} f(s)ds, \ n-1 < q < n, t > 0.
$$

Definition 2.3. [17] The Caputo derivative of order q for a function $f : [0, \infty) \longrightarrow \mathbb{R}$ can be written as

$$^{c}D^{q}f(t) = {}^{L}D^{q}\left(f(t) - \sum_{k=0}^{k}\frac{t^k}{k!}f^k(0)\right),\ n-1<q<n,\ t>0.$$

Definition 2.4. [21] A function $x \in C(J,X)$ is said to be a mild solution of the following problem:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = Ax(t) + y(t),\ t \in (0,T], \\ x(0) = x_0, \end{cases}$$

if it satisfies the integral equation
$x(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)y(s)ds,$

Here
$P_\alpha(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta$, $Q_\alpha(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta$,

$\xi_\alpha(\theta) = \dfrac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\overline{\omega}_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$
$\overline{\omega_\alpha}(\theta) = \dfrac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\dfrac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha), \theta \in (0,\infty)$, and $\xi_\alpha(\theta) \geq 0$, $\theta \in (0,\infty)$, $\int_0^\infty \xi_\alpha(\theta)d\theta = 1.$

It is easy to verify that $\int_0^\infty \theta\xi_\alpha(\theta)d\theta = \dfrac{1}{\Gamma(1+\alpha)}$.

Theorem 2.5. *(Darbo-Sadovskii* [8]*) if $D \subset X$ is bounded, closed and convex, the continuous map $S : D \to D$ is β- condensing, then S has a fixed point in D.*

Lemma 2.6. [4, 5, 20] *The operators P_α and Q_α have the following properties:*

(1) For any fixed $t \geq 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are linear and bounded operators, and for any $x \in X$,
$$\| P_\alpha(t)x \| \leq M_A \| x \|,\ \| Q_\alpha(t)x \| \leq \frac{\alpha M_A}{\Gamma(1+\alpha)} \| x \|,$$
(2) $\{P_\alpha(t), t \geq 0\}$ and $\{Q_\alpha(t), t \geq 0\}$ are strongly continuous,
(3) if $(T(t))_{t\geq 0}$ is an equicontinuous semigroup, then $P_\alpha(t)$ and $Q_\alpha(t)$ are continuous in $(0,\infty)$ by the norm, which means that for $0 < t' < t'' < T$ we have:
$$\| P_\alpha(t'') - P_\alpha(t') \| \longrightarrow 0 \text{ and } \| Q_\alpha(t'') - Q_\alpha(t') \| \longrightarrow 0 \text{ as } t'' \longrightarrow t'.$$

Definition 2.7. [8] Let X be a Banach space and Ω_X be the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\beta : \Omega_X \longrightarrow [0,\infty)$ defined by :
$\beta(B) = \inf\{\epsilon > 0 : B = \cup_{i=1}^{n}B_i \text{ and } diam(B_i) \leq \epsilon \text{ for } i = 1,2,..,n\}$
with $diamB_i = \sup\{|x-y| : x,y \in B_i\}$ and $B \in \Omega_X$

Remark 2.8. It is clear that $\beta(B) \leq diam(B)$.

Next, we are going to look back on some properties of the measure of noncompactness that will be used in the proof of our main results.

Lemma 2.9. [8] *Let A and B be bounded sets of X and λ be a real number. Then the measure of noncompactness has the following properties:*

1) $\beta(A) = 0$ if and only if A is a relatively compact set,
2) $A \subset B$ implies that $\beta(A) \leq \beta(B)$,

3) $\beta(\overline{A}) = \beta(A)$,
4) $\beta(A \cup B) = max\,\{\beta(A), \beta(B)\}$,
5) $\beta(\lambda A) = |\lambda|\beta(A)$,
6) $\beta(A + B) \leq \beta(A) + \beta(B)$,
7) $\beta(\overline{co}A) = \beta(A)$.

Where $\overline{co}$ *means the closure of the convex hull.*

Lemma 2.10. [2] *Let* X *be a Banach space,* $W \subset X$ *be bounded. Then there exists a countable set* $W_1 \subset W$ *such that*

$$\beta(W) \leq 2\beta(W_1).$$

Lemma 2.11. [8] *Let* X *be a Banach space,* $W \subset C(J, X)$ *be bounded and equicontinuous. Then* $\beta(W(t))$ *is continuous on* J *and*

$$\beta(W) = \max_{t\in J} \beta(W(t)) = \beta(W(J)).$$

Lemma 2.12. [10] *Let* X *be a Banach space,* $W = \{u_n\} \in C(J, X)$ *be bounded and countable set. Then* $\beta(W(t))$ *is a Lebesgue integral on* J *and*

$$\beta\left(\left\{\int_J u_n(t)dt : n \in \mathbb{N}\right\}\right) \leq 2\int_J \beta(W(t))dt.$$

3. The construction of mild solutions

Let $x \in PC(J, X)$. We first consider the following fractional impulsive problem:

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t), Bx(t)) + C(t)u(t),\ t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., m, u \in U_{ad}, \\ {}^cD^\beta x(t) = g_i(t, x(t)),\ t \in (t_i, s_i], i = 1, 2, ..., m, \\ x(0) = x_0. \end{cases}$$

From the property of the Caputo derivative, a general solution can be written as

$$x(t) = \begin{cases} x_0 + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[Ax(s) + f(s, x(s), Bx(s)) + C(s)u(s)\right] ds,\ t \in (0, t_1], \\ d_{1x} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_1(s, x(s))ds,\ t \in (t_1, s_1], \\ K_{1x} + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[Ax(s) + f(s, x(s), Bx(s)) + C(s)u(s)\right] ds,\ t \in (s_1, t_2], \\ . \\ . \\ . \\ d_{ix} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_i(s, x(s))ds,\ t \in (t_i, s_i], \\ K_{ix} + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[Ax(s) + f(s, x(s), Bx(s)) + C(s)u(s)\right] ds,\ t \in (s_i, t_{i+1}], \end{cases}$$

where d_{ix} and K_{ix}, $i = 1, 2, ..., m$, are elements of X.
We obtain

$$x(t)=\begin{cases} d_{ix}+\dfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g_i(s,x(s))ds,\ t\in(t_i,s_i],1\leq i\leq m,\\ P_\alpha(t-s_i)K_{ix}+\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds,\\ \qquad\qquad t\in(s_i,t_{i+1}],0\leq i\leq m,\\ K_{0x}=x_0.\end{cases}$$

