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Notes on Some Recent Papers Concerning F-Contractions in b-Metric Spaces

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ABSTRACT. In several recent papers, attempts have been made to apply Wardowski's method of F-contractions in order to obtain fixed point results for single and multivalued mappings in b-metric spaces. In this article, it is shown that in most cases the conditions imposed on respective mappings are too strong and that the results can be obtained directly, i.e., without using most of the properties of auxiliary function F.

Keywords: b-Metric space, F-Contraction, α-Admissible mappings

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1. INTRODUCTION AND PRELIMINARIES

b-metric spaces, as a generalization of metric spaces, were introduced by Bakhtin [3] and Czerwik [6]. If X is a nonempty set and $s \geq 1$ is a fixed real number, a function $b : X \times X \to [0, +\infty)$ is called a b-metric on X with parameter s if the following holds for all $x, y, z \in X$:

(1) $b(x, y) = 0$ if and only if $x = y$,
(2) $b(x, y) = b(y, x)$,
(3) $b(x, z) \leq s[b(x, y) + b(y, z)]$.

Then, (X, b, s) is called a b-metric space.

Further on, several researchers obtained a lot of fixed point and common fixed point results, both for single and multivalued mappings in such spaces.

On the other hand, F-contractions were introduced by Wardowski [19] and several genuine generalizations of Banach Contraction Principle were produced using this concept. In [19], the class $\mathcal{F}$ of all functions $F : (0, +\infty) \to \mathbb{R}$ was used, satisfying the conditions:

(F1) F is strictly increasing,
(F2) $\lim_{t\to+0} F(t) = -\infty$,
(F3) for each sequence $\{t_n\}$ of positive reals with $\lim_{n\to\infty} t_n = 0$ there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} t_n^k F(t_n) = 0$.

In the paper [5], Cosentino et al. attempted to apply Wardowski's method in the context of b-metric spaces, by using the following additional assumption for the class of auxiliary functions that are used (they denoted the new class by $\mathcal{F}_s$):

(F4) if t_n is a sequence of positive reals satisfying $\tau + F(st_n) \leq F(t_{n-1})$ for some $\tau > 0$ and each $n \in \mathbb{N}$, then $\tau + F(s^n t_n) \leq F(s^{n-1} t_{n-1})$ for each $n \in \mathbb{N}$.

Further, their approach was used in the papers [1, 2, 8, 10, 11, 12, 13, 14, 15, 16] to obtain various fixed point results, mostly for multivalued mappings.

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However, as we are going to show using the following result, most of the conditions used in all these articles are too strong. In fact, with these conditions, just the property $(F1)$ of functions $F \in \mathcal{F}$ is sufficient to obtain the desired results.

Lemma 1.1. [18, 9] *Let (X, b, s) be a b-metric space and let $\{x_n\}$ be a sequence in X. If there exists $r \in [0, 1)$ satisfying*

$$b(x_n, x_{n+1}) \le r b(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N},$$

then $\{x_n\}$ is a Cauchy sequence.

Moreover, we will show that some conditions of admissibility, used in [12] and some other papers, can be replaced by easier ones.

2. Main results

We will assume in this section that $s > 1$ (otherwise, the results are already known).

The notion of α-admissibility of mappings was introduced and used in fixed point results by Samet et al. in [17]. It can be used as a unified approach to problems in spaces endowed with partial order, graph and alike. The notion was modified in several papers; we will use here the following version.

Definition 2.1. [4, Definitions 1.4 and 1.7] *Let X be a non-empty set and $f, g, h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$, and let $\alpha : X \times X \to [0, +\infty)$ be a function. The pair (f, g) is said to be*

(1) partially weakly α-admissible with respect to h if $\alpha(fx, gy) \ge 1$ for all $y \in X$ with $hy = fx$,
(2) triangular partially weakly α-admissible with respect to h if, moreover, $\alpha(x, z) \ge 1$ and $\alpha(z, y) \ge 1$ imply that $\alpha(x, y) \ge 1$ for all $x, y, z \in X$.

