

# An effective numerical technique for the Rosenau-KdV-RLW equation

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## Abstract

*In this study, the Rosenau-Korteweg-de Vries-Regular Longwave (Rosenau-KdV-RLW) equation has been converted into a partial differential equation system consisting of two equations using a splitting technique. Then, numerical solutions for the Rosenau-KdV-RLW equation system have been obtained using separately both cubic and quintic B-spline finite element collocation method. For the unknowns in those equations, B-spline functions at  $x$ -position and Crank-Nicolson type finite difference approaches at time positions are used. A test problem has been chosen to check the accuracy of the proposed discretized scheme. The basic conservation properties of the Rosenau-KdV-RLW equation have been shown to be protected by the proposed numerical scheme. The results are compared with the analytical solution of the problem and the results given in the literature. For the reliability of the method the error norms  $L_2$  and  $L_\infty$  are calculated. It is seen that the proposed method gives harmonious results with exact solutions.*

**Keywords:** Rosenau-KdV-RLW, B-spline functions, collocation method, splitting technique.

## Rosenau-KdV-RLW denklemi için etkin bir sayısal teknik

## Özet

*Bu çalışmada, Rosenau Korteweg-de Vries düzenli uzun dalga (Rosenau-KdV-RLW) denklemi bir parçalama tekniği kullanılarak iki denklemden oluşan bir kısmi diferansiyel denklem sistemine dönüştürülmüştür. Daha sonra, Rosenau-KdV-RLW denklem sistemi için kübik ve kuintik B-spline sonlu eleman kollakasyon yöntemi kullanılarak sayısal çözümler önerilmiştir. Bu denklemlerdeki bilinmeyenler için  $x$ -konumunda B-spline fonksiyonlar ve zaman konumunda Crank-Nicolson tipi sonlu fark*

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yaklaşimleri kullanılmıştır. Önerilen sayısal şemaların doğruluğunu kontrol etmek için bir test problemi seçilmiştir. Rosenau-KdV-RLW denkleminin temel korunum özelliklerinin önerilen sayısal şemalar ile korunduğu görülmüştür. Elde edilen sonuçlar problemin analitik çözümü ve literatürde verilen sonuçlarla karşılaştırılmıştır. Yöntemin güvenilirliği için  $L_2$  ve  $L_\infty$  hata normları hesaplanmıştır. Önerilen yöntemin tam çözümlerle uyumlu sonuçlar verdiği görülmüştür.

**Anahtar kelimeler:** Rosenau-KdV-RLW, B-spline fonksiyonlar, kollakasyon metodu, parçalama tekniği.

## 1. Introduction

In this paper, we will deal with the numerical solution of the Rosenau-KdV-RLW equation given by

$$u_t + au_x + b(u^p)_x - cu_{xxt} + du_{xxx} + eu_{xxxxt} = 0 \quad (1)$$

subject to the initial condition

$$u(x, 0) = u_0(x), \quad x \in [x_L, x_R] \quad (2)$$

and the boundary conditions

$$\begin{aligned} u(x_L, t) = u(x_R, t) &= 0 \\ u_x(x_L, t) = u_x(x_R, t) &= 0, \quad t \in (0, T], \\ u_{xx}(x_L, t) = u_{xx}(x_R, t) &= 0 \end{aligned} \quad (3)$$

where  $x$  and  $t$  denote the spatial and time variables, respectively,  $T$  is final time,  $u(x, t)$  is the nonlinear wave profile,  $a, b, c, d$  and  $e$  are non-negative real numbers,  $u_0(x)$  is a given smooth function and  $x_L \leq 0$  and  $x_R \geq 0$  which are both large [1]. If  $u_0(x)$  tends to zero when  $x_L \leq 0$  and  $x_R \geq 0$ , then the above initial-boundary value problem is consistent, so the given boundary conditions are meaningful for the solitary solution of the Rosenau-KdV-RLW Eq. (1) is obtained by combining the Rosenau-KdV equation and Rosenau-RLW equation. For  $c=0$ , Eq. (1) becomes

