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DNA CODES OVER A NON-CHAIN RING

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Abstract

We study the algebraic structure of DNA codes constructed over the ring $R = F_2[u, v, w]\,\langle u^2 = v^2, uv = 0,\ w^2 = w\rangle$, which is a commutative local Frobenius non-chain ring. We define a gray map over R and generate DNA codes using the images of the gray map. We define reversible DNA codes and reversible complement DNA codes over the ring.

Keywords: Linear codes, DNA codes

1. Introduction

Nucleotides are found in the structure of DNA chains; these are Adenine (A), Guanine (G), Cytosine (C) and Thymine (T). These chains have a double-stranded structure. The formation of the double helix structure follows a specific rule. This rule is known as the Watson-Crick Complement (WCC) principle. According to this principle, Adenine pairs with Thymine, and Guanine pairs with Cytosine. For example, if a strand contains the nucleotides A, G, C, and T, the complementary strand will contain nucleotides T, C, G, and A. The double helix structure is formed by the bonds created between the complementary nucleotides.

In 1994, Adleman [2] conducted groundbreaking research. He solved the NP-complete Hamilton path problem using the structural properties of DNA sequences. This pioneering work revealed that DNA sequences could be utilized in computations, attracting the attention of many scientists due to their potential for massive parallelization. Some of these applications break the Data Encryption Standard (DES) cryptographic system [3], solve the NP-complete SAT problem [17], and solve combinatorial optimization problems [7].

DNA computation is based on the principle of DNA hybridization. DNA hybridization occurs when the reverse complement of an existing DNA sequence, via Watson-Crick base pairing, binds to it to form a double helical structure. This feature, used in Adleman's work, clearly demonstrates the importance of reversible and reversible complementary DNA codes. However, errors may occur during the hybridization process. Therefore, error correction techniques are significant in DNA computing. Using DNA with algebraic approaches yields numerous new computational methods in various fields, including computing, storage, error correction, and cryptography [6, 8, 9, 10, 11, 16]. For successful DNA computing, DNA codes must satisfy certain constraints, such as the reverse constraint, the reverse complement constraint, the GC-constraint, and so on.

In recent years, DNA codes have been defined on algebraic structures, including finite fields [1, 12, 13], finite chain rings [14, 15, 18], and non-chain rings [4, 5, 19]. Reversible DNA codes were produced in these studies. In this paper, we will construct a reversible and reversible complement DNA code over the non-chain ring, which is a three-variable residue ring introduced in [20]. Furthermore, the dual codes of the obtained DNA codes are reversible.

2. Preliminaries

In this section, our goal is to generate reversible and reversible complement DNA codes via cyclic codes over R by solving the reversibility problem. To generate the related DNA codes, we establish a correspondence between the elements of R and DNA double bases.

Throughout, let

$$R = F_2 + v F_2 + v^2 F_2 + w F_2 + u F_2 + wu F_2 + wv F_2 + wv^2 F_2$$

be the quotient ring $F_2[u, v, w] \ / \ \langle u^2 = v^2, uv = 0, w^2 = w \rangle$, which is a commutative non-chain ring. We can express the ring R more simply.

$$R = F_2 + u F_2 + v F_2 + w F_2 + uw F_2 + vw F_2 + u^2 F_2 + wu^2 F_2,$$
$$u^2 = v^2, uv = 0, w^2 = w.$$
$$R = (F_2 + u F_2 + v F_2 + v^2 F_2) + w(F_2 + u F_2 + v F_2 + v^2 F_2), u^2 = v^2, uv = 0, w^2 = w.$$
$$R = \mathcal{R} + w\mathcal{R} \, , \ w^2 = w.$$

Here $\mathcal{R}$ is a local Frobenius non-chain ring $F_2 + u F_2 + v F_2 + v^2 F_2 \ with \ u^2 = v^2$ and $uv = 0$.