In [12], the authors modified the previous definition, putting s^2 instead of 1 on the right-hand sides of the respective inequalities (the idea was to use them for mappings acting in b-metric spaces with parameter s). However, it is clear that if one puts $\alpha_1(x, y) = \frac{1}{s^2}\alpha(x, y)$, all of their definitions reduce to the ones from [4]. In particular, [12, Definition 7] reduces to Definition 2.1. Similarly, instead of [12, Definitions 8, 9 and 10], it is enough to use the following ones.

Definition 2.2. *Let (X, b, s) be a b-metric space, $\alpha : X \times X \to [0, +\infty)$ be a function, and f, g be self-mappings on X.*

(1) [7] The space (X, b, s) is called α-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ converges in X.
(2) [4] The space (X, b, s) is called α-regular if

$$\lim_{n\to\infty} x_n = x \text{ and } \alpha(x_n, x_{n+1}) \ge 1 \text{ for all } n \in \mathbb{N} \text{ imply that } \alpha(x_n, x) \ge 1 \text{ for all } n \in \mathbb{N}.$$

(3) [7] The mapping f is α-b-continuous if, for given x and sequence $\{x_n\}$ in X,

$$\lim_{n\to\infty} x_n = x \text{ and } \alpha(x_n, x_{n+1}) \ge 1 \text{ for all } n \in \mathbb{N} \text{ implies that } \lim_{n\to\infty} f x_n = f x.$$

(4) [4] The pair (f, g) is α-compatible if $\lim_{n\to\infty} b(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$.

As a sample, we formulate and prove first of all an improved version of [12, Theorems 1 and 2] (since the conditions $(F2)$–$(F4)$ for functions $F \in \mathcal{F}$ will not be assumed and conditions of admissibility will be formulated in an easier way); moreover the proof will be much shorter than in [12].

Theorem 2.1. *Let (X, b, s) be an α-complete b-metric space, f, g, S, T be self-mappings on X such that $f(X) \subseteq T(X)$, $g(X) \subset S(X)$ and let $\alpha : X \times X \to [0, +\infty)$. Suppose that*

(1) there exist $\tau > 0$ and $F : (0, +\infty) \to \mathbb{R}$ satisfying $(F1)$ such that

$$\tau + F(sb(fx, gy)) \leq F(M(x, y)) \tag{2.1}$$

holds for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $b(fx, gy) > 0$, where

$$M(x, y) = \max\left\{b(Sx, Ty), b(fx, Sx), b(gy, Ty), \frac{b(Sx, gy) + b(fx, Ty)}{2s}\right\},$$

(2) the pairs (f, S) and (g, T) are α-compatible,

(3) the pairs (f, g) and (g, f) are triangular partially weakly α-admissible with respect to T and S, respectively.

If

$(4')$ f, g, S, T are α-b-continuous, or

$(4'')$ $T(X)$ and $S(X)$ are closed subsets of X and X is α-regular,

then the pairs (f, S), (g, T) have a common coincidence point $z \in X$. If, moreover, $\alpha(Sz, Tz) \geq 1$, then z is a common fixed point of the mappings f, g, S, T.

Proof. Take an arbitrary point $x_0 \in X$. Since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, we can form Jungck sequences $\{x_n\}$, $\{y_n\}$ in a standard way, satisfying

$$y_{2n+1} = f(x_{2n}) = T(x_{2n+1}) \text{ and } y_{2n+2} = g(x_{2n+1}) = S(x_{2n+2})$$

for $n = 0, 1, 2, \dots$ Moreover, using the assumption (3), we have that

$$\alpha(Tx_{2n+1}, Sx_{2n+2}) = \alpha(y_{2n+1}, y_{2n+2}) \geq 1 \text{ and } \alpha(Sx_{2n+2}, Tx_{2n+3}) = \alpha(y_{2n+2}, y_{2n+3}) \geq 1,$$

i.e., $\alpha(y_n, y_{n+1}) \geq 1$ for $n = 0, 1, 2, \dots$

Assume that $b(y_n, y_{n+1}) > 0$ for each $n = 0, 1, 2, \dots$ (otherwise the conclusions follow easily). As $\alpha(Sx_{2n}, Tx_{2n+1}) \geq 1$ and $b(fx_{2n}, gx_{2n-1}) > 0$, we get by (2.1) that

