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*Research Article*

# Taylor Collocation Method for Nonlinear System of Second-Order Boundary Value Problems

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## ABSTRACT

In this study, a numerical approach is proposed to obtain approximate solutions of nonlinear system of second-order boundary value problem. This technique is essentially based on the truncated Taylor series and its matrix representations with collocation points. Using the matrix method, we reduce the problem system of nonlinear algebraic equations. Numerical examples are also given to demonstrate the validity and applicability of the presented technique. The method is easy to implement and produces accurate results. All numerical computations have been performed on the computer algebraic system Maple 9.

***Keywords:*** *Nonlinear system, Second-order boundary value problem, Taylor polynomials and series, Collocation points*

# Lineer Olmayan Sistemlerin İkinci Mertebe Sınır-Değer Problemleri İçin Taylor Sıralama Metodu

## ÖZET

Bu çalışmada, lineer olmayan ikinci mertebe sınır değer probleminin yaklaşık çözümünü elde etmek için bir nümerik yaklaşım önerilmiştir. Bu teknik, temel olarak sıralama noktaları ile birlikte kesilmiş Taylor serisi ve onun matris gösterimlerini esas almaktadır. Matris metodu ile problem, lineer olmayan cebirsel denklem sistemine indirgenir. Ayrıca, sunulan tekniğin geçerliliği ve uygulanabilirliğini göstermek için nümerik örnekler verilmiştir. Metodun uygulanması kolaydır ve uygulama sonucunda doğru sonuçlar elde edilir. Çalışmadaki bütün sayısal hesaplamalar Maple 9 programında yapılmıştır.

***Anahtar Kelimeler:*** *Lineer olmayan sistem, İkinci mertebe sınır değer problemi, Taylor polinomu ve serisi, Sıralama noktaları*

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# I. INTRODUCTION

ORDINARY differential systems are encountered in scientific fields such as biology, medicine, physics and engineering. Many real life phenomena are modelled by second order ordinary differential systems [1-4]. Although, there are many classical methods to solve second-order initial value problems [5,6], these methods cannot be applied to second-order boundary value problems (BVPs). Therefore, it is required to numerical methods.

In this paper, we will consider the following nonlinear system of second-order differential equations:

$$\begin{cases} y_1'' + a_1(t)y_1' + a_2(t)y_1 + a_3(t)y_2'' + a_4(t)y_2' + a_5(t)y_2 + N_1(y_1, y_2) = f_1(t), \\ y_2'' + b_1(t)y_2' + b_2(t)y_2 + b_3(t)y_1'' + b_4(t)y_1' + b_5(t)y_1 + N_2(y_1, y_2) = f_2(t), \end{cases} \tag{1}$$

with boundary conditions

$$\begin{aligned} y_1(0) = y_1(1) = 0, \\ y_2(0) = y_2(1) = 0, \end{aligned} \tag{2}$$

where $0 \le t \le 1$, $N_1$ and $N_2$ are nonlinear functions of $y_1$ and $y_2$. Also $a_i(t)$, $b_i(t)$ for $i = 1, \ldots, 5$ are given continuous functions and $f_1$ and $f_2$ are known functions.

In [7], the analytical solution of problem (1)-(2) is represented in the form of series in the reproducing kernel space under the assumption that the solution to problem (1)-(2). In [8] Lu, proposed a variational iteration method, in [9] Dehghan et al. presented a numerical method based on Sinc-collocation method, in [10] Saadatmandi et al. solved this problem by using the Chebyshev finite difference method. Dehghan et al. suggested a numerical method base on the cubic B-spline scaling functions to find the solutions of the system [11]. Also, in [12] He's homotopy perturbation method is introduced to solve problem (1)-(2). The existence and uniqueness of solutions of second-order systems have been discussed, including the approximation of solutions via finite difference method [13-19].

Since the beginning of 1994, Taylor, Chebyshev, Legendre, Laguerre, Berstein and Bessel collocation and matrix methods have been used by Sezer et al.[20-25] to solve differential, difference, integral , integro-differential, delay differential equations and their systems. In the present work, by modifying and developing matrix and collocation methods studied in [20-25], we will find the approximate solutions of the system (1) with boundary conditions (2) in the truncated Taylor series form

$$y_i(t) = \sum_{n=0}^{N} y_{in}(t-c)^n, \quad y_{in} = \frac{y_i^{(n)}(c)}{n!}, \quad i = 1, 2, \qquad a \le t \le b. \tag{3}$$

where $y_{in}$, $(n = 0, 1, ..., N, \quad i = 1, 2)$ are unknown coefficients to be determined.

The organization of this paper is as follows: In the next section we describe the matrix representations of each term in the system (1)-(2). In Section 3, we find the fundamental matrix relation of this system. In Section 4, the Taylor collocation method is performed. In Section 5, the accuracy of solution is given and in Section 6, some computational results are given to clarify the method. Section 7 ends this paper with a brief conclusion.