We define a gray map

$$\Phi: R \longrightarrow \mathcal{R}^2$$
$$a + bw \longrightarrow (a, a + b)$$

where $R = F_2 + u F_2 + v F_2 + v^2 F_2 + w F_2 + wu F_2 wv F_2 + wv^2 F_2 \ and \ \mathcal{R} = F_2 + u F_2 + v F_2 + v^2 F_2 \ and \ a, b \in \mathcal{R}.$

We define an θ automorphism over R as follows:

$$\theta : \mathcal{R} + w\mathcal{R} \longrightarrow \mathcal{R} + w\mathcal{R}$$
$$a + bw \longrightarrow a + (1 + w)b$$

If A and B are codes, the tensor product of these two codes is defined as $A \otimes B = \{(a, b) | a \in A, b \in B \}$, and direct sum is defined as $A \oplus B = \{ a + b \ | a \in A, b \in B\}$. For a linear code

C with length n over $F_2 + u F_2 + v F_2 + w F_2 + uw F_2 + vw F_2 + v^2 F_2 + wv^2 F_2$, we define

$$C_1 = \{ a + b \in \mathcal{R} \mid w(a+b) + (w+1)a \in C, a, b \in \mathcal{R} \},$$
$$C_2 = \{ a \in \mathcal{R} \mid w(a+b) + (w+1)a \in C, b \in \mathcal{R} \}.$$

Then C_1 and C_2 are linear codes over $\mathcal{R}$ and $C = wC_1 \oplus (w+1)C_2$.

Massey introduced the notion of reversible codes over finite fields by establishing a connection with self-reciprocal polynomials [21]. He showed that a linear code is reversible if and only if it can be generated by a monic self-reciprocal polynomial.
For a codeword

$$c = (c_0, c_1, \dots, c_{n-1}) \in R,$$

the reverse of c is defined as

$$c^r = (c_{n-1}, c_{n-2}, \dots, c_0).$$

Similarly, for a polynomial

$$c(x) = c_0 + c_1 x + \cdots + c_{rx}^r \quad (c_r \neq 0),$$

the reciprocal polynomial is given by

$$c^*(x) = x^r\, c(1/x) = c_r + c_{r-1} + \cdots + c_0 x^r.$$

Observe that $deg\big(c^*(x)\big) \leq deg(c(x))$, and if $c_0 \neq 0$, then $c(x)$ and $c^*(x)$ have the same degree.
A polynomial $c(x)$ is said to be *self-reciprocal* if

$$c(x) = c^*(x).$$

3. Reversibility over *R*

The structure of cyclic codes is determined by the generator polynomial $g(x)$. In this case, where

$$g(x) = wg_1(x) + (w+1)g_2(x).$$

Now, if both $g_1(x)$ and $g_2(x)$ are self-reciprocal polynomials over $\mathcal{R}$[x], one might expect that $g(x)$ would also be self-reciprocal over $R[x]$. However, this implication does not hold in general.

- Let $g(x) = wg_1(x) + (1+w)g_2(x)$ over R, where $g_1(x)$ and $g_2(x)$ are self-reciprocal polynomials over $\mathcal{R}[x]$. In general, $g(x)$ need not be self-reciprocal.

For example, if

$$g_1(x) = x^6 + x^3 + 1 \text{ and } g_2(x) = x^2 + x + 1,$$

then

$$g(x) = w\,x^6 + w\,x^3 + w\,x^2 + w\,x + 2\,w + x^2 + x + 1,$$

which is not a self-reciprocal polynomial over R.

- Let $g(x) = wg_1(x) + (1+w)g_2(x)$ over R, where $g_1(x)$ and $g_2(x)$ are not self-reciprocal polynomials over $\mathcal{R}[x]$. In this case, $g(x)$ may also fail to be self-reciprocal.

For instance, if

$$g_1(x) = x^3 + x + 1 \text{ and } g_2(x) = x^3 + x^2 + 1,$$

then

$$g(x) = w\,x^2 + w\,x + x^3 + x^2 + 1,$$

which is not self-reciprocal over R.

Motivated by these observations, we now focus on characterizing self-reciprocal polynomials over the ring R. The following theorem will characterize the reversible cyclic code $x^n - 1$ using its divisors.

Theorem 3.1 Let

$$C = \langle\, g(x)\rangle, g(x) = wg_1(x) + (1+w)g_2(x),$$

be a cyclic code over R. Then C is reversible if and only if $g_1(x)$ and $g_2(x)$ are self-reciprocal polynomials over $\mathcal{R}[x]$ and both divide $x^n - 1$.