$$\tau + F(sb(y_{2n}, y_{2n+1})) \leq F(M(y_{2n-1}, y_{2n})), \tag{2.2}$$

and, similarly,

$$\tau + F(sb(y_{2n-1}, y_{2n})) \leq F(M(y_{2n-2}, y_{2n-1})). \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\tau + F(sb(y_n, y_{n+1})) \leq F(M(y_{n-1}, y_n)), \tag{2.4}$$

for $n = 1, 2, \dots$ However, in a standard way, we have that, in this case, $M(y_{n-1}, y_n) = b(y_{n-1}, y_n)$. Hence, from (2.4), we have

$$F(sb(y_n, y_{n+1})) < \tau + F(sb(y_n, y_{n+1})) \leq F(b(y_{n-1}, y_n)),$$

i.e., since F is strictly increasing,

$$b(y_n, y_{n+1}) < \frac{1}{s} b(y_{n-1}, y_n) \quad \text{for all } n \in \mathbb{N}.$$

Since $s > 1$, applying Lemma 1.1, we get that $\{y_n\}$ is a Cauchy sequence in X with $\alpha(y_n, y_{n+1}) \geq 1$. Thus, there exists $z \in X$ such that

$$\lim_{n\to\infty} b(y_{2n+1}, z) = \lim_{n\to\infty} b(Tx_{2n+1}, z) = \lim_{n\to\infty} b(fx_{2n}, z) = 0$$

and

$$\lim_{n\to\infty} b(y_{2n}, z) = \lim_{n\to\infty} b(Sx_{2n}, z) = \lim_{n\to\infty} b(gx_{2n-1}, z) = 0.$$

Hence, $Sx_{2n} \to z$ and $fx_{2n} \to z$ as $n \to \infty$.

The rest of the proof is the same as for [12, Theorems 1 and 2] (note that it does not use further properties of the function F). □

Corollaries 1–6 of the paper [12], formulated in an easier way, follow similarly. The same is true for Theorem 3 of that paper, as well as for the results in ordered b-metric spaces and for b-metric spaces endowed with a graph.

Further on, we will formulate and prove an improved version of [5, Theorem 3.4] (since again just the condition $(F1)$ of function F will be assumed); moreover the proof will again be much shorter than in [5]. First, we recall the following notions.

If (X, b, s) is a b-metric space, $CB(X)$ will denote the family of all non-empty, closed and bounded subsets of X. The Pompeiu-Hausdorff b-metric H on $CB(X)$ is defined by

$$H(C, D) = \max\{\sup_{c\in C} b(c, D), \sup_{d\in D} b(d, C)\},$$

for $C, D \in CB(X)$, where $b(x, A) = \inf_{a\in A} b(x, a)$ for $x \in X$ and $A \in CB(X)$.

Theorem 2.2. *Let (X, b, s) be a complete b-metric space and let $T : X \to CB(X)$. Assume that there exist $\tau > 0$ and a continuous from the right function $F : (0, +\infty) \to \mathbb{R}$ satisfying $(F1)$ such that*

$$2\tau + F(sH(Tx, Ty)) \leq F(b(x, y)) \tag{2.5}$$

for all $x, y \in X$, $Tx \neq Ty$. Then T has a fixed point.

Proof. As in the proof of [5, Theorem 3.4] (this part of the proof does not use the conditions $(F2)$–$(F4)$), starting from arbitrary $x_0 \in X$, we can form a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \notin Tx_n$ and

$$\tau + F(sb(x_{n+1}, x_{n+2})) \leq F(b(x_n, x_{n+1}))$$

for all $n = 0, 1, 2, \ldots$, and hence

$$F(sb(x_{n+1}, x_{n+2})) < F(b(x_n, x_{n+1})).$$

Using the condition $(F1)$, we get that

$$b(x_{n+1}, x_{n+2}) < \frac{1}{s} b(x_n, x_{n+1}).$$

Since $\frac{1}{s} < 1$, applying Lemma 1.1, we conclude that $\{x_n\}$ is a Cauchy sequence in X and, hence, it converges to some $z \in X$. The proof that $z \in Tz$ is the same as in [5, Theorem 3.4] (this part again does not use any other conditions of function F). □

Open question 1. *Does Theorem 2.1 remain valid if the condition (2.1) is replaced by*

$$\tau + F(b(fx, gy)) \leq F(b(Sx, Ty))$$

(which is the case for $s = 1$)? Similarly for Theorem 2.2.

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