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Homotopies of Crossed Modules of Lie Algebras

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Abstract

In this paper we will define a notion of homotopy of Lie crossed module morphisms. Then we construct a groupoid structure of Lie crossed module morphisms and their homotopies.

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1. Introduction

Crossed modules were firstly introduced by J.H.C Whitehead in his work on combinatorial homotopy theory [8]. They have found important role in many areas of mathematics (including homotopy theory, homology and cohomology of groups, algebraic K-theory, cyclic homology, combinatorial group theory and differential geometry). Kassel and Loday [6] introduced crossed modules of Lie algebras as computational algebraic objects equivalent to simplicial Lie algebras with associated Moore complex of length 1.
The homotopy relation between crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$ can be equivalently addressed either by considering natural functorial path objects for $\mathscr{P}'$ or cylinder objects for $\mathscr{P}$. It yields, given any two crossed modules $\mathscr{P}$ and $\mathscr{P}'$, a groupoid of morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$ and their homotopies. In addition , the homotopy relation between crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$ is an equivalence relation in the general case, with no restriction on $\mathscr{P}$ or $\mathscr{P}'$. This should be compared with what would be guaranteed from the model category [4] point of view, where we would expect homotopy of maps $\mathscr{P} \longrightarrow \mathscr{P}'$ to be an equivalence relation only when $\mathscr{P}$ is a cofibrant (given that any object is fibrant). The well-known model category structure in the category of crossed modules [3], obtained by transporting the usual model category structure of the category of simplicial sets, $\mathscr{P} = (\partial : M \longrightarrow P)$ is cofibrant if and only if P is a free group ([7]).
Whitehead [8] explored homotopies of morphisms of his "homotopy systems" and this was put in the general context of crossed complexes of groupoids by Brown and Higgins [2]. In this paper we will define a notion of homotopy for morphisms of crossed modules of Lie algebras and we will show that if $\mathscr{P}$ and $\mathscr{P}'$ are crossed modules of Lie algebras, without any restriction on $\mathscr{P}$ or $\mathscr{P}'$, then we have a groupoid of crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$ and their homotopies, similar to the group case [5] and the commutative algebra case [1].

2. Crossed Modules

J.H.C Whitehead [8] described crossed modules in his investigations into the algebraic structure of relative homotopy groups. In this section, we recall the definition of crossed modules of Lie algebras given by Kassel and Loday [6].
Let M and P be two Lie algebras. An action of P on M is a bilinear map $P \times M \longrightarrow M$, $(p,m) \longmapsto p \cdot m$ satisfying

$$[p,p'] \cdot m = p \cdot (p' \cdot m) - p' \cdot (p \cdot m)$$

$$p \cdot [m,m'] = [p \cdot m, m'] + [m, p \cdot m']$$

for all $m,m' \in M$, $p,p' \in P$. For instance, if P is a subalgebra of a Lie algebra Q (including possibly the case $P = Q$) and if M is an ideal of Q, then Lie bracket of Q yields an action of P on M.
A crossed module of Lie algebras is a Lie homomorphism $\partial : M \longrightarrow P$ together with an action of P on M such that for all $m,m' \in M$, $p \in P$

CM1) $\partial(p \cdot m) = [p, \partial(m)]$

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and

CM2) $\partial m \cdot m' = [m,m']$.

The last condition is called the Peiffer identity. We can denote such a crossed module by $\mathscr{P} = (M,P,\partial)$ and say Lie crossed module, shortly. A Lie crossed module morphism from $\mathscr{P} = (M,P,\partial)$ to $\mathscr{P}' = (M',P',\partial')$ is a pair of Lie algebra morphisms,

$\theta : M \longrightarrow M'$, $\psi : P \longrightarrow P'$

such that

$\theta(p \cdot m) = \psi(p) \cdot \theta(m)$ and $\partial'\theta(m) = \psi\partial(m)$.

Therefore we can obtain a category of Lie crossed modules. It will usually be denoted by **LXmod.**
A standart example of a Lie crossed module is any ideal I in Lie algebra P giving an inclusion map

$inc : I \longrightarrow P.$

Then (I,P,inc) is a Lie crossed module. Conversely, given any crossed module $\partial : M \longrightarrow P$, one can easily verify that $\partial M = I$ is an ideal of P.

3. Homotopies of Lie crossed module morphisms

Whitehead [8] explored homotopies of morphisms of his "homotopy systems" and this was put in the general context of crossed complexes of groupoids by Brown and Higgins [2]. In this section, we define a notion of homotopy for Lie crossed module morphism .