And using the fact that x is continuous at the points t_i, we get :

$$\begin{aligned} x(t_i) &= P_\alpha(t_i-s_{i-1})K_{(i-1)x}+\int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds\\ &= d_{ix}+\frac{1}{\Gamma(\beta)}\int_0^{t_i}(t_i-s)^{\beta-1}g_i(s,x(s))ds.\end{aligned}$$

Which implies that:

$$\begin{aligned} d_{ix} &= P_\alpha(t_i-s_{i-1})K_{(i-1)x}+\int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds\\ &\quad -\frac{1}{\Gamma(\beta)}\int_0^{t_i}(t_i-s)^{\beta-1}g_i(s,x(s))ds.\end{aligned}$$

Using the fact that x is continuous at the points s_i, we get :

$$\begin{aligned} x(s_i) &= d_{ix}+\frac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i-s)^{\beta-1}g_i(s,x(s))ds\\ &= K_{ix}+\int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds.\end{aligned}$$

Which implies that:

$$\begin{aligned} K_{ix} &= d_{ix}+\frac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i-s)^{\beta-1}g_i(s,x(s))ds\\ &\quad -\int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds.\end{aligned}$$

Therefore, a mild solution of problem (1.2) is given by

$$x(t)=\begin{cases} P_\alpha(t)x_0+\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds,\ t\in(0,t_1],\\ d_1+\dfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g_1(s,x(s))ds,\ t\in(t_1,s_1],\\ P_\alpha(t-s_1)K_1+\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds,\ t\in(s_1,t_2],\\ .\\ .\\ .\\ d_{ix}+\dfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g_i(s,x(s))ds,\ t\in(t_i,s_i],1\leq i\leq m,\\ P_\alpha(t-s_i)K_{ix}+\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds,\\ \qquad\qquad t\in(s_i,t_{i+1}],1\leq i\leq m,\end{cases}$$

where

$K_{0x}=x_0,$

$$\begin{aligned} d_{ix} &= P_\alpha(t_i-s_{i-1})K_{(i-1)x}+\int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds \\ &\quad -\frac{1}{\Gamma(\beta)}\int_0^{t_i}(t_i-s)^{\beta-1}g_i(s,x(s))ds, \\ K_{ix} &= d_{ix}-\int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds \\ &\quad +\frac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i-s)^{\beta-1}g_i(s,x(s))ds. \end{aligned}$$

Definition 3.1. A function $x\in PC(J,X)$ is said to be a mild solution of problem (1.2) if it satisfies the following relation:

$$x(t)=\begin{cases} P_\alpha(t)K_{0x}+\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds, \\ \qquad\qquad t\in(0,t_1],u\in U_{ad}, \\ d_{ix}+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g_i(s,x(s))ds,\ t\in(t_i,s_i],1\le i\le m, \\ P_\alpha(t-s_i)K_{ix}+\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds, \\ \qquad\qquad t\in(s_i,t_{i+1}],1\le i\le m. \end{cases}$$

Where

$K_{0x}=x_0,$

$$\begin{aligned} d_{ix} &= P_\alpha(t_i-s_{i-1})K_{(i-1)x}+\int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds \\ &\quad -\frac{1}{\Gamma(\beta)}\int_0^{t_i}(t_i-s)^{\beta-1}g_i(s,x(s))ds, \\ K_{ix} &= d_{ix}-\int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)\left[f(s,x(s),Bx(s))+C(s)u(s)\right]ds \\ &\quad +\frac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i-s)^{\beta-1}g_i(s,x(s))ds. \end{aligned}$$

4. Existence results

This section deals with the existence results of the problem (1.2).

To prove our first existence result we introduce the following assumptions.

(H_1) A generates an equicontinuous and uniformly bounded strongly continuous semigroup $T(t)_{t\ge0}$ on a Banach space X such that $\| T(t)\|\le M_A$ for all $t\in J$,

(H_2) $C:[0,T]\longrightarrow L(Y,X)$ is essentially bounded, ie $C\in L^\infty([0,T],L(Y,X))$,

(H_3) The function $f\in C(J\times X\times X,X)$,
(H_4)there exists $\tau\in(0,\alpha)$ and a positive function $m\in L^{\frac{1}{\tau}}(J,\mathbb{R})$ such that

$\| f(t,u,v)\|\le m(t)$, for $u,v\in X$ and $t\in J$,

(H_5) For $i=1,2,...,m$, the function $g_i\in C([t_i,s_i]\times X;X)$ and there exists constants $K_i>0$ such that:

$\| g_i(t,u)-g_i(t,v)\|\le K_i\| u-v\|$ for all $u,v\in X$, and $K=\max\limits_{1\le i\le m}\{K_i\}$

(H_5') For $i = 1, 2, ..., m$, $g_i \in C(J \times X; X)$ is completely continuous, and there exists constants $b, d > 0$ such that:

$\| g_i(t, u) \| \leq b_i \| u \| + d_i$, for all $u \in B_r$. B_r is a set that will be defined later, and $b = \max\limits_{1 \leq i \leq m} \{b_i\}$, $d = \max\limits_{1 \leq i \leq m} \{d_i\}$,

(H_6) There exists constants $L_1, L_2 > 0$ such that:

$\beta(f(t, D_1, D_2) \leq L_1 \beta(D_1) + L_2 \beta(D_2)$ for all $t \in J$ and D_1, D_2 bounded and countable sets in X.