$$u_t + au_x + b(u^p)_x + du_{xxx} + eu_{xxxxt} = 0 \quad (4)$$

which is the well known Rosenau-KdV equation. Eq.(4) has been solved numerically by some methods [2-4]. For  $d=0$ , Eq. (1) takes the form

$$u_t + au_x + b(u^p)_x - cu_{xxt} + eu_{xxxxt} = 0 \quad (5)$$

which is the well known Rosenau-RLW equation. The numerical solution of this equation has been studied in the past years [1, 5-7].

The Rosenau-KdV-RLW equation (1) with the initial (2) and boundary conditions given by (3) has been solved by several authors. Pan and et.al. [8] presented a new Crank-Nicolson pseudo-compact conservative numerical scheme for the Rosenau-KdV-RLW equation. Korkmaz and Dereli [9] proposed a meshfree method based on the

collocation with radial basis functions to solve the Rosenau-KdV-RLW equation. Wang and Dai [4] proposed a three-level linear conservative implicit finite difference scheme. Ghilongi and Omrani [10] introduced some high-order accurate finite difference schemes for the Rosenau-KdV-RLW equation. Foroutan and Ebadian [11] proposed the modified Chebyshev rational approximations for the Rosenau-KdV-RLW equation on the infinite intervals. Fernandez and Ramos [12] analyzed numerically Rosenau-KdV-RLW equation with second and fourth-order dissipative terms subject to homogeneous boundary conditions and initial Gaussian conditions by means of a second-order accurate trapezoidal procedure in time where the first, second, third and fourth order spatial derivatives are considered as unknowns and have been discretized by means of three-point, fourth-order accurate, compact finite difference formulae.

## 2. Mathematical model

In the present study, the numerical solutions of the Rosenau-KdV-RLW equation (1) are going to be sought using separately both cubic and quintic B-spline finite element collocation method together with the initial (2) and the boundary conditions (3). First of all, the Rosenau-KdV-RLW equation (1) is converted into a system consisting of two partial differential equations as follows

$$u_t + au_x + pbu^{p-1}u_x - cv_t + dv_x + ev_{xxt} = 0 \quad (6)$$

$$u_{xx} - v = 0 \quad (7)$$

The resulting equations (6)-(7) are going to be called the Rosenau-KdV-RLW equation system. Under these conditions, the Rosenau-KdV-RLW equation system is converted into a new system of equation given by the following initial and boundary conditions.

$$u(x, 0) = u_0(x), \quad v(x, 0) = u_0''(x), \quad x \in [x_L, x_R] \quad (8)$$

$$u(x_L, t) = u(x_R, t) = 0, \quad v(x_L, t) = v(x_R, t) = 0, \quad t \in (0, T] \quad (9)$$

In the Rosenau-KdV-RLW equations system (6)-(7), writing forward difference equations in place of derivatives with respect to time variable  $t$ , and Crank-Nicolson type finite difference approximations with respect to space variable  $x$ , and assuming  $Z = a + pbu^{p-1}$ , we obtain the following system of equations

$$\frac{u^{n+1} - u^n}{k} + Z \frac{u_x^{n+1} + u_x^n}{2} - c \frac{v^{n+1} - v^n}{k} + d \frac{v_x^{n+1} + v_x^n}{2} + e \frac{v_{xx}^{n+1} - v_{xx}^n}{k} = 0 \quad (10)$$

$$\frac{u_{xx}^{n+1} + u_{xx}^n}{2} - \frac{v^{n+1} + v^n}{2} = 0. \quad (11)$$

If we reorganize the above system of equations, we obtain the following relations between the  $n$  and  $n+1$  time levels