Definition 3.1. *Let $\mathscr{P} = (M,P,\partial)$ and $\mathscr{P}' = (M',P',\partial')$ be Lie crossed modules, $f = (f_1, f_0)$ and $g = (g_1, g_0)$ be Lie crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$. If there is a linear map $d : P \longrightarrow M'$ such that,*

$$\begin{aligned} g_0(p) &= f_0(p) + \partial' d(p) \\ g_1(m) &= f_1(m) + d\partial(m) \end{aligned}$$

for $m \in M$ and $p \in P$, then we say that d is a homotopy connecting f to g and denoted by $d : f \simeq g$ or $f \xrightarrow{d} g$.

Definition 3.2. *Let $\mathscr{P} = (M,P,\partial)$ and $\mathscr{P}' = (M',P',\partial')$ be Lie crossed modules and $f = (f_1, f_0)$ be a Lie crossed module morphism $\mathscr{P} \longrightarrow \mathscr{P}'$. Then a linear map $d : P \longrightarrow M'$ satisfying for all $p, p' \in P$*

$$d[p, p'] = f_0(p) \cdot d(p') - f_0(p') \cdot d(p) + [d(p), d(p')]$$

is called an f_0-derivation .

Proposition 3.1. *Let $\mathscr{P} = (M,P,\partial)$ and $\mathscr{P}' = (M',P',\partial')$ be Lie crossed modules and $f = (f_1, f_0)$ be a Lie crossed module morphism $\mathscr{P} \longrightarrow \mathscr{P}'$. If $d : P \longrightarrow M'$ is an f_0-derivation, then the maps $g_0 : P \longrightarrow P'$, $g_0(p) = f_0(p) + \partial' d(p)$ for all $p \in P$ and $g_1 : M \longrightarrow M'$, $g_1(m) = f_1(m) + d\partial(m)$ for all $m \in M$ are Lie algebra morphisms.*

Proof. For $p, p' \in P$,

$$\begin{aligned} g_0(p + p') &= f_0(p + p') + \partial' d(p + p') \\ &= f_0(p) + f_0(p') + \partial'(d(p) + d(p')) \\ &= f_0(p) + f_0(p') + \partial' d(p) + \partial' d(p') \\ &= g_0(p) + g_0(p'), \end{aligned}$$

and

$$\begin{aligned} g_0[p, p'] &= f_0[p, p'] + \partial' d[p, p'] \\ &= [f_0(p), f_0(p')] + \partial'(f_0(p) \cdot d(p') - f_0(p') \cdot d(p) \\ &\quad + [d(p), d(p')]) \\ &= [f_0(p), f_0(p')] + \partial'(f_0(p) \cdot d(p')) - \partial'(f_0(p') \cdot d(p)) \\ &\quad + \partial'([d(p), d(p')]) \\ &= [f_0(p), f_0(p')] + [f_0(p), \partial' d(p')] - [f_0(p'), \partial' d(p)] \\ &\quad + [\partial' d(p), \partial' d(p')] \\ &= [f_0(p), f_0(p')] + [f_0(p), \partial' d(p')] + [\partial' d(p), f_0(p')] \\ &\quad + [\partial' d(p), \partial' d(p')] \\ &= [f_0(p) + \partial' d(p), f_0(p') + \partial' d(p')], \\ &= [g_0(p), g_0(p')]. \end{aligned}$$

Thus g_0 is a Lie algebra morphism.
For $m, m' \in M$,

$$\begin{aligned} g_1(m + m') &= f_1(m + m') + d\partial(m + m') \\ &= f_1(m) + f_1(m') + d(\partial(m) + \partial(m')) \\ &= f_1(m) + f_1(m') + d\partial(m) + d\partial(m') \\ &= g_1(m) + g_1(m'), \end{aligned}$$

and

$$\begin{array}{rcl} g_1[m,m'] & = & f_1[m,m'] + d\partial[m,m'] \\ & = & [f_1(m), f_1(m')] + d[\partial(m), \partial(m')] \\ & = & [f_1(m), f_1(m')] + f_0(\partial(m)) \cdot d(\partial(m')) \\ & & -f_0(\partial(m')) \cdot d(\partial(m)) + [d(\partial(m)), d(\partial(m'))] \\ & = & [f_1(m), f_1(m')] + \partial'(f_1(m)) \cdot d(\partial(m')) \\ & & -\partial'(f_1(m')) \cdot d(\partial(m)) + [d(\partial(m)), d(\partial(m'))] \\ & = & [f_1(m), f_1(m')] + [f_1(m), d(\partial(m'))] - [f_1(m'), d(\partial(m))] \\ & & +[d(\partial(m)), d(\partial(m'))] \\ & = & [f_1(m), f_1(m')] + [f_1(m), d(\partial(m'))] + [d(\partial(m)), f_1(m')] \\ & & +[d(\partial(m)), d(\partial(m'))] \\ & = & [f_1(m) + d(\partial(m)), f_1(m') + d(\partial(m'))] \\ & = & [g_1(m), g_1(m')] \end{array}$$

thus g_1 is a Lie algebra morphism. □

Proposition 3.2.