Proof. Assume that $C = \langle\, g(x)\rangle$ is reversible. By the definition of reversibility,

$$g^*(x) \in \langle\, g(x)\rangle.$$

Since

$$g(x) = w\,g_1(x) + (1+w)g_2(x),$$

the reciprocal polynomial of $g(x)$ is given by

$$g^*(x) = w\,g_1^*(x) + (1+w)g_2^*(x).$$

Because w and $1+w$ are orthogonal idempotents in R, the inclusion

$$g^*(x) \in \langle\, g(x)\rangle$$

implies that

$$g_1^*(x) \in \langle\, g_1(x)\rangle \text{ and } g_2^*(x) \in \langle\, g_2(x)\rangle.$$

Hence, $g_1(x)$ and $g_2(x)$ are self-reciprocal polynomials. Moreover, since $g(x) \mid x^n - 1$, it follows that both $g_1(x)$ and $g_2(x)$ divide $x^n - 1$ over R .

Conversely, suppose that $g_1(x)$ and $g_2(x)$ are self-reciprocal divisors of $x^n - 1$. Then

$$g^*(x) = w\, g_1^*(x) + (1+w) g_2^*(x) = w\, g_1(x) + (1+w) g_2(x) = g(x).$$

Thus, $g(x)$ is self-reciprocal, and consequently the cyclic code $C = \langle g(x) \rangle$ is reversible.

Corollary 3.2 If a polynomial $g(x)$ dividing $x^n - 1$ over R is self-reciprocal, then the cyclic code $C = \langle g(x) \rangle$ reversible.

This follows directly from previous Theorem when $g(x) | x^n - 1$.

3.1. DNA codes over R

As communication system bandwidth increases, emerging technologies enable the transmission of larger amounts of data per channel. Consequently, DNA k-bases, rather than single DNA bases, can be employed to encode higher data volumes over wide-bandwidth systems. This motivates the fundamental problem of establishing a robust correspondence between elements of algebraic structures (such as rings or fields) and DNA k-bases.

The reversibility problem arises when defining a correspondence between double strings and elements of a set with more than four elements. For example, the codeword $(1 + u + v + vw, u + w, u + v^2)$, which corresponds to the DNA string ``$GAACGCACCGCG$" (see Table 1), has reverse $(u + v^2, u + w, 1 + u + v + vw)$, corresponding to ``$CGCGGCACGAAC$". However, ``$CGCGGCACGAAC$" is not the reverse of ``$GAACGCACCGCG$", whose true reverse is ``$CGCGGTGCGTTC$". To overcome this issue, we introduce a novel structure called θ-sets over R. This method is an improved version of the method presented in [5].

Let $S_{D_4} = \{A, T, G, C\}$ denote the DNA alphabet and the Watson-Crick complement is defined by

$$A^c = T,\ T^c = A,\ C^c = G, G^c = C.$$

Now, let $\omega = (\omega_0, \dots, \omega_{n-1})$ be a codeword. Then its complement is

$$\omega^c = (\omega_0^c, \dots, \omega_{n-1}^c),$$

and its reverse-complement is

$$\omega^{rc} = (\omega_{n-1}^c, \dots, \omega_0^c).$$

The elements of R are transformed into pairs over $\mathcal{R}$ using the Gary map defined over R. To find the DNA fragment counterparts of these elements, an automorphism, such as the following, is used.

Define,

$$\psi: \mathcal{R} \longrightarrow \{A, T, G, C\}^2$$
$$a \longrightarrow \psi(a).$$

Table 1. Shows the DNA $2-$mers ($2-$basis) corresponding to the elements over $\mathcal{R}$ [19].

AA	0	AT	v	AG	1	GA	$1+u+v$
TT	v^2	TA	$v+v^2$	TC	$1+v^2$	CT	$1+u+v+v^2$
GG	$u+v$	GC	u	GT	$1+v$	TG	$1+u+v^2$
CC	$u+v+v^2$	CG	$u+v^2$	CA	$1+v+v^2$	AC	$1+u$

This mapping can be extended for n-coordinates, for $c = (c_0, c_1, \dots, c_{n-1}) \in R^n, \psi(c) = (\psi(c_0), \psi(c_1), \dots, \psi(c_{n-1}))$. The correspondences in the table involve a situation that requires solving the reversibility problem using the properties of the θ-set over R.