## II. FUNDAMENTAL RELATIONS

Let us consider the nonlinear system in the form (1) and find the matrix representations of each term in the system. First we convert the solution defined by (3) and its derivatives, for $n = 0,1,\dots,N$ to the following matrix forms:

$$\mathbf{y_i}(t) = \mathbf{T}(t)\,\mathbf{Y}_i, \quad i = 1,2 \tag{4}$$

$$\mathbf{y_i}^{(\mathbf{n})}(t) = \mathbf{T}(t)\mathbf{B}^{\mathbf{n}}\,\mathbf{Y}_i, \quad i = 1,2 \tag{5}$$

where

$$\mathbf{T}(t) = \begin{bmatrix} 1 & t & t^2 & \dots & t^N \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$\mathbf{Y}_i = \begin{bmatrix} y_{i,0} & y_{i,1} & y_{i,2} & \dots & y_{i,N} \end{bmatrix}^T.$$

Nonlinear part of the system (1), $N_i(y_1, y_2),\ i = 1,2$ can be found as $y_i^{\,2}(t)$ or $y_i(t)y_j(t),\ i \neq j,\ \ i,j = 1,2.$ Also, we can write the matrix form of these nonlinear expressions, respectively,

$$y_1^{\,2}(t) = \mathbf{T}(t)\mathbf{T}^*(t)\overline{\mathbf{Y}}_{\mathbf{1,1}} \tag{6}$$

$$y_2^{\,2}(t) = \mathbf{T}(t)\mathbf{T}^*(t)\overline{\mathbf{Y}}_{\mathbf{2,2}} \tag{7}$$

$$y_1(t)y_2(t) = \mathbf{T}(t)\mathbf{T}^*(t)\overline{\mathbf{Y}}_{\mathbf{2,1}} \tag{8}$$

$$y_2(t)y_1(t) = \mathbf{T}(t)\mathbf{T}^*(t)\overline{\mathbf{Y}}_{\mathbf{1,2}} \tag{9}$$

where

$$\mathbf{T}^*(t) = \begin{bmatrix} \mathbf{T}(t) & 0 & \dots & 0 \\ 0 & \mathbf{T}(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{T}(t) \end{bmatrix}_{(N+1)\times(N+1)^2},$$

$$\overline{\mathbf{Y}}_{\mathbf{1,1}} = \begin{bmatrix} y_{1,0}\mathbf{Y_1} & y_{1,1}\mathbf{Y_1} & \dots & y_{1,N}\mathbf{Y_1} \end{bmatrix}^{\mathbf{T}}_{(N+1)^2\times 1}$$
$$\overline{\mathbf{Y}}_{\mathbf{2,2}} = \begin{bmatrix} y_{2,0}\mathbf{Y_2} & y_{2,1}\mathbf{Y_2} & \dots & y_{2,N}\mathbf{Y_2} \end{bmatrix}^{\mathbf{T}}_{(N+1)^2\times 1}$$
$$\overline{\mathbf{Y}}_{\mathbf{2,1}} = \begin{bmatrix} y_{1,0}\mathbf{Y_2} & y_{1,1}\mathbf{Y_2} & \dots & y_{1,N}\mathbf{Y_2} \end{bmatrix}^{\mathbf{T}}_{(N+1)^2\times 1}$$
$$\overline{\mathbf{Y}}_{\mathbf{1,2}} = \begin{bmatrix} y_{2,0}\mathbf{Y_1} & y_{2,1}\mathbf{Y_1} & \dots & y_{2,N}\mathbf{Y_1} \end{bmatrix}^{\mathbf{T}}_{(N+1)^2\times 1}.$$

## III. FUNDAMENTAL MATRIX EQUATIONS FOR SYSTEM (1)

We are now ready to construct the fundamental matrix equations for the nonlinear system of second-order boundary value problem (1). For this purpose, substituting the matrix relations (4)-(9) into system (1) and simplifying, we obtain the system of matrix equations

$$\begin{cases} \mathbf{T}(t)[\mathbf{B}^2 + a_1(t)\mathbf{B} + a_2(t)\mathbf{I}]\mathbf{Y_1} + \mathbf{T}(t)[a_3(t)\mathbf{B}^2 + a_4(t)\mathbf{B} + a_5(t)\mathbf{I}]\mathbf{Y_2} + \mathbf{N_1}(t)\overline{\mathbf{Y}}_{i,j} = f_1(t), \\ \mathbf{T}(t)[\mathbf{B}^2 + b_1(t)\mathbf{B} + b_2(t)\mathbf{I}]\mathbf{Y}_2 + \mathbf{T}(t)[b_3(t)\mathbf{B}^2 + b_4(t)\mathbf{B} + b_5(t)\mathbf{I}]\mathbf{Y_1} + \mathbf{N_2}(t)\overline{\mathbf{Y}}_{i,j} = f_2(t). \end{cases} \tag{10}$$