Theorem 4.1. *Assume that $(H_1) - (H_6)$ hold. In addition, let's suppose that the following property is verified:*

$$max\{A, B\} < 1,$$

with

$$\begin{aligned} A &= \left[\frac{(s_m^\beta + t_m^\beta) + M_A(s_{m-1}^\beta + t_{m-1}^\beta) + ... + M_A^{m-1}(s_1^\beta + t_1^\beta)}{\Gamma(\beta + 1)} \right] K \\ &\quad + 4M_A(L_1 + \omega L_2 K_0) \left[\frac{t_m^\alpha + M_A(t_{m-1}^\alpha + s_{m-1}^\alpha) + ... + M_A^{m-1}(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha + 1)} \right] \end{aligned}$$

and

$$\begin{aligned} B &= \left[\frac{M_A(s_m^\beta + t_m^\beta) + M_A^2(s_{m-1}^\beta + t_{m-1}^\beta) + ... + M_A^m(s_1^\beta + t_1^\beta)}{\Gamma(\beta + 1)} \right] K \\ &\quad + 4M_A(L_1 + \omega L_2 K_0) \left[\frac{t_{m+1}^\alpha + M_A(t_m^\alpha + s_m^\alpha) + ... + M_A^m(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha + 1)} \right] \end{aligned}$$

Then the problem (1.2) has at least one mild solution.

Proof. We introduce the composition $Q = Q_1 + Q_2$ where :

$$Q_1 x(t) = \begin{cases} P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) C(s) u(s) ds, \ t \in [0, t_1], \\ d_{i1x} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_i(s, x(s)) ds, \ t \in (t_i, s_i], i = 1, 2, .., m, \\ P_\alpha(t - s_i) K_{i1x} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) C(s) u(s) ds, \\ \qquad\qquad\qquad\qquad t \in (s_i, t_{i+1}], i = 1, 2, .., m, \end{cases}$$

$$Q_2 x(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x(s), Bx(s)) ds, \ t \in [0, t_1], \\ d_{i2x}, \ t \in (t_i, s_i], i = 1, 2, .., m, \\ P_\alpha(t - s_i) K_{i2x} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x(s), Bx(s)) ds, \\ \qquad\qquad\qquad\qquad t \in (s_i, t_{i+1}], i = 1, 2, .., m, \end{cases}$$

with

$$\begin{cases} d_{i1x} = P_\alpha(t_i - s_{i-1})K_{(i-1)1x} + \int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)C(s)u(s)ds \\ \qquad -\dfrac{1}{\Gamma(\beta)}\int_0^{t_i}(t_i-s)^{\beta-1}g_i(s,x(s))ds, \qquad i=1,2,..,m, \\ K_{i1x} = d_{i1x} + \dfrac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i-s)^{\beta-1}g_i(s,x(s))ds \\ \qquad -\int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)C(s)u(s)ds, i=1,2,..,m, \\ K_{01x} = x_0, \end{cases}$$

and

$$\begin{cases} d_{i2x} = P_\alpha(t_i - s_{i-1})K_{(i-1)2x} + \int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)f(s,x(s),Bx(s))ds, i=1,2,..,m, \\ K_{i2x} = d_{i2x} - \int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)f(s,x(x),Bx(s))ds, i=1,2,..,m, \\ K_{02x} = 0. \end{cases}$$

Our proof will be divided into several steps.

Step1: We show that $QB_r(J) \subset B_r(J)$

where $B_r = \{x \in PC(J,X); \| x \| \leq r\}$ the ball with radius $r > 0$;

$$K_{\alpha,\tau} = \left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} \| Cu \|_{L^{1/\tau}} \text{ and } S_{\alpha,\tau} = \left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} \| m \|_{L^{1/\tau}},$$

$$\gamma_1 = M_A^{m+1} \| x_0 \| \frac{\Gamma(\beta+1)}{\Gamma(\beta+1) - M_A K(t_1^\beta + s_1^\beta)},$$

$$\gamma_2 = \frac{(M_A t_{m+1}^{\alpha-\tau} + M_A^2(t_m^{\alpha-\tau} + s_m^{\alpha-\tau}) + ... + M_A^{m+1}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau}))\Gamma(\beta+1)}{(\Gamma(\beta+1) - M_A K(t_1^\beta + s_1^\beta))\Gamma(\alpha)}(K_{\alpha,\tau} + S_{\alpha,\tau}),$$

$$\gamma_3 = \frac{M_A^2(t_{m-1}^\beta + s_{m-1}^\beta) + ... + M_A^m(t_1^\beta + s_1^\beta)}{(\Gamma(\beta+1) - M_A K(t_1^\beta + s_1^\beta))} \| g_m(t,0) \|,$$

here $\gamma_1 + \gamma_1 + \gamma_1 \leq r$.

For any $x \in B_r$, we have:

Case1.For $t \in [0, t_1]$

$$\begin{aligned} \| Qx(t) \| &\leq \| P_\alpha(t)K_{0x} \| + \| \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Bx(s)) + C(s)u(s)\right]ds \| \\ &\leq M_A \| x_0 \| + \frac{M_A t_1^{\alpha-\tau}}{\Gamma(\alpha)}\left(K_{\alpha,\tau} + S_{\alpha,\tau}\right). \end{aligned}$$

Case 2.For $t \in (t_i, s_i], i = 1, 2, ..., m$.

For $t \in (t_1, s_1]$

$$\begin{aligned}
\| Qx(t) \| &\leq \| d_{1x} \| + \| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_1(s,x(s))ds \| \\
&\leq \| P_\alpha(t_1) K_{0x} \\
&\quad + \int_0^{t_1} (t_1-s)^{\alpha-1} Q_\alpha(t_1-s) \left[f(s,x(s),Bx(s)) + C(s)u(s) \right] ds \| \\
&\quad + \frac{1}{\Gamma(\beta)} \| \int_0^{t_1} (t_1-s)^{\beta-1} g_1(s,x(s))ds \| + \| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_1(s,x(s)ds \| \\
&\leq M_A \| x_0 \| + \frac{M_A t_1^{\alpha-\tau}}{\Gamma(\alpha)} (K_{\alpha,\tau} + S_{\alpha,\tau}) + \frac{t_1^\beta + s_1^\beta}{\Gamma(\beta+1)} (K \| x \| + \| g_1(t,0) \|) \\
&\leq r.
\end{aligned}$$

For $t \in (s_1, t_2]$

$$\begin{aligned}
\| Qx(t) \| &\leq \| P_\alpha(t-s_1) K_{1x} \| + \| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) \left[f(s,x(s),Bx(s)) + C(s)u(s) \right] ds \| \\
&\leq \| P_\alpha(t-s_1) d_{1x} \| \\
&\quad + \| P_\alpha(t-s_1) (\int_0^{s_1} (s_1-s)^{\alpha-1} Q_\alpha(t-s) \left[f(s,x(s),Bx(s)) + C(s)u(s) \right] ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{s_1} (s_1-s)^{\beta-1} g_1(s,x(s))ds) \| \\
&\quad + \| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) \left[f(s,x(s),Bx(s)) + C(s)u(s) \right] ds \| \\
&\leq M_A^2 \| x_0 \| \\
&\quad + \frac{M_A^2 (t_1^{\alpha-\tau} + s_1^{\alpha-\tau}) + M_A t_2^{\alpha-\tau}}{\Gamma(\alpha)} (K_{\alpha,\tau} + S_{\alpha,\tau}) \\
&\quad + \frac{M_A (t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)} (K \| x \| + \| g_1(t,0) \|) \\
&\leq r.
\end{aligned}$$

We suppose that: for $1 \leq j \leq i$.