$$u^{n+1} + \frac{kZ}{2}u_x^{n+1} - cv^{n+1} + \frac{kd}{2}v_x^{n+1} + ev_{xx}^{n+1} = u^n - \frac{kZ}{2}u_x^n - cv^n - \frac{kd}{2}v_x^n + ev_{xx}^n \quad (12)$$

$$u_{xx}^{n+1} - v^{n+1} = -u_{xx}^n + v^n. \quad (13)$$

The approximate solutions corresponding to the exact solutions  $u(x, t)$  and  $v(x, t)$  of the Rosenau-KdV-RLW equation system (6)-(7) are going to be denoted by  $u_N(x, t)$  and  $v_N(x, t)$ , respectively. The solution domain of the problem is taken as  $[x_L, x_R] \times [0, T]$ . An uniform grid structure is constructed on the solution domain of the problem by taking  $h = x_{m+1} - x_m$  for  $m = 0(1)N - 1$  on the space domain  $[x_L, x_R]$  as  $x_L = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = x_R$  and  $k = t_{n+1} - t_n$  for  $n = 0(1)M - 1$  on the time domain  $[0, T]$  as  $0 = t_0 < t_1 < t_2 < \dots < t_{M-1} < t_M = T$ . Under these conditions, the values of  $u_N(x_m, t_n)$  and  $v_N(x_m, t_n)$  at the nodal points  $(x_m, t_n)$  are going to be denoted by  $u_m^n$  and  $v_m^n$ , respectively.

### 2.1. Scheme-I: Cubic B-spline collocation method

The first numerical scheme is going to be obtained by cubic B-spline finite element collocation method. Cubic B-spline basis functions  $\phi_m(x)$  for  $m = -1(1)N + 1$  are defined as follows

$$\phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3 & , [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3 & , [x_{m-1}, x_m] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3 & , [x_m, x_{m+1}] \\ (x_{m+2} - x)^3 & , [x_{m+1}, x_{m+2}] \\ 0 & , otherwise \end{cases} \quad (14)$$

[13]. Since the set of cubic B-spline basis functions  $\{\phi_{-1}(x), \phi_0(x), \dots, \phi_{N+1}(x)\}$  constitutes a base for the smooth functions defined over the domain  $[x_L, x_R]$ , the approximate solutions  $u_N(x, t_n)$  and  $v_N(x, t_n)$  can be written as follows in terms of the cubic B-spline basis functions

$$u_N(x, t) = \sum_{i=-1}^{N+1} \phi_i(x) \delta_i(t), \quad v_N(x, t) = \sum_{i=-1}^{N+1} \phi_i(x) \sigma_i(t). \quad (15)$$

Here  $\delta_i(t)$  and  $\sigma_i(t)$  are time dependent parameters to be determined. Since the cubic B-spline basis functions and their derivatives are zero outside the domain  $[x_{m-2}, x_{m+2}]$ , the approximations over the typical element  $[x_m, x_{m+1}]$  can be written in the following form

$$u_N(x, t) = \sum_{i=m-1}^{m+2} \phi_i(x) \delta_i(t), \quad v_N(x, t) = \sum_{i=m-1}^{m+2} \phi_i(x) \sigma_i(t). \quad (16)$$

If we apply the local coordinate transformation  $h\xi = x - x_m$ ,  $0 \leq \xi \leq 1$  on the typical element  $[x_m, x_{m+1}]$ , the B-spline basis functions on the new interval  $[0, 1]$  in terms of local variable  $\xi$  can be written as follows

$$\begin{aligned} \phi_{m-1} &= (1 - \xi)^3, \\ \phi_m &= 1 + 3(1 - \xi) + 3(1 - \xi)^2 - 3(1 - \xi)^3, \\ \phi_{m+1} &= 1 + 3\xi + 3\xi^2 - 3\xi^3, \\ \phi_{m+2} &= \xi^3. \end{aligned} \quad (17)$$