Therefore, we can write the matrix representation of the system (10) in the form

$$\begin{cases} \mathbf{D_1}(t)\mathbf{Y_1} + \mathbf{D_2}(t)\mathbf{Y_2} + \mathbf{N_1}(t)\overline{\mathbf{Y}}_{i,j} = f_1(t) \\ \mathbf{E_1}(t)\mathbf{Y_1} + \mathbf{E_2}(t)\mathbf{Y_2} + \mathbf{N_2}(t)\overline{\mathbf{Y}}_{i,j} = f_2(t) \end{cases} \tag{11}$$

where

$$\mathbf{D_1}(t) = \mathbf{T}(t)[\mathbf{B}^2 + a_1(t)\mathbf{B} + a_2(t)\mathbf{I}],$$

$$\mathbf{D_2}(t) = \mathbf{T}(t)[a_3(t)\mathbf{B}^2 + a_4(t)\mathbf{B} + a_5(t)\mathbf{I}],$$

$$\mathbf{E_1}(t) = \mathbf{T}(t)[b_3(t)\mathbf{B}^2 + b_4(t)\mathbf{B} + b_5(t)\mathbf{I}],$$

and

$$\mathbf{E_2}(t) = \mathbf{T}(t)[\mathbf{B}^2 + b_1(t)\mathbf{B} + b_2(t)\mathbf{I}].$$

Consequently, the fundamental matrix equations of the system (11) can be written in the following compact form

$$\mathbf{P}(t)\mathbf{Y} + \mathbf{N}(t) = \mathbf{f}(t) \tag{12}$$

where

$$\mathbf{P}(t)=\begin{bmatrix}\mathbf{D}_1(t) & \mathbf{D}_2(t)\\ \mathbf{E}_1(t) & \mathbf{E}_2(t)\end{bmatrix}_{2\times 2(N+1)},\quad \mathbf{Y}=\begin{bmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{bmatrix}_{2(N+1)\times 1},\quad \mathbf{f}(t)=\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}_{2\times 1},$$

$$\mathbf{N}(t)=\begin{bmatrix}\mathbf{N}_1(t) & 0\\ 0 & \mathbf{N}_2(t)\end{bmatrix}_{2\times 2(N+1)^2},\quad \overline{\mathbf{Y}}=\begin{bmatrix}\overline{\mathbf{Y}}_{i,j}\\ \overline{\mathbf{Y}}_{i,j}\end{bmatrix}_{2(N+1)^2\times 1}.$$

## IV. TAYLOR COLLOCATION METHOD

In this section, by substituting the collocation points defined by

$$t_s=\frac{b}{N}s,\quad s=0,1,\ldots,N,$$

into the fundamental matrix equation (12), we obtain the new system

$$\mathbf{P}(t_s)\mathbf{Y}+\mathbf{N}(t_s)\overline{\mathbf{Y}}=\mathbf{F}(t_s),\quad s=0,1,\ldots,N \tag{13}$$

and therefore, the new fundamental matrix equation

$$\mathbf{W}\mathbf{Y}^*+\mathbf{V}\overline{\overline{\mathbf{Y}}}=\mathbf{F} \tag{14}$$

where

$$\mathbf{W}=\begin{bmatrix}\mathbf{P}(t_0) & 0 & \dots & 0\\ 0 & \mathbf{P}(t_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{P}(t_N)\end{bmatrix}_{2(N+1)\times 2(N+1)^2},\quad \mathbf{Y}^*=\begin{bmatrix}\mathbf{Y}\\ \mathbf{Y}\\ \vdots\\ \mathbf{Y}\end{bmatrix}_{2(N+1)^2\times 1},$$

$$\mathbf{V}=\begin{bmatrix}\mathbf{N}(t_0) & 0 & \dots & 0\\ 0 & \mathbf{N}(t_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{N}(t_N)\end{bmatrix}_{2(N+1)\times 2(N+1)^3},\quad \overline{\overline{\mathbf{Y}}}=\begin{bmatrix}\overline{\mathbf{Y}}\\ \overline{\mathbf{Y}}\\ \vdots\\ \overline{\mathbf{Y}}\end{bmatrix}_{2(N+1)^3\times 1}.$$

To find matrix representation of boundary conditions given with (2), by using Eq. 4 we can write row matrices as

$$\mathbf{T}(0)\mathbf{Y}_1=0,\quad \mathbf{T}(0)\mathbf{Y}_2=0$$

and

$$\mathbf{T}(1)\mathbf{Y}_1=0,\quad \mathbf{T}(1)\mathbf{Y}_2=0.$$

Thus, we obtain the matrix forms of the conditions, respectively,

$$\mathbf{J}_0\mathbf{Y}=\mathbf{0} \tag{15}$$

and

$$\mathbf{J}_1\mathbf{Y}=\mathbf{0} \tag{16}$$

where

$$\mathbf{J}_0=\begin{bmatrix}\mathbf{T}(0) & 0\\ 0 & \mathbf{T}(0)\end{bmatrix}_{2\times 2(N+1)}, \qquad \mathbf{J}_1=\begin{bmatrix}\mathbf{T}(1) & 0\\ 0 & \mathbf{T}(1)\end{bmatrix}_{2\times 2(N+1)}$$

$$\mathbf{Y}=\begin{bmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{bmatrix}_{2(N+1)\times 1}.$$