Definition 3.3 Let C be a code over R with length n and $c = (c_0, c_1, \dots, c_{n-1})$ be a codeword in C. We define Ψ as follows:

$$\Psi: C \longrightarrow \{A, T, G, C\}^4$$
$$(c_0, c_1, \dots, c_{n-1}) \longrightarrow (\psi(c_0), \psi(c_1), \dots, \psi(c_{n-1}))$$

Then, $\Psi(C)$ is a DNA code.

When examining Table 1, let a be an element of $\mathcal{R}$. The complement of $\psi(a)$ is $\psi(a+v^2)$, and its DNA inverse is $\psi((1+u+v)a)$. Expanding this expression, if $c = a + bw$ is an element of the ring R, where a, b in $\mathcal{R}$, then $\Phi(c) = (a, a+b)$ is its image over $\mathcal{R}$. Continuing with the above expression, the DNA mapping of the element c over the ring R is obtained as $\psi(a)\psi(a+b)$ [19].

Example 3.4 Let 1 be an element in $\mathcal{R}$. Then, $\psi(1) = AG$. The complement of $\psi(1)$ is $\psi(1+v^2) = TC$ and DNA inverse of $\psi(1)$ is $\psi((1+u+v)1) = GA$.

Let $c = 1 + u + vw$. Then, its image over $\mathcal{R}$ is $\Phi(c) = (1+u, 1+u+v)$. For $(1+u, 1+u+v)$, the corresponding $4-$mers is $\psi(1+u, 1+u+v) = \psi(1+u)\psi(1+u+v) = ACGA$.

Now, we want to define a generator matrix for a DNA code over R to solve the DNA reversibility problem.

The θ transformation is required when creating linearly independent rows used in the generator matrix. The θ transformation mentioned in the definition only affects the polynomial coefficients.

Definition 3.5 Let $g(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ be a polynomial over R. Then $\theta(g(x)) = \theta(a_0) + \theta(a_1)x + \dots + \theta(a_{n-1})x^{n-1}$.

Definition 3.6 Let $g_1(x)$ and $g_2(x)$ be two polynomials with $deg\ g_1(x) = t_1,\ deg\ g_2(x) = t_2$, both dividing $x^n - 1$ over $\mathcal{R}$.
Let $t = min\{n - t_1, n - t_2\}$ and define

$$g(x) = w\, g_1(x) + (1+w)g_2(x)$$

over R. The set $\mathcal{E}(g)$ is called a θ −set and is defined as $\mathcal{E}(g) = \mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{t-1}$
where

$$\mathcal{E}_i = \begin{cases} x^i g(x), & if\ i\ is\ even, \\ x^i \theta\big(g(x)\big), & if\ i\ is\ odd. \end{cases}$$

Then, $\mathcal{E}(g)$ generates a linear code C over R, denoted by

$$C = \langle \mathcal{E}(g) \rangle.$$

In this paper, the notation $\langle \mathcal{E}(g) \rangle$ denotes the R module generated by $\mathcal{E}(g)$.

Let $g(x) = a_0 + a_1 x + \cdots + a_t\, x^t$ be a polynomial over R. The R-submodule generated by $\mathcal{E}(g)$ can be represented by the rows of the following matrix:

$$\mathcal{E}(g) = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_t & 0 & 0 & \ldots & 0 \\ 0 & \theta(a_0) & \theta(a_1) & \ldots & \ldots & \theta(a_t) & 0 & \ldots & 0 \\ 0 & 0 & a_0 & a_1 & \ldots & a_t & 0 & \ldots & 0 \\ 0 & 0 & 0 & \theta(a_0) & \theta(a_0) & \ldots & \ldots & \theta(a_0) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Theorem 3.7 Let $g(x)$ be a self-reciprocal polynomial and $deg\big(g(x)\big) = t \geq 2$ and let t be an even integer. If $C = \langle \mathcal{E}(g) \rangle$ is a code, then $\Psi(C)$ is a DNA code.

Theorem 3.8 Let $g_1(x)$ and $g_2(x)$ be self-reciprocal polynomials that divide $x^n - 1$ over $\mathcal{R}$, with degrees t_1 and t_2, respectively. Let $t \geq 2$ be an even integer.