For $t \in (t_j, s_j]$

$$\begin{aligned}
\| Qx(t) \| &\leq M_A^j \| x_0 \| + \frac{M_A t_j^{\alpha-\tau} + M_A^2 (t_{j-1}^{\alpha-\tau} + s_{j-1}^{\alpha-\tau}) + ... + M_A^j (t_1^{\alpha-\tau} + s_1^{\alpha-\tau})}{\Gamma(\alpha)} (K_{\alpha,\tau} + S_{\alpha,\tau}) \\
&\quad + \frac{(t_j^\beta + s_j^\beta) + M_A (t_{j-1}^\beta + s_{j-1}^\beta) + ... + M_A^{j-1} (t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)} (K \| x \| + \| g_j(t,0) \|) \\
&\leq r.
\end{aligned}$$

For $t \in (s_j, t_{j+1}]$

$$\begin{aligned}
\| Qx(t) \| &\leq M_A^{j+1} \| x_0 \| + \frac{M_A t_{j+1}^{\alpha-\tau} + M_A^2 (t_j^{\alpha-\tau} + s_j^{\alpha-\tau}) + ... + M_A^{j+1} (t_1^{\alpha-\tau} + s_1^{\alpha-\tau})}{\Gamma(\alpha)} (K_{\alpha,\tau} + S_{\alpha,\tau}) \\
&\quad + \frac{M_A (t_j^\beta + s_j^\beta) + M_A^2 (t_{j-1}^\beta + s_{j-1}^\beta) + ... + M_A^j (t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)} (K \| x \| + \| g_j(t,0) \|) \\
&\leq r.
\end{aligned}$$

And we prove the relations for $j = i + 1$.

For $t \in (t_{i+1}, s_{i+1}]$

$$\begin{aligned}
\| Qx(t) \| &\leq \| d_{(i+1)x} \| + \| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_{i+1}(s, x(s) ds \| \\
&\leq \| P_\alpha(t_{i+1}-s_i)) K_{ix} + \int_0^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} Q_\alpha(t_{i+1}-s) \left[f(s, x(s), Bx(s)) + C(s)u(s) \right] ds \| \\
&\quad + \| \frac{1}{\Gamma(\beta)} \int_0^{t_{i+1}} (t_{i+1}-s)^{\beta-1} g_{i+1}(s, x(s)) ds \\
&\quad + \| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_{i+1}(s, x(s) ds \| \\
&\leq M_A^{i+1} \| x_0 \| + \frac{M_A t_{i+1}^{\alpha-\tau} + M_A^2(t_i^{\alpha-\tau} + s_i^{\alpha-\tau}) + \ldots + M_A^{i+1}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau})}{\Gamma(\alpha)} (K_{\alpha,\tau} + S_{\alpha,\tau}) \\
&\quad + \frac{(t_{i+1}^\beta + s_{i+1}^\beta) + M_A(t_i^\beta + s_i^\beta) + \ldots + M_A^i(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)} (K \| x \| + \| g_{i+1}(t, 0) \|) \\
&\leq r.
\end{aligned}$$

For $t \in (s_{i+1}, t_{i+2}]$

$$\begin{aligned}
\| Qx(t) \| &\leq M_A^{i+2} \| x_0 \| + \frac{M_A t_{i+2}^{\alpha-\tau} + M_A^2(t_{i+1}^{\alpha-\tau} + s_{i+1}^{\alpha-\tau}) + \ldots + M_A^{i+2}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau})}{\Gamma(\alpha)} (K_{\alpha,\tau} + S_{\alpha,\tau}) \\
&+ \frac{M_A(t_{i+1}^\beta + s_{i+1}^\beta) + M_A^2(t_i^\beta + s_i^\beta) + \ldots + M_A^{i+1}(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)} (K \| x \| + \| g_{i+1}(t, 0) \|) \\
&\leq r.
\end{aligned}$$

We proved that $QB_r(J) \subset B_r(J)$.

Step2: Q_1 is lipschitz. Let $x, y \in PC(J, X)$,

Case 1. For $t \in [0, t_1]$, we have:

$\| Q_1x(t) - Q_1y(t) \| = 0.$

Similar to the proof on Step1, we prove that:

Case 2. For $t \in [t_i, s_i], 1 \leq i \leq m$,

$$\| Q_1x(t) - Q_1y(t) \| \leq \left[\frac{(s_i^\beta + t_i^\beta) + M_A(s_{i-1}^\beta + t_{i-1}^\beta) + \ldots + M_A^{i-1}(s_1^\beta + t_1^\beta)}{\Gamma(\beta+1)} \right] K \| x - y \|_{PC} .$$

For $t \in [s_i, t_{i+1}], 1 \leq i \leq m$,

$$\| Q_1x(t) - Q_1y(t) \| \leq \left[\frac{M_A(s_i^\beta + t_i^\beta) + M_A^2(s_{i-1}^\beta + t_{i-1}^\beta) + \ldots + M_A^i(s_1^\beta + t_1^\beta)}{\Gamma(\beta+1)} \right] K \| x - y \|_{PC} .$$

This implies that Q_1 is Lipschitz.

Step3: Q_2 is continuous.