By replacing the matrices (15) and (16) into any rows of the part $\mathbf{W}$ in Eq. 14, we get the new coefficient matrix $\tilde{\mathbf{W}}$ for system based on the conditions. Thus, the fundamental matrix equation of the system (1) under boundary conditions (2) corresponds to a system of $2(N+1)$ nonlinear algebraic equations with the unknown coefficients $y_{1,n}$ and $y_{2,n}$, $(n=0,1,\ldots,N)$.

Finally, the unknown coefficients are computed by solving this system and they are substituted in Eq. 3. Hence, the Taylor polynomial solutions

$$y_{i,N}(t)=\sum_{n=0}^{N} y_{i,n}t^n\,, \quad i=1,2 \tag{17}$$

can be obtained.

## V. Accuracy of Solutions

We can easily check the accuracy of the above solutions. Since truncated Taylor series (3) is the approximate solution of system (1), when the function $y_{i,N}(t),\ i=1,2,$ and its derivatives are substituted in system (1), the resulting equation must be satisfied approximately; that is, for $t=t_q\in[0,1],\ q=0,1,2,\ldots,$

$$E_{1,N}(t_q)=\left|\begin{array}{l} y'''_{1,N}(t_q)-a_1(t_q)y'_{1,N}(t_q)+a_2(t_q)y_{1,N}(t_q)+a_3(t_q)y'''_{2,N}(t_q)+a_4(t_q)y'_{2,N}(t_q)\\ +a_5(t_q)y_{2,N}(t_q)+N_1(y_{1,N}(t_q),y_{2,N}(t_q))-f_1(t_q)\end{array}\right|\cong 0,$$

$$E_{2,N}(t_q)=\left|\begin{array}{l} y'''_{2,N}(t_q)-b_1(t_q)y'_{2,N}(t_q)+b_2(t_q)y_{2,N}(t_q)+b_3(t_q)y'''_{1,N}(t_q)+b_4(t_q)y'_{1,N}(t_q)\\ +b_5(t_q)y_{1,N}(t_q)+N_2(y_{1,N}(t_q),y_{2,N}(t_q))-f_2(t_q)\end{array}\right|\cong 0,$$

and $E_{i,N}(t_q)\le 10^{-k_q}$, $\quad i=1,2$ ($k_q$ positive integer).

If $\max 10^{-k_q} = 10^{-k}$ ($k_q$ positive integer) is prescribed, then the truncation limit $N$ is increased until the difference $E_{i,N}(t_q)$ at each of the points becomes smaller than the prescribed $10^{-k}$, see [20-25].

## VI. NUMERICAL EXPERIMENTS

In this section we show the efficiency of the presented method by solving the following examples. In tables and figures, we give the values of the exact solutions $y_i(t),\ i=1,2,$ and the absolute error functions $e_{i,N}(t) = \left| y_i(t) - y_{i,N}(t) \right|,\ i=1,2,$ are presented at selected points of the given interval. Results are shown with tables and figures. All numerical computations have been made in Maple 9.

**Example 1.** Let us first consider the linear system of second-boundary value problems [8,12].

$$\begin{cases} y_1''(t) + t\, y_1(t) + t\, y_2(t) = 2, \\ y_2''(t) + 2t\, y_2(t) + 2t\, y_1(t) = -2 \end{cases} \qquad 0 \le t \le 1,$$

with the boundary conditions $y_1(0) = y_1(1) = 0,\quad y_2(0) = y_2(1) = 0.$ The exact solutions of this problem are $y_1(t) = t^2 - t$ and $y_2(t) = t - t^2$. Now, let us apply the procedure in Section 4 to obtain this approximate solution. Firstly, we note that

$a_1(t) = a_3(t) = a_4(t) = b_1(t) = b_3(t) = b_4(t) = 0,\quad a_2(t) = a_5(t) = t,\quad b_2(t) = b_5(t) = 2t.$ The set of collocation points for $N = 2$ is computed as $\left\{ t_0 = 0,\ t_1 = \frac{1}{2},\ t_2 = 1 \right\}$ and the fundamental matrix equation of the problem from Eq. 10 is

$$\begin{cases} [\mathbf{T}(t)\mathbf{B}^{\mathbf{2}} + t\mathbf{T}(t)]\mathbf{Y}_1 + t\mathbf{T}(t)\mathbf{Y}_2 = 2, \\ 2t\mathbf{T}(t)\mathbf{Y}_1 + [\mathbf{T}(t)\mathbf{B}^{\mathbf{2}} + 2t\mathbf{T}(t)]\mathbf{Y}_2 = -2. \end{cases}$$