Let

$$g(x) = w\, g_1(x) + (1+w)g_2(x).$$

- If $deg\ g_1(x) = deg\ g_2(x)$, then

$$|\mathcal{E}(g)| = 256^{t}$$

- If $deg\ g_1\ (x) = 0\ or\ deg\ g_2(x) = 0$, then

$$|\mathcal{E}(g)| = 16^{t},$$

- If $deg\ g_1(x) \neq\ deg\ g_2(x)$ and $2 \leq\ s = t_1 - t_2$ is even, then

$g(x) = w\, g_1(x) + (1+w)x^{\frac{s}{2}} g_2\ (x)$ for $deg\ g_1(x) > deg\ g_2(x)$,

$g(x) = w\, x^{\frac{s}{2}} g_1(x) + (1+w) g_2(x)$ for $deg\; g_1(x) < deg\; g_2(x)$,

and

$|\mathcal{E}(g)| = 256^{\mathrm{t}}$.

In both cases, $C = \langle\, \mathcal{E}(g) \rangle$ is a linear code over R. Then, $\Psi(C)$ is a reversible DNA code.

Proof. Most of the assertions follow directly from the algebraic structures introduced earlier. In particular, for a DNA code $C = \langle \mathcal{E}(g) \rangle$ over R, the reverse of any codeword is again contained in C.
This follows from the identity

$$\left(\varphi\left(\sum_i \alpha_i \mathcal{E}_i\right)\right)^r = \alpha\left((1+u+v)\left(\sum_i \theta(\alpha_i)\, \mathcal{E}_{t-1-i}\right)\right)$$

where $\alpha_i \in R$ and $0 \leq i \leq t-1$.

The factor $x^{s/2}$ is introduced to balance the degree difference between $g_1(x)$ and $g_2(x)$, ensuring that the resulting polynomial $g(x)$ remains self-reciprocal, which is necessary for the reversibility of the associated DNA code.

Example 3.9 Let $g_1(x) = 1 + x^3 + x^6$ and $g_2(x) = 1 + x + x^2$ be two polynomials, both of which divide $x^9 - 1$ over $\mathcal{R}$. Then,

$$g(x) = w\, g_1(x) + (w+1)x^2\, g_2(x) = w\,(w+1)x^2 + x^3 + (w+1)x^4 + w\, x^6.$$

The matrix $\mathcal{E}(g)$ is obtained as follows:

$$\mathcal{E}(g) = \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix} = \begin{pmatrix} w & 0 & w+1 & 1 & w+1 & 0 & w & 0 & 0 \\ 0 & w+1 & 0 & w & 1 & w & 0 & w+1 & 0 \\ 0 & 0 & w & 0 & w+1 & 1 & w+1 & 0 & w \end{pmatrix}$$

The code C generated by $\mathcal{E}(g)$ yields a linear code over R. Now, let us consider the generator matrix of C. Let $\alpha_0 = 1+v,\ \alpha_1 = u$, and $\alpha_2 = 1$. Then, let us generate a codeword as

$$\begin{aligned} &\alpha_0 \mathcal{E}_0 + \alpha_1 \mathcal{E}_1 + \alpha_2 \mathcal{E}_2 \\ &\qquad = (1+\mathrm{v})\mathrm{w} + (\mathrm{u}+\mathrm{uw})\mathrm{x} + (1+\mathrm{v}+\mathrm{vw})\mathrm{x}^2 + (1+\mathrm{v}+\mathrm{uw})\mathrm{x}^3 \\ &\qquad + (\mathrm{u}+\mathrm{v}+\mathrm{vw})\mathrm{x}^4 + (1+\mathrm{uw})\mathrm{x}^5 + (1+\mathrm{vw})\mathrm{x}^6 + (\mathrm{u}+\mathrm{uw})\mathrm{x}^7 + \mathrm{wx}^8 \end{aligned}$$

The corresponding codeword is

$$c_1 = ((1+v)w\,, u+uw, 1+v+vw, 1+v+uw, u+v+vw, 1+uw, 1+vw, u + uw, w).$$

Hence,

$$\Psi(c_1) = AAGTGCAAGTAGGTGAGGGCAGACAGGTGCAAAAAG.$$

Now, we know that the reverse of $\Psi(c_1)$ is

$$(\Psi(\alpha_0\,\mathcal{E}_0 + \alpha_1\,\mathcal{E}_1 + \alpha_2\,\mathcal{E}_2\,))^r = \Psi((1+u+v)(\theta(\alpha_0)\mathcal{E}_2 + \theta(\alpha_1)\mathcal{E}_1 + \theta(\alpha_0)\mathcal{E}_2))$$