Let $(x_n)_{n\geq 0}$ be a sequence such that $\lim\limits_{x\to\infty} \parallel x_n - x \parallel_{PC}= 0$, we have :

Case 1. For $t\in[0,t_1]$

$$\begin{aligned} \parallel Q_2x_n(t)-Q_2x(t)\parallel &\leq \parallel \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\left(f(s,x_n(s),Bx_n(s))-f(s,x(s),Bx(s))\right)ds \parallel \\ &\leq \frac{M_A t_1^\alpha}{\Gamma(\alpha+1)} \parallel f(.,x_n(.),Bx_n(.))-f(.,x(.),Bx(.)) \parallel_{PC} . \end{aligned}$$

Similar to the proof we did in Step1, we prove that:

Case 2. For $t\in(t_i,s_i], i=1,2,...,m,$

$\parallel Q_2x_n(t)-Q_2x(t)\parallel \leq \left[\dfrac{M_At_i^\alpha + M_A^2(t_{i-1}^\alpha+s_{i-1}^\alpha)+...+M_A^i(t_1^\alpha+s_1^\alpha)}{\Gamma(\alpha+1)}\right] \parallel f(.,x_n(.),Bx_n(.))- f(.,x(.),Bx(.)) \parallel_{PC}$.

Case 3. For $t\in(s_i,t_{i+1}), i=1,2,...,m,$

$\parallel Q_2x_n(t)-Q_2x(t)\parallel \leq \left[\dfrac{M_At_{i+1}^\alpha + M_A^2(t_i^\alpha+s_i^\alpha)+...+M_A^{i+1}(t_1^\alpha+s_1^\alpha)}{\Gamma(\alpha+1)}\right] \parallel f(.,x_n(.),Bx_n(.))- f(.,x(.),Bx(.)) \parallel_{PC}$.

Step4: Q_2 is equicontinuous, which means $\parallel Q_2x(t_2)-Q_2x(t_1)\parallel\to 0$ as $t_2\to t_1$.

For $0\leq t'<t''\leq t_1$, we have:

$$\begin{aligned} \parallel Q_2x(t'')-Q_2x(t')\parallel &\leq \parallel \int_0^{t''}(t''-s)^{\alpha-1}Q_\alpha(t''-s)f(s,x(s),Bx(s))ds \\ &\quad -\int_0^{t'}(t'-s)^{\alpha-1}Q_\alpha(t''-s)f(s,x(s),Bx(s))ds \parallel \\ &\leq I_1+I_2+I_3, \end{aligned}$$

where $I_1 = \parallel \int_{t'}^{t''}(t''-s)^{\alpha-1}Q_\alpha(t''-s)f(s,x(s),Bx(s))ds \parallel$,

$I_2 = \parallel \int_0^{t'}(t'-s)^{\alpha-1}\left[Q_\alpha(t''-s)-Q_\alpha(t'-s)\right]f(s,x(s),Bx(s))ds \parallel$,

$I_3 = \parallel \int_0^{t'}\left[(t''-s)^{\alpha-1}-(t'-s)^{\alpha-1}\right]Q_\alpha(t''-s)f(s,x(s),Bx(s))ds \parallel$,

$$\begin{aligned} I_1 &\leq \frac{\alpha M_A}{\Gamma(\alpha+1)}\int_{t'}^{t''}\parallel (t''-s)^{\alpha-1}f(s,x(s),Bx(s))\parallel ds \\ &\leq \frac{M_AS_{\alpha,\tau}}{\Gamma(\alpha)}(t''-t')^{\alpha-\tau}\longrightarrow 0 \text{ as } t''-t'\longrightarrow 0, \end{aligned}$$

$I_1\longrightarrow 0$ as $t''-t'\longrightarrow 0$.

For $t'=0$, $0<t''<t_1$, it is easy to see that $I_2=0$.

For $t'>0$ and $\epsilon>0$ small enough, we have

$$\begin{aligned}
I_2 &\leq \parallel \int_0^{t'-\epsilon}(t'-s)^{\alpha-1}\left[Q_\alpha(t''-s)-Q_\alpha(t'-s)\right]f(s,x(s),Bx(s))ds \parallel \\
&\quad + \parallel \int_{t'-\epsilon}^{t'}(t'-s)^{\alpha-1}\left[Q_\alpha(t''-s)-Q_\alpha(t'-s)\right]f(s,x(s),Bx(s))ds \parallel \\
&\leq \sup_{s\in[0,t'-\epsilon]} \parallel Q_\alpha(t''-s)-Q_\alpha(t'-s) \parallel \int_0^{t'-\epsilon} \parallel (t'-s)^{\alpha-1}f(s,x(s),Bx(s)) \parallel ds \\
&\quad + \frac{2M_A}{\Gamma(\alpha)}\int_{t'-\epsilon}^{t'} \parallel (t'-s)^{\alpha-1}f(s,x(s),Bx(s)) \parallel ds \\
&\leq S_{\alpha,\tau}\left(t'^{\frac{\alpha-\tau}{1-\tau}}-\epsilon^{\frac{\alpha-\tau}{1-\tau}}\right)^{1-\tau}\sup_{s\in[0,t'-\epsilon]} \parallel Q_\alpha(t''-s)-Q_\alpha(t'-s) \parallel + \frac{2M_A}{\Gamma(\alpha)}S_{\alpha,\tau}\epsilon^{\alpha-\tau}
\end{aligned}$$

$I_2 \longrightarrow 0$ as $t''-t' \longrightarrow 0$ and $\epsilon \longrightarrow 0$.

$$\begin{aligned}
I_3 &\leq \int_0^{t'} \parallel \left[(t''-s)^{\alpha-1}-(t'-s)^{\alpha-1}\right]Q_\alpha(t''-s)f(s,x(s),Bx(s)) \parallel ds \\
&\leq \frac{M_A S_{\alpha,\tau}}{\Gamma(\alpha)}\left((t''-t')^{\frac{\alpha-\tau}{1-\tau}}+t'^{\frac{\alpha-\tau}{1-\tau}}+t''^{\frac{\alpha-\tau}{1-\tau}}\right)^{1-\tau} \\
&\leq \frac{M_A S_{\alpha,\tau}}{\Gamma(\alpha)}(t''-t')^{\alpha-\tau} \longrightarrow 0; t''-t' \longrightarrow 0
\end{aligned}$$

$I_3 \longrightarrow 0$ as $t''-t' \longrightarrow 0$.