We can find the compact form of this system from Eq. 12 as

$$\mathbf{P}(t)\mathbf{Y} = \mathbf{f}(t)$$

where

$$\mathbf{P}(t) = \begin{bmatrix} t & t^2 & 2+t^3 & t & t^2 & t^3 \\ 2t & 2t^2 & 2t^3 & 2t & 2t^2 & 2+2t^3 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$
[\mathbf{W};\mathbf{F}]=\begin{bmatrix}
0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&;&2\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&;&-2\\
0&0&0&0&0&0&\frac{1}{2}&\frac{1}{4}&\frac{17}{8}&\frac{1}{2}&\frac{1}{4}&\frac{1}{8}&0&0&0&0&0&0&;&2\\
0&0&0&0&0&0&1&\frac{1}{2}&\frac{1}{4}&1&\frac{1}{2}&\frac{9}{4}&0&0&0&0&0&0&;&-2\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&3&1&1&1&;&2\\
0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&2&2&4&;&-2
\end{bmatrix}.
$$

From Eq. 15 and 16, the matrix forms of the boundary conditions are written as

$$
[\mathbf{J}_0;0]=\begin{bmatrix}
1&0&0&0&0&0&;&0\\
0&0&0&1&0&0&;&0
\end{bmatrix},
$$

$$
[\mathbf{J}_1;0]=\begin{bmatrix}
1&1&1&0&0&0&;&0\\
0&0&0&1&1&1&;&0
\end{bmatrix}.
$$

Therefore, the new augmented matrix based on the conditions becomes

$$
[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix}
0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&;&2\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&;&-2\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&;&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&;&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&;&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&;&0
\end{bmatrix}.
$$

By solving this system, the Taylor coefficients matrix is gained as

$$
\mathbf{Y}=\begin{bmatrix}0&-1&1&0&1&-1\end{bmatrix}^{\mathbf{T}} \text{ or}
$$

$$
\mathbf{Y}_1=\begin{bmatrix}0&-1&1\end{bmatrix}^{\mathbf{T}},\quad \mathbf{Y}_2=\begin{bmatrix}0&1&-1\end{bmatrix}^{\mathbf{T}}.
$$

Substituting the elements of these column matrixes into Eq. 4, we obtain the solution set in terms of Taylor polynomials as

$$
y_1(t)=t^2-t,\quad y_2(t)=t-t^2
$$

which are the exact solutions.

**Example 2.** In the second example, consider the nonlinear system [8]

$$\begin{cases} y_1''(t) - t\,y_2'(t) + y_1(t) = f_1(t), \\ y_2''(t) + t\,y_1'(t) + y_1(t)y_2(t) = f_2(t), \end{cases} \qquad 0 \le t \le 1,$$

with the boundary conditions $y_1(0) = y_1(1) = 0,\quad y_2(0) = y_2(1) = 0.$ Here, $f_1(t) = t^3 - 2t^2 + 6t$ and $f_2(t) = t^5 - t^4 + 2t^3 + t^2 - t + 2.$ The exact solutions of the problem are $y_1(t) = t^3 - t$ and $y_2(t) = t^2 - t.$ When we apply the procedure in Section 4 for $N = 3$, we obtain the solutions $\hat{y}_1(t) = t^3 - t,\quad \hat{y}_2(t) = t^2 - t$ which are the exact solutions.

**Example 3.** [7,9] In this example, consider the linear system of second-order boundary value problem

$$\begin{cases} y_1''(t) + y_1'(t) + t y_1(t) + y_2'(t) + 2t y_2(t) = f_1(t), \\ y_2''(t) + y_2(t) + 2t y_1'(t) + t^2 y_1(t) = f_2(t), \\ y_1(0) = y_1(1) = 0, \qquad y_2(0) = y_2(1) = 0 \end{cases}$$

where $0 \le t \le 1,\quad f_1(t) = -2(1+t)\cos(t) + \pi\cos(\pi t) + 2t\sin(\pi t) + (4t - 2t^2 - 4)\sin(t),$ and $f_2(t) = -4(t-1)\cos(t) + 2(-t^3 + t^2 - 2)\sin(t) + (1-\pi^2)\sin(\pi t).$ The exact solutions are $y_1(t) = 2(1-t)\sin(t)$ and $y_2(t) = \sin(\pi t).$ We obtain the approximate solutions by Taylor polynomials of the problem for $N = 5, 10, 20.$ In Tables 1-2, the absolute errors obtained by the present method for $N = 10, 20$ are compared with the results obtained by the method using in [7] and the sinc-collocation method in [9] with the same number of points in $[0,1]$. It is seen from these tables that the present method is closer to exact solution than the other methods.