Then,

$$\begin{aligned}\alpha_0\mathcal{E}_2 + \alpha_1\mathcal{E}_1 + \alpha_2\mathcal{E}_0 &= w + (u+uw)x + (1+vw)x^2 + (1+uw)x^3 + (u+v+vw)x^4 \\ &+ (1+v+uw)x^5 + (1+v+vw)x^6 + (u+uw)x^7 + (1+v)wx^8.\end{aligned}$$

The corresponding codeword is

$$c_2 = (w, u+uw, 1+vw, 1+uw, u+v+vw, 1+v+uw, 1+v+vw, u+uw, (1+v)).$$

Thus,

$$\Psi(c_2) = GAAAAACGTGGACAGACGGGAGTGGATAAACGTGAA.$$

Corollary 3.10 Let $C = \langle\mathcal{E}(g)\rangle$ be a linear code over R and $\Psi(C)$ be a reversible DNA code. If

$$v^2\left(\frac{x^n-1}{x-1}\right) \in C,$$

then $\Psi(C)$ is a reversible-complement DNA code.

Example 3.11 We know that $g(x) = 1 + x^3 + x^6$ is a self-reciprocal polynomial over R and the code $C = \langle\mathcal{E}(g)\rangle$ contains the codeword $v^2\mathbf{1}$. Therefore, $\Psi(C)$ is a reversible-complement DNA code.

Corollary 3.12 Suppose that $g_1(x)$ and $g_2(x)$ are self-reciprocal polynomials over $\mathcal{R}$ such that $g_1(x)$ and $g_2(x)$ divide $x^n - 1$. Let $C = \langle\mathcal{E}(g)\rangle$ be a linear code over R and assume that $\Psi(C)$ is a reversible DNA code. If we add the polynomial

$$v^2\left(\frac{x^n-1}{x-1}\right) \in C$$

to the generating set $\mathcal{E}(g)$, then $\Psi(C)$ is also a reversible-complement DNA code.

Theorem 3.13 Let $g_1(x)$be a self-reciprocal polynomial over R such that $g_1(x)|x^n-1$, and define

$$g(x) = wg_1(x) + (w+1)g_1(x)$$

over R. Assume that $C = \langle g(x)\rangle$ is a reversible cyclic code over R and that $\Psi(C)$ is a reversible DNA code. If $g(x)$ is not divisible by $x-1$, then $\Psi(C)$ is a reversible-complement DNA code.

Proof. Let $dim(C) = k$. Since C is a linear cyclic code, it admits a generator matrix whose rows are

$$g(x), xg(x), \ldots, x^{k-1}g(x).$$

Using the θ-set $\mathcal{E}(g)$ we compute

$$\Psi\left(\sum_i \alpha_i x^i g(x)\right)^r = \Psi\left(\sum_i \theta(\alpha_i) x^{k-1-i} g(x)\right)$$

where $\alpha_i \in R$ and $0 \leq i \leq k-1$. Since $g(x) = wg_1(x) + (w+1)g\_1(x)$ and all coefficients of $g(x)$ lie in $\mathcal{R}$, the reversibility property in the DNA sense follows immediately. Consequently, the generator matrix of the linear cyclic code may be replaced by the θ-set $\mathcal{E}(g)$, as θ leaves the coefficients unchanged.

Finally, the condition $g(x)$ is not divisible by $x - 1$ implies that

$$v^2(1 + x + \cdots + x^{n-1}) \in C.$$

Hence, by Corollary 3.10, $\Psi(C)$ yields a reversible-complement DNA code.

4. Conclusion

In this study, DNA codes over a finite ring R are investigated. By introducing a θ-automorphism defined on the ring $R = F_2[u, v, w]/\langle u^2 = v^2, uv = 0, w^2 = w \rangle$, a new algebraic framework is developed, through which generator matrices for linear codes over R are constructed. Using the proposed structure, reversible DNA codes, reversible-complement DNA codes, and cyclic reversible DNA codes are systematically obtained. The results demonstrate the effectiveness of ring-based coding theory in DNA code design and highlight the role of the θ-automorphism in the construction and characterization of these classes of DNA codes. These results provide a basis for developing structured DNA codes over algebraic rings and may stimulate further progress in DNA computing and coding theory.

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