$g_0\partial = \partial' g_1$.

Proof. For $m \in M$, we can write

$$\begin{array}{rcl} (g_0\partial)(m) & = & g_0(\partial(m)) \\ & = & f_0(\partial(m)) + \partial' d(\partial(m)) \\ & = & \partial'(f_1(m)) + \partial'(d\partial(m)) \\ & = & \partial'(f_1(m) + d\partial(m)) \\ & = & \partial'(g_1(m)) \\ & = & (\partial' g_1)(m), \end{array}$$

and so we have

$g_0\partial = \partial' g_1$.

□

Proposition 3.3. *For $p \in P$ and $m \in M$,*

$g_1(p \cdot m) = g_0(p) \cdot g_1(m)$.

Proof.

$$\begin{array}{rcl} g_1(p\cdot m) & = & f_1(p\cdot m) + d\partial(p\cdot m) \\ & = & f_0(p)\cdot f_1(m) + d[p, \partial(m)] \\ & = & f_0(p)\cdot f_1(m) + f_0(p)\cdot d\partial(m) - f_0\partial(m)\cdot d(p) + [d(p), d\partial(m)] \\ & = & f_0(p)\cdot f_1(m) + f_0(p)\cdot d\partial(m) - \partial' f_1(m)\cdot d(p) + [d(p), d\partial(m)] \\ & = & f_0(p)\cdot f_1(m) + f_0(p)\cdot d\partial(m) - [f_1(m), d(p)] + [d(p), d\partial(m)] \\ & = & f_0(p)\cdot f_1(m) + f_0(p)\cdot d\partial(m) + [d(p), f_1(m)] + [d(p), d\partial(m)] \\ & = & f_0(p)\cdot f_1(m) + f_0(p)\cdot d\partial(m) + [d(p), f_1(m) + d\partial(m)] \\ & = & f_0(p)\cdot f_1(m) + f_0(p)\cdot d\partial(m) + \partial' d(p)\cdot (f_1(m) + d\partial(m)) \\ & = & (f_0(p) + \partial' d(p))\cdot (f_1(m) + d\partial(m)) \\ & = & g_0(p)\cdot g_1(m). \end{array}$$

□

By the above propositions, we can give the following theorem:

Corollary 3.1. *Let $\mathscr{P} = (M, P, \partial)$ and $\mathscr{P}' = (M', P', \partial')$ be Lie crossed modules and $f = (f_1, f_0)$ be a Lie crossed module morphism $\mathscr{P} \longrightarrow \mathscr{P}'$. If $d : P \longrightarrow M'$ is an f_0-derivation, then the map $g = (g_1, g_0) : \mathscr{P} \longrightarrow \mathscr{P}'$ is a Lie crossed module morphism.*

Corollary 3.2. *Let $\mathscr{P} = (M, P, \partial)$ and $\mathscr{P}' = (M', P', \partial')$ be crossed modules of Lie algebras and $f = (f_1, f_0), g = (g_1, g_0)$ be crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$. Then the f_0-derivation $d : P \longrightarrow M'$ satisfying for all $p, p' \in P$*

$d[p, p'] = f_0(p) \cdot d(p') - f_0(p') \cdot d(p) + [d(p), d(p')]$

is a homotopy connecting f to g.

4. Groupoid Structure of Lie Crossed module morphisms and their homotopies

In this section we construct a groupoid structure whose objects are the Lie crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$, with morphisms being the homotopies between them.

Lemma 4.1. *Let $\mathscr{P} = (M,P,\partial)$ and $\mathscr{P}' = (M',P',\partial')$ be Lie crossed modules and $f = (f_1, f_0)$ be a Lie crossed module morphism $\mathscr{P} \longrightarrow \mathscr{P}'$. Then the null function $0 : P \longrightarrow M'$, $0(p) = 0_{M'}$ defines an f_0 -derivation connecting f to f.*

Lemma 4.2. *Let $f = (f_1, f_0)$ and $g = (g_1, g_0)$ be Lie crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$ and d be an f_0 -derivation connecting f to g. Then the linear map $\overline{d} = -d : P \longrightarrow M'$ with $\overline{d}(p) = -d(p)$ is a g_0 -derivation connecting g to f.*