The nodal values of the approximate functions  $u_N(x_m, t) = u_m$ ,  $v_N(x_m, t) = v_m$  and their derivatives up to second order with respect to space variable  $x$  for  $m = 0(1)N$  in terms of parameters  $\delta_m$  and  $\sigma_m$  are obtained as follows

$$\begin{aligned}
 u_m &= \delta_{m-1} + 4\delta_m + \delta_{m+1}, & v_m &= \sigma_{m-1} + 4\sigma_m + \sigma_{m+1}, \\
 u'_m &= \frac{3}{h}(-\delta_{m-1} + \delta_{m+1}), & v'_m &= \frac{3}{h}(-\sigma_{m-1} + \sigma_{m+1}), \\
 u''_m &= \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}), & v''_m &= \frac{6}{h^2}(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}).
 \end{aligned} \tag{18}$$

Here the superscript denotes the derivative with respect to variable  $x$ . If we write these pointwise values in Eqs. (12)-(13) and rearrange them for  $m = 0(1)N$ , we obtain the following set of algebraic equations

$$\begin{aligned}
 A_1\delta_{m-1}^{n+1} + A_2\delta_m^{n+1} + A_3\delta_{m+1}^{n+1} + B_1\sigma_{m-1}^{n+1} + B_2\sigma_m^{n+1} + B_3\sigma_{m+1}^{n+1} = \\
 C_1\delta_{m-1}^n + C_2\delta_m^n + C_3\delta_{m+1}^n + D_1\sigma_{m-1}^n + D_2\sigma_m^n + D_3\sigma_{m+1}^n
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 E_1\delta_{m-1}^{n+1} + E_2\delta_m^{n+1} + E_3\delta_{m+1}^{n+1} + F_1\sigma_{m-1}^{n+1} + F_2\sigma_m^{n+1} + F_3\sigma_{m+1}^{n+1} = \\
 G_1\delta_{m-1}^n + G_2\delta_m^n + G_3\delta_{m+1}^n + H_1\sigma_{m-1}^n + H_2\sigma_m^n + H_3\sigma_{m+1}^n.
 \end{aligned} \tag{20}$$

The values of coefficients  $A_i, B_i, C_i, D_i, E_i, F_i, G_i$  and  $H_i$  for  $i = 1(1)3$  are given in the Table 1. This system of equations consists of  $(2N + 6)$  unknowns and  $(2N+2)$  equations. If the unknowns  $\delta_{-1}, \delta_{N+1}, \sigma_{-1}, \sigma_{N+1}$  encountered for values of  $m=0,N$

Tablo 1. The values of the coefficients of the equation systems given by Eqs. (19)-(20)

$i$	$A_i$	$B_i$	$C_i$	$D_i$	$E_i$	$F_i$	$G_i$	$H_i$
1	$1 - \frac{3kZ_m}{2h}$	$-c - \frac{3kd}{2h} + \frac{6e}{h^2}$	$A_{4-i}$	$B_{4-i}$	$\frac{6}{h^2}$	-1	$-\frac{6}{h^2}$	1
2	4	$-4c - \frac{12e}{h^2}$	$A_{4-i}$	$B_{4-i}$	$-\frac{12}{h^2}$	-4	$\frac{12}{h^2}$	4
3	$1 - \frac{3kZ_m}{2h}$	$-c + \frac{3kd}{2h} + \frac{6e}{h^2}$	$A_{4-i}$	$B_{4-i}$	$\frac{6}{h^2}$	-1	$-\frac{6}{h^2}$	1