***Table 1.*** *Absolute errors of* $y_1(t)$ *for Example 3.*

| | **Method of [7]** | | **Sinc-Collocation Method [9]** | | **The Present Method** | |
|---|---|---|---|---|---|---|
| $t_i$ | $N=10$ | $N=20$ | $N=10$ | $N=20$ | $N=10$ | $N=20$ |
| 0.08 | 3.3e-003 | 8.0e-004 | 3.2e-003 | 3.0e-004 | 0.5e-007 | 0.1e-009 |
| 0.24 | 7.7e-003 | 1.9e-003 | 9.2e-004 | 8.5e-005 | 0.12e-006 | 0 |
| 0.40 | 9.7e-003 | 2.4e-003 | 2.0e-003 | 3.5e-004 | 0.16e-006 | 0.1e-009 |
| 0.56 | 9.5e-003 | 2.4e-003 | 2.2e-004 | 2.6e-004 | 0.16e-006 | 0.1e-009 |
| 0.72 | 7.3e-003 | 1.8e-003 | 4.1e-003 | 2.0e-004 | 0.14e-006 | 0.3e-009 |
| 0.88 | 3.4e-003 | 8.0e-004 | 1.0e-002 | 2.6e-004 | 0.9e-007 | 0.1e-009 |
| 0.96 | 1.1e-003 | 2.0e-003 | 2.1e-003 | 2.6e-003 | 0.4e-007 | 0.2e-010 |

*__Table 2.__ Absolute errors of $y_2(t)$ for Example 3.*

| $t_i$ | Method of [7] | | Sinc-Collocation Method [9] | | The Present Method | |
|---|---|---|---|---|---|---|
| | $N=10$ | $N=20$ | $N=10$ | $N=20$ | $N=10$ | $N=20$ |
| 0.08 | 7.7e-003 | 1.9e-003 | 1.5e-003 | 2.0e-003 | 0.86e-007 | 0 |
| 0.24 | 2.0e-002 | 5.1e-003 | 7.0e-003 | 9.8e-004 | 0.23e-006 | 0 |
| 0.40 | 2.7e-002 | 7.1e-003 | 7.4e-003 | 1.1e-003 | 0.35e-006 | 0.5e-009 |
| 0.56 | 2.7e-002 | 6.9e-003 | 1.0e-002 | 1.4e-003 | 0.45e-006 | 0 |
| 0.72 | 2.0e-002 | 5.2e-003 | 4.4e-003 | 5.5e-005 | 0.55e-006 | 0.7e-009 |
| 0.88 | 9.4e-003 | 2.4e-003 | 2.1e-002 | 7.7e-004 | 0.63e-006 | 0.14e-008 |
| 0.96 | 3.1e-003 | 8.0e-003 | 6.9e-003 | 8.3e-004 | 0.44e-006 | 0.18e-008 |

**Example 4.** [12] Our last example is the non-linear system

$$\begin{cases} y_1''(t) + t\,y_2(t) + t\,y_1^2(t) = f_1(t), \\ y_2''(t) + t\,y_1'(t) + y_2(t) = f_2(t), \end{cases} \qquad 0 \le t \le 1,$$

with the boundary conditions $y_1(0) = y_1(1) = 0,\ \ y_2(0) = y_2(1) = 0,$ where $f_1(t) = -\pi^2 sin(\pi t) + t\,sin^2(\pi t) + t^4 - 3t^3 + 2t$ and $f_2(t) = t\,\pi cos(\pi t) + t^3 - 3t^2 + 8t - 6.$ The exact solutions of this problem are $y_1(t) = sin(\pi t)$ and $y_2(t) = t^3 - 3t^2 + 2t.$ Using the procedure in Section 4, we calculate the approximate solutions $y_{1,N}(t)$ and $y_{2,N}(t)$ for $N = 6,8,10.$ In Tables 3-4, the exact solutions and approximate solutions obtained by the present method are compared. On the other hand, in Fig. 1-2, the absolute errors for the present method are shown for different values of $N.$ Additionally, in Table 5, the accuracy of solutions are stated. These results show that if $N$ increases, than the absolute errors decrease more rapidly.