Case 1. For $t_i \leq t' < t'' \leq s_i$,

$\parallel Q_2x(t'')-Q_2x(t') \parallel = 0$.

Case 2. For $s_i \leq t' < t'' \leq t_{i+1}$,

$$\parallel Q_2x(t'')-Q_2x(t') \parallel \leq I_1+I_2+I_3 + \parallel \left(P_\alpha(t''-s_i)-P_\alpha(t'-s_i)\right)K_{i2x} \parallel . \tag{4.1}$$

The right-hand side of (4.1) tends to 0 independently of $x \in B_r$ as $t'' \to t'$.

Case 3. For $t_i \leq t' < s_i < t'' \leq t_{i+1}$,

$\parallel Q_2x(t'')-Q_2x(t') \parallel \leq \parallel P_\alpha(t''-s_i)K_{i2x}+\int_0^{t''}(t''-s)^{\alpha-1}Q_\alpha(t''-s)f(s,x(s),Bx(s))ds - d_{i2x} \parallel \longrightarrow 0$

indenpendently of $x \in B_r$, as $t'' \longrightarrow t'$ we have $(t'' \longrightarrow s_i)$.

In conclusion, $\parallel Q_2x(t'')-Q_2x(t') \parallel \longrightarrow 0$, as $t''-t' \longrightarrow 0$, which implies that $Q_2(B_r(J))$ is equicontinuous.

We have $Q_2B_r \subseteq B_r$, where $Q_2B_r(t) = \{Q_2x(t); x \in B_r\}$ for $t \in J$.

Step5: Q is β- condensing in B_r.

For any $W \subset B_r$, $Q_2(W)$ is bounded and equicontinuous. Hence, by lemma(2.10), there exists a countable set $W_1 = \{u_n\}_{n=1}^{\infty} \subset W$ such that $\beta(Q_2(W)) \leq 2\beta(Q_2(W_1))$. Since $Q_2(W_1) \subset B_r$ is equicontinuous, lemma(2.11) implies $Q_2(W_1) = max\beta Q_2(W_1(t))$.

We have: $\beta\left(\left\{\int_0^t B(t,s)u(s)ds \Big/ u\in B_r, t\in J\right\}_{n=1}^{\infty}\right) \leq \omega K_0 \beta\left(\{u(t)/u\in B_r, t\in J\}_{n=1}^{\infty}\right).$

Case1. For $t\in[0,t_1]$, we have:

$$\begin{aligned}
\beta(Q_2(W_1(t))) &= \beta\left(\left\{\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f(s,u_n(s),Bu_u(s))ds\right\}_{n=1}^{\infty}\right)\\
&\leq \frac{2M_A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta\left(\{f(s,u_n(s),Bu_n(s))\}_{n=1}^{\infty}\right)ds\\
&\leq \frac{2M_A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(L_1\beta(W_1(s))+L_2\beta(B(W_1)(s))\right)ds\\
&\leq \frac{2M_AL_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(W_1(s))ds+\frac{2M_AL_2\omega K_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(W_1(s))ds\\
&\leq \frac{2M_A(L_1+\omega L_2K_0)t_1^\alpha}{\Gamma(\alpha+1)}\beta(W).
\end{aligned}$$

Since $Q_2(W_1)$ is bounded and equicontinuous, by lemma (2.11)

$$\begin{aligned}
\beta(Q_2(W)) &\leq 2\beta(Q_2(W_1)) = 2\max_{t\in J}\beta(Q_2(W_1(t)))\\
&\leq \frac{4M_A(L_1+\omega L_2K_0)t_1^\alpha}{\Gamma(\alpha+1)}\beta(W)\\
&< \beta(W).
\end{aligned}$$

In the other hand we have:

$\| Q_1x(t)-Q_1y(t) \|=0$ which implies that $\beta(Q_1(W))=0$.

Then

$$\begin{aligned}
\beta(Q(W)) &\leq \beta(Q_1(W))+\beta(Q_2(W))\\
&\leq \frac{4M_A(L_1+\omega L_2K_0)t_1^\alpha}{\Gamma(\alpha+1)}\beta(W)<\beta(W).
\end{aligned}$$

Case2. For $t\in(t_i,s_i]$,$i=1,2,...,m$, we have:

$$\| Q_1x(t)-Q_1y(t) \|\leq \left[\frac{(s_i^\beta+t_i^\beta)+M_A(s_{i-1}^\beta+t_{i-1}^\beta)+...+M_A^{i-1}(s_1^\beta+t_1^\beta)}{\Gamma(\beta+1)}\right]K\| x-y \|_{PC}.$$

Hence, by definition (2.7) we get:

$$\beta(Q_1(W))\leq \left[\frac{(s_i^\beta+t_i^\beta)+M_A(s_{i-1}^\beta+t_{i-1}^\beta)+...+M_A^{i-1}(s_1^\beta+t_1^\beta)}{\Gamma(\beta+1)}\right]K\beta(W).$$

On the other hand:

$$\beta(Q_2(W))\leq 4M_A(L_1+\omega L_2K_0)\left[\frac{t_i^\alpha+M_A(t_{i-1}^\alpha+s_{i-1}^\alpha+...+M_A^{i-1}(t_1^\alpha+s_1^\alpha)}{\Gamma(\alpha+1)}\right]\beta(W).$$

Then

$$\begin{aligned}\beta(Q(W)) &\leq \beta(Q_1(W))+\beta(Q_2(W))\\ &\leq \left[\frac{(s_i^{\beta}+t_i^{\beta})+M_A(s_{i-1}^{\beta}+t_{i-1}^{\beta})+...+M_A^{i-1}(s_1^{\beta}+t_1^{\beta})}{\Gamma(\beta+1)}\right]K\beta(W)\\ &\quad +4M_A(L_1+\omega L_2K_0)\left[\frac{t_i^{\alpha}+M_A(t_{i-1}^{\alpha}+s_{i-1}^{\alpha})+...+M_A^{i-1}(t_1^{\alpha}+s_1^{\alpha})}{\Gamma(\alpha+1)}\right]\beta(W)\\ &< \beta(W).\end{aligned}$$