Proof. Since d is an f_0-derivation connecting f to g, we have

$$g_0(p) = f_0(p) + \partial' d(p) \text{ and } g_1(m) = f_1(m) + d\partial(m) \tag{4.1}$$

on the other hand, we can write

$$\begin{aligned}
\overline{d}[p,p'] &= -d[p,p'] \\
&= -(f_0(p)\cdot d(p') - f_0(p')\cdot d(p) + [d(p),d(p')]) \\
&= -f_0(p)\cdot d(p') + f_0(p')\cdot d(p) - [d(p),d(p')] \\
&\quad -[d(p),d(p')] + [d(p),d(p')] \\
&= -f_0(p)\cdot d(p') - \partial' d(p)\cdot d(p') + f_0(p')\cdot d(p) \\
&\quad +\partial' d(p')\cdot d(p) + [d(p),d(p')] \\
&= f_0(p)\cdot(-d(p')) + \partial' d(p)\cdot(-d(p')) - f_0(p')\cdot(-d(p)) \\
&\quad -\partial' d(p')\cdot(-d(p)) + [-d(p),-d(p')] \\
&= (f_0(p) + \partial' d(p))\cdot(-d(p')) - (f_0(p') + \partial' d(p'))\cdot(-d(p)) \\
&\quad +[-d(p),-d(p')] \\
&= g_0(p)\cdot(-d(p')) - g_0(p')\cdot(-d(p)) + [-d(p),-d(p')] \\
&= g_0(p)\cdot\overline{d}(p') - g_0(p')\cdot\overline{d}(p) + [\overline{d}(p),\overline{d}(p')].
\end{aligned}$$

Thus $\overline{d}$ is a g_0-derivation. □

Lemma 4.3. *(Concatenation of derivations) Let f, g and h be Lie crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$, d be an f_0-derivation connecting f to g, and d' be a g_0 -derivation connecting g to h. Then the linear map $(d+d') : P \longrightarrow M'$ such that $(d+d')(p) = d(p) + d'(p)$, defines an f_0-derivation (therefore a homotopy) connecting f to h.*

Proof. Since d is an f_0-derivation connecting f to g, we have

$$\begin{aligned}
g_0(p) &= f_0(p) + \partial' d(p) \\
g_1(m) &= f_1(m) + d\partial(m).
\end{aligned}$$

Since $d^{'}$ is an g_0-derivation connecting g to h, we have

$$\begin{aligned}
h_0(p) &= g_0(p) + \partial' d^{'}(p) \\
h_1(m) &= g_1(m) + d^{'}\partial(m).
\end{aligned}$$

Let us see that $d+d'$ satisfies the condition for it to be an f_0 derivation:

$$\begin{aligned}
(d+d')[p,p'] &= d[p,p'] + d'[p,p'] \\
&= f_0(p)\cdot d(p') - f_0(p')\cdot d(p) + [d(p),d(p')] \\
&\quad +g_0(p)\cdot d'(p') - g_0(p')\cdot d'(p) + [d'(p),d'(p')] \\
&= f_0(p)\cdot d(p') - f_0(p')\cdot d(p) + [d(p),d(p')] \\
&\quad +(f_0(p) + \partial' d(p))\cdot d'(p') - (f_0(p') + \partial' d(p'))\cdot d'(p) \\
&\quad +[d'(p),d'(p')] \\
&= f_0(p)\cdot d(p') - f_0(p')\cdot d(p) + [d(p),d(p')] + f_0(p)\cdot d'(p') \\
&\quad +\partial' d(p)\cdot d'(p') - f_0(p')\cdot d'(p) - \partial' d(p')\cdot d'(p) \\
&\quad +[d'(p),d'(p')] \\
&= f_0(p)\cdot d(p') - f_0(p')\cdot d(p) + [d(p),d(p')] + f_0(p)\cdot d'(p') \\
&\quad +[d(p),d'(p')] - f_0(p')\cdot d'(p) - [d(p'),d'(p)] + [d'(p),d'(p')] \\
&= f_0(p)\cdot(d(p') + d'(p')) - f_0(p')\cdot(d(p) + d'(p)) \\
&\quad +[d(p) + d'(p), d(p') + d'(p')] \\
&= f_0(p)\cdot(d+d')(p') - f_0(p')\cdot(d+d')(p) + [(d+d')(p),(d+d')(p')].
\end{aligned}$$

for all $p, p' \in P$. Therefore $(d+d')$ is an f_0 derivation connecting f to h. □

Theorem 4.4. *Let $\mathscr{P}$ and $\mathscr{P}'$ be two arbitrary Lie crossed modules. We have a groupoid $HOM(\mathscr{P},\mathscr{P}')$, whose objects are the Lie crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$, the morphisms being their homotopies. In particular the relation below, for Lie crossed module morphisms $\mathscr{P} \longrightarrow \mathscr{P}'$, is an equivalence relation:*

"$f \simeq g \Leftrightarrow$ there exists an f_0-derivation d connecting f to g ".

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