*__Table 3.__ Numerical results of solutions $y_1(t)$ of Example 4.*

| $t_i$ | Exact Solution | Present method | | |
|---|---|---|---|---|
| | $y_1(t_i) = \sin(\pi t_i)$ | $N=6,\ \ y_{1,6}(t_i)$ | $N=8,\ \ y_{1,8}(t_i)$ | $N=10,\ \ y_{1,10}(t_i)$ |
| 0.1 | 0.3090169944 | 0.3085841093 | 0.3090243256 | 0.3090169105 |
| 0.2 | 0.5877852524 | 0.5869635288 | 0.5877994680 | 0.5877850874 |
| 0.3 | 0.8090169944 | 0.8077989920 | 0.8090382049 | 0.8090167485 |
| 0.4 | 0.9510565165 | 0.9494390161 | 0.9510845314 | 0.9510561908 |
| 0.5 | 1.0 | 0.9980106604 | 1.000034626 | 0.9999995978 |
| 0.6 | 0.9510565163 | 0.9487139849 | 0.9510973706 | 0.9510560420 |
| 0.7 | 0.8090169941 | 0.8063201981 | 0.8090635500 | 0.8090164533 |
| 0.8 | 0.5877852522 | 0.5848734949 | 0.5878372288 | 0.5877846485 |
| 0.9 | 0.3090169936 | 0.3065966020 | 0.3090671066 | 0.3090163656 |

***Table 4.*** *Numerical results of solutions* $y_2(t)$ *of Example 4.*

| $t_i$ | Exact Solution $y_2(t_i)=t_i^3-3t_i^2+2t_i$ | Present method $N=6,\ y_{2,6}(t_i)$ | $N=8,\ y_{2,8}(t_i)$ | $N=10,\ y_{2,10}(t_i)$ |
|---|---|---|---|---|
| 0.1 | 0.171 | 0.1709254664 | 0.1710012938 | 0.1709999851 |
| 0.2 | 0.288 | 0.2878551222 | 0.2880025096 | 0.2879999712 |
| 0.3 | 0.357 | 0.3567941626 | 0.3570035593 | 0.3569999591 |
| 0.4 | 0.384 | 0.3837474041 | 0.3840043670 | 0.3839999500 |
| 0.5 | 0.375 | 0.3747185358 | 0.3750048635 | 0.3749999444 |
| 0.6 | 0.336 | 0.3357103322 | 0.3360049891 | 0.3359999430 |
| 0.7 | 0.273 | 0.2727258226 | 0.2730047107 | 0.2729999466 |
| 0.8 | 0.192 | 0.1917704189 | 0.1920039970 | 0.1919999550 |
| 0.9 | 0.099 | 0.09885500963 | 0.09900267529 | 0.09899996937 |