Case3. For $t\in[s_i,t_{i+1}]$,$i=1,2,...,m$ we have:

$$\| Q_1x(t)-Q_1y(t)\| \leq \left[\frac{M_A(s_i^{\beta}+t_i^{\beta})+M_A^2(s_{i-1}^{\beta}+t_{i-1}^{\beta})+...+M_A^{i}(s_1^{\beta}+t_1^{\beta})}{\Gamma(\beta+1)}\right]K\| x-y\|_{PC}.$$

Hence, by definition (2.7) we get:

$$\beta(Q_1(W))\leq \left[\frac{M_A(s_i^{\beta}+t_i^{\beta})+M_A^2(s_{i-1}^{\beta}+t_{i-1}^{\beta})+...+M_A^{i}(s_1^{\beta}+t_1^{\beta})}{\Gamma(\beta+1)}\right]K\beta(W).$$

On the other hand:

$$\beta(Q_2(W))\leq 4M_A(L_1+\omega L_2K_0)\left[\frac{t_{i+1}^{\alpha}+M_A(t_i^{\alpha}+s_i^{\alpha})+...+M_A^{i}(t_1^{\alpha}+s_1^{\alpha})}{\Gamma(\alpha+1)}\right]\beta(W).$$

Then

$$\begin{aligned}\beta(Q(W)) &\leq \beta(Q_1(W))+\beta(Q_2(W))\\ &\leq \left[\frac{M_A(s_i^{\beta}+t_i^{\beta})+M_A^2(s_{i-1}^{\beta}+t_{i-1}^{\beta})+...+M_A^{i}(s_1^{\beta}+t_1^{\beta})}{\Gamma(\beta+1)}\right]K\beta(W)\\ &+ 4M_A(L_1+\omega L_2K_0)\left[\frac{t_{i+1}^{\alpha}+M_A(t_i^{\alpha}+s_i^{\alpha}+...+M_A^{i}(t_1^{\alpha}+s_1^{\alpha})}{\Gamma(\alpha+1)}\right]\beta(W)\\ &< \beta(W).\end{aligned}$$

Conclusion: in all cases we have:

$\beta(Q(W))\leq\beta(Q_1(W))+\beta(Q_2(W))\leq C\beta(W)<\beta(W)$ with $C>0$.

Since the operator Q is continuous and β-condensing. According to Darbo-Sadovskii's fixed point theorem, Q has a fixed point in B_r. Therefore, the problem(1.2) has at least one mild solution in B_r. This completes the proof.

□

Theorem 4.2. *Assume that $(H_1)-(H_4)$,and $(H_5')-(H_6)$ hold. In addition, let's suppose that the following property is verified:*

$$max\{C,D\}<1$$

Where $C = 4M_A(L_1+\omega L_2K_0)\left[\dfrac{t_m^\alpha + M_A(t_{m-1}^\alpha + s_{m-1}^\alpha) + ... + M_A^{m-1}(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)}\right]$,
and $D = 4M_A(L_1+\omega L_2K_0)\left[\dfrac{t_{m+1}^\alpha + M_A(t_m^\alpha + s_m^\alpha) + ... + M_A^{m}(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)}\right]$.
Then the problem (1.2) *has at least one mild solution.*

Proof. We introduce the composition $Q = Q_1 + Q_2$ where :

$$Q_1x(t) = \begin{cases} P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)C(s)u(s)ds, t \in [0,t_1], \\ d_{i1x} + \dfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g_i(s,x(s))ds,\ t \in (t_i,s_i],\ i = 1,2,..,m, \\ P_\alpha(t-s_i)K_{i1x} + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)C(s)u(s)ds, \\ \qquad\qquad\qquad t \in (s_i,t_{i+1}],\ i = 1,2,..,m, \end{cases}$$

$$Q_2x(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f(s,x(s),Bx(s))ds,\ t \in [0,t_1], \\ d_{i2x},\ t \in (t_i,s_i],\ i = 1,2,..,m, \\ P_\alpha(t-s_i)K_{i2x} + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f(s,x(s),Bx(s))ds, \\ \qquad\qquad\qquad t \in (s_i,t_{i+1}],\ i = 1,2,..,m, \end{cases}$$

with

$$\begin{cases} d_{i1x} = P_\alpha(t_i - s_{i-1})K_{(i-1)1x} + \int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)C(s)u(s)ds \\ \qquad -\dfrac{1}{\Gamma(\beta)}\int_0^{t_i}(t_i-s)^{\beta-1}g_i(s,x(s))ds, i = 1,2,..,m, \\ K_{i1x} = d_{i1x} + \dfrac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i-s)^{\beta-1}g_i(s,x(s))ds \\ \qquad -\int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)C(s)u(s)ds, i = 1,2,..,m, \\ K_{01x} = x_0, \end{cases}$$

and

$$\begin{cases} d_{i2x} = P_\alpha(t_i - s_{i-1})K_{(i-1)2x} + \int_0^{t_i}(t_i-s)^{\alpha-1}Q_\alpha(t_i-s)f(s,x(s),Bx(s))ds, i = 1,2,..,m, \\ K_{i2x} = d_{i2x} - \int_0^{s_i}(s_i-s)^{\alpha-1}Q_\alpha(s_i-s)f(s,x(x),Bx(s))ds, i = 1,2,..,m, \\ K_{02x} = 0. \end{cases}$$

Our proof will be divided into several steps.

Step1: We show that $QB_r(J) \subset B_r(J)$,

where $B_r = \{x \in PC(J,X); \| x \| \leq r\}$ the ball with radius $r > 0$;

$K_{\alpha,\tau} = \alpha\left(\dfrac{1-\tau}{\alpha-\tau}\right)^{1-\tau} \| Cu \|_{L^{1/\tau}}$ and $S_{\alpha,\tau} = \alpha\left(\dfrac{1-\tau}{\alpha-\tau}\right)^{1-\tau} \| m \|_{L^{1/\tau}}$,

$$\gamma_1 = M_A^{m+1} \parallel x_0 \parallel \frac{\Gamma(\beta+1)}{\Gamma(\beta+1) - M_A K(t_1^\beta + s_1^\beta)},$$

$$\gamma_2 = \frac{(M_A t_{m+1}^{\alpha-\tau} + M_A^2(t_m^{\alpha-\tau} + s_m^{\alpha-\tau}) + ... + M_A^{m+1}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau}))\Gamma(\beta+1)}{(\Gamma(\beta+1) - M_A K(t_1^\beta + s_1^\beta))\Gamma(\alpha)}(K_{\alpha,\tau} + S_{\alpha,\tau}),$$

$$\gamma_3 = \frac{M_A^2(t_{m-1}^\beta + s_{m-1}^\beta) + ... + M_A^m(t_1^\beta + s_1^\beta)}{(\Gamma(\beta+1) - M_A K(t_1^\beta + s_1^\beta))} d.$$

Here $\gamma_1 + \gamma_1 + \gamma_1 \leq r$.

For any $x \in B_r$, we have:

Case1. For $t \in [0, t_1]$

$$\begin{aligned} \parallel Qx(t) \parallel &\leq \parallel P_\alpha(t) K_{0x} \parallel + \parallel \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) \left[f(s, x(s), Bx(s)) + C(s)u(s) \right] ds \parallel \\ &\leq M_A \parallel x_0 \parallel + \frac{M_A t_1^{\alpha-\tau}}{\Gamma(\alpha)} \left(K_{\alpha,\tau} + S_{\alpha,\tau} \right). \end{aligned}$$

Similar to the proof of the previous theorem we show that:
for $t \in (t_i, s_i]$

$$\begin{aligned} \parallel Qx(t) \parallel &\leq M_A^i \parallel x_0 \parallel + \frac{M_A t_i^{\alpha-\tau} + M_A^2(t_{i-1}^{\alpha-\tau} + s_{i-1}^{\alpha-\tau}) + ... + M_A^i(t_1^{\alpha-\tau} + s_1^{\alpha-\tau})}{\Gamma(\alpha)}(K_{\alpha,\tau} + S_{\alpha,\tau}) \\ &\quad + \frac{(t_i^\beta + s_i^\beta) + M_A(t_{i-1}^\beta + s_{i-1}^\beta) + ... + M_A^{i-1}(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)}(b \parallel x \parallel + d) \\ &\leq r. \end{aligned}$$

For $t \in (s_i, t_{i+1}]$

$$\begin{aligned} \parallel Qx(t) \parallel &\leq M_A^{i+1} \parallel x_0 \parallel + \frac{M_A t_{i+1}^{\alpha-\tau} + M_A^2(t_i^{\alpha-\tau} + s_i^{\alpha-\tau}) + ... + M_A^{i+1}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau})}{\Gamma(\alpha)}(K_{\alpha,\tau} + S_{\alpha,\tau}) \\ &\quad + \frac{M_A(t_i^\beta + s_i^\beta) + M_A^2(t_{i-1}^\beta + s_{i-1}^\beta) + ... + M_A^i(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)}(b \parallel x \parallel + d) \\ &\leq r. \end{aligned}$$

We proved that $QB_r(J) \subset B_r(J)$.

Step2: Q_2 is continuous.

Let $(x_n)_{n\geq 0}$ be a sequence such that $\lim_{n\to\infty} \parallel x_n - x \parallel_{PC} = 0$, we have by (H_5') : $g_i(t, x_n(t)) \longrightarrow g_i(t, x(t))$.

Case 1. For $t \in [0, t_1]$, we have:

$\parallel Q_1 x_n(t) - Q_1 x(t) \parallel = 0$

Case 2. For $t \in [t_i, s_i], 1 \leq i \leq m$,

$$\| Q_1x(t) - Q_1y(t) \| \leq \left[\frac{(s_i^\beta + t_i^\beta) + M_A(s_{i-1}^\beta + t_{i-1}^\beta) + ... + M_A^{i-1}(s_1^\beta + t_1^\beta)}{\Gamma(\beta + 1)} \right] \| g_i(t, x_n(t)) - g_i(t, x(t)) \|_{PC} .$$

For $t \in [s_i, t_{i+1}], 1 \leq i \leq m$,

$$\| Q_1x(t) - Q_1y(t) \| \leq \left[\frac{M_A(s_i^\beta + t_i^\beta) + M_A^2(s_{i-1}^\beta + t_{i-1}^\beta) + ... + M_A^i(s_1^\beta + t_1^\beta)}{\Gamma(\beta + 1)} \right] \| g_i(t, x_n(t)) - g_i(t, x(t)) \|_{PC} .$$

Thus we get $\| Q_1x_n(t) - Q_1x(t) \|_{PC} \longrightarrow 0$ as $n \longrightarrow \infty$.
Then we can say that Q_1 is continuous.
We already have that Q_2 is continuous. Finally Q is continuous.
Step3: Q is β- condensing in B_r.

Case1. For $t \in [0, t_1]$, we have:

Considering the condition (H_5') and using the same method in the previous theorem we get:

$$\begin{aligned} \beta(Q(W)) &\leq \beta(Q_2(W)) \\ &\leq \frac{4M_A(L_1 + \omega L_2 K_0)t_1^\alpha}{\Gamma(\alpha + 1)}\beta(W) \\ &< \beta(W). \end{aligned}$$

Case2. For $t \in (t_i, s_i]$,$i = 1, 2, ..., m$, we have:

$$\begin{aligned} \beta(Q(W)) &\leq \beta(Q_2(W)) \\ &\leq 4M_A(L_1 + \omega L_2 K_0) \left[\frac{t_i^\alpha + M_A(t_{i-1}^\alpha + s_{i-1}^\alpha) + ... + M_A^{i-1}(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha + 1)} \right] \beta(W). \end{aligned}$$

Case3: For $t \in [s_i, t_{i+1}]$,$i = 1, 2, ..., m$, we have:

$$\begin{aligned} \beta(Q(W)) &\leq \beta(Q_2(W)) \\ &\leq 4M_A(L_1 + \omega L_2 K_0) \left[\frac{t_{i+1}^\alpha + M_A(t_i^\alpha + s_i^\alpha) + ... + M_A^i(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha + 1)} \right] \beta(W). \end{aligned}$$

Conclusion: in all cases we have:

$\beta(Q(W)) \leq \beta(Q_2(W)) \leq C\beta(W) < \beta(W)$ with $C > 0$.

Since the operator Q is continuous and β-condensing. According to Darbo-Sadovskii's fixed point theorem, Q has a fixed point in B_r. Therefore, the problem(1.2) has at least one mild solution in B_r. This completes the proof.

□

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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