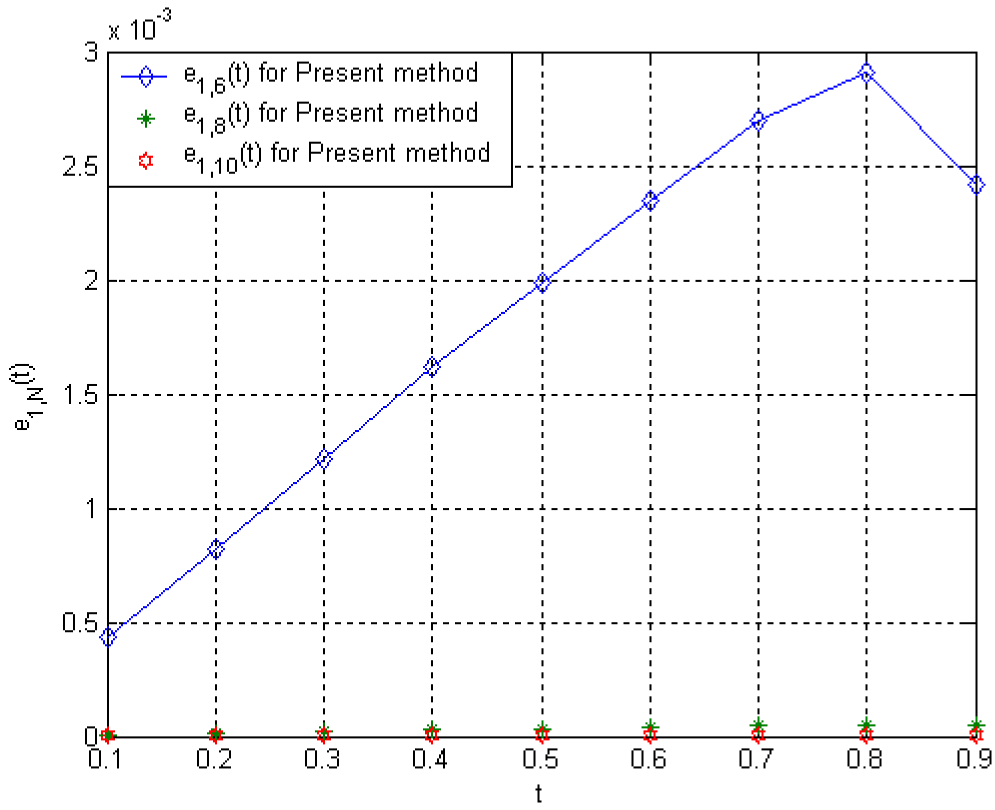


***Figure. 1.*** *Comparison of absolute error functions* $e_{1,N}(t)$ *of Example 4 for* $N=6,8,10$.

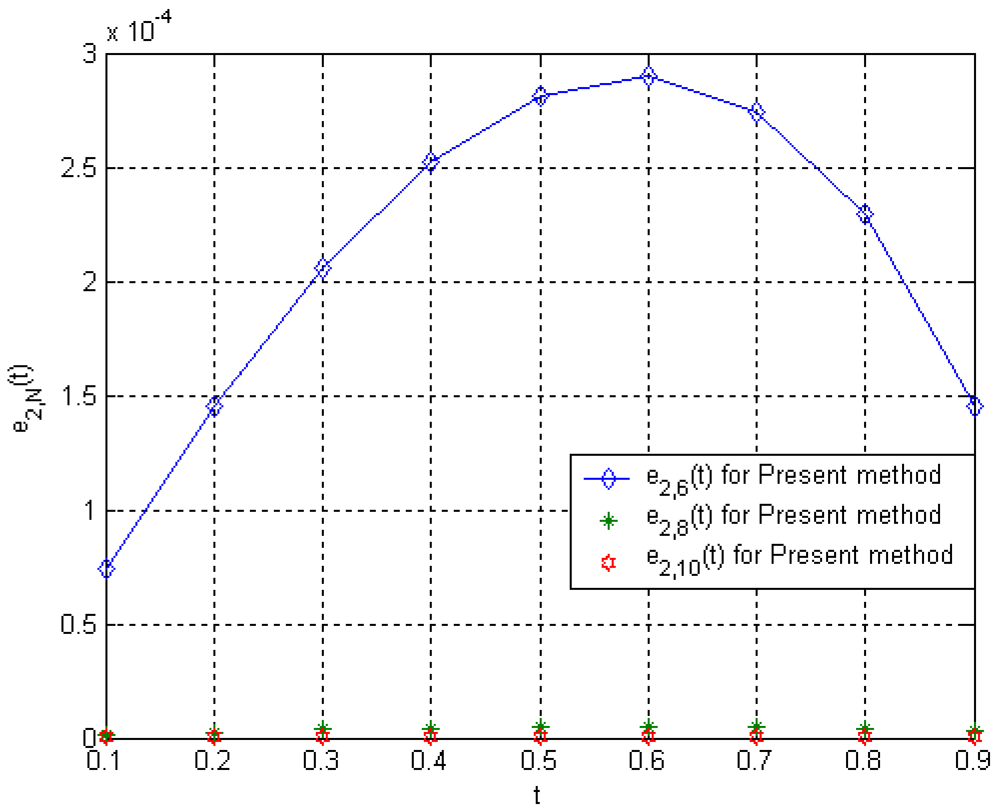


***Figure. 2.*** *Comparison of absolute error functions* $e_{2,N}(t)$ *of Example 4 for* $N = 6,8,10.$

***Table 5.*** *Accuracies of the solutions of Example 4 for* $N = 8,10.$

| | Present method | | | |
|---|---|---|---|---|
| $t_i$ | $E_{1,8}(t_i)$ | $E_{1,10}(t_i)$ | $E_{2,8}(t_i)$ | $E_{2,10}(t_i)$ |
| 0.1 | 0.38636283e-004 | 0.2717e-008 | 0.3897e-006 | 0.29997344e-009 |
| 0.2 | 0.2348542e-004 | 0.442e-008 | 0.2575e-006 | 0.28676425e-009 |
| 0.3 | 0.135312e-004 | 0.58e-008 | 0.1271e-006 | 0.50503154e-009 |
| 0.4 | 0.69530e-005 | 0.150e-007 | 0.360e-007 | 0.169243191e-008 |
| 0.5 | 0.2e-008 | 0.24e-007 | 0.171288e-009 | 0.193190512e-008 |
| 0.6 | 0.143913e-004 | 0.187e-007 | 0.1609e-006 | 0.196237761e-008 |
| 0.7 | 0.658243e-004 | 0.183e-007 | 0.1564e-005 | 0.52748838e-008 |
| 0.8 | 0.4209972e-003 | 0.218e-007 | 0.16807e-004 | 0.79202062e-008 |
| 0.9 | 0.41214521e-002 | 0.444069e-004 | 0.24904e-003 | 0.200696419e-005 |

## VII. CONCLUSION

In this study, a new Taylor matrix-collocation method is proposed for nonlinear system of second-order boundary value problems. It is observed from Figures and Tables that the method is a simple and powerful tool to obtain the approximate solution. When the numerical experiments are analyzed and the results are compared, it is seen that, the present method is quite effective. Additionally, if $N$ is increased, it can be seen that approximate solutions obtained by the mentioned method are close to the exact solutions. One of the considerable advantages of the method is finding the approximate solutions very easily by using the computer program written in Maple 9. Shorter computation time and lower operation count results in a reduction of cumulative truncation errors and improvement of overall accuracy. In addition, the method can also be extended to other models in the future.

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