are eliminated using the boundary conditions given by Eq.(9), a system of  $(2N+2)$  equations in  $(2N+2)$  unknowns are obtained. First of all, we write the unknowns of this system of equations in the form of  $\mathbf{d}^T = [\delta_0 \sigma_0 \delta_1 \sigma_1 \dots \delta_N \sigma_N]^T$  and arrange the both sides of the equation in such a way that the coefficients matrices are in agreement with  $\mathbf{d}^T$ . The newly obtained  $(2N+2)$  dimensional square matrices  $\mathbf{A}$  and  $\mathbf{B}$  are used in the system of equations  $\mathbf{A}\mathbf{d}^{n+1} = \mathbf{B}\mathbf{d}^n$  and finally they are solved using an appropriate algorithm. The matrices  $\mathbf{A}$  and  $\mathbf{B}$  can be easily handled since they are six-band matrices. In order to be able to calculate the parameters  $\mathbf{d}^{n+1}$ , first of all, the initial parameter  $\mathbf{d}^0$  should be known. Using the initial conditions given together with Eq.(8)

$$\begin{aligned}
 (u_N)_x(x_L, 0) &= \frac{3}{h}(-\delta_{-1}^0 + \delta_1^0) = u'_0(x_L) \\
 u_N(x_m, 0) &= \delta_{m-1}^0 + 4\delta_m^0 + \delta_{m+1}^0 = u_0(x_m), & m &= 0(1)N \\
 (u_N)_x(x_R, 0) &= \frac{3}{h}(-\delta_{N-1}^0 + \delta_{N+1}^0) = u'_0(x_R)
 \end{aligned} \tag{21}$$

and

$$\begin{aligned}
 (v_N)_x(x_L, 0) &= \frac{3}{h}(-\sigma_{-1}^0 + \sigma_1^0) = u_0'''(x_L) \\
 v_N(x_m, 0) &= \sigma_{m-1}^0 + 4\sigma_m^0 + \sigma_{m+1}^0 = u_0'(x_m), \quad m = 0(1)N \\
 (v_N)_x(x_R, 0) &= \frac{3}{h}(-\sigma_{N-1}^0 + \sigma_{N+1}^0) = u_0'''(x_R)
 \end{aligned} \tag{22}$$

the above system of equations is obtained. The solution of these systems of equations is found by the initial parameter  $\mathbf{d}^0$ . The following inner iteration has been applied to the nonlinear terms of the equation 3 or 5 times to improve the approximations

$$\delta_m^* = \delta_m^n + \frac{1}{2}(\delta_m^{n+1} - \delta_m^n) \quad \text{and} \quad \sigma_m^* = \sigma_m^n + \frac{1}{2}(\sigma_m^{n+1} - \sigma_m^n). \tag{23}$$

## 2.2. Stability analysis

The stability analysis of the numerical scheme resulting from the application of the cubic B-spline finite element collocation method to the Rosenau-KdV-RLW equation system is going to be implemented by the von Neumann method. Because of this reason, in place of  $u$  in the nonlinear term  $u^{p-1}u_x$  in Eq. (6) a local constant  $\hat{Z}$  is taken. Under this condition, the term  $Z_m$  found in the coefficients  $A_i, C_i$  in Eq. (19) is going to be a constant in the form of  $a + pb\hat{Z}^{p-1}$ . Where  $i$  is the imaginary unit,  $\varphi$  is an arbitrary real number, the amplification factor  $q = q(\varphi)$  is a complex number  $\delta_m^n = Pq^n e^{im\varphi}$ ,  $\sigma_m^n = Wq^n e^{im\varphi}$  special solutions are written in Eqs. (19)-(20) and the Euler formula  $e^{i\varphi} = \cos\varphi + i\sin\varphi$  is used and the following homogenous equations systems is obtained

$$[(A + iZB)q - (A - iZB)]P + [(D + iBd)q - (D - iBd)]W = 0 \tag{24}$$

$$(q + 1)(CP - AW) = 0 \tag{25}$$

where  $A = 2(\cos\varphi + 2)$ ,  $B = \frac{3k}{h}\sin\varphi$ ,  $C = \frac{12}{h^2}(\cos\varphi - 1)$ ,  $D = -Ac + Ce$ ,  $Z = \max Z_m$ . It is known that this homogeneous equation system has at least one nonzero solution when the determinant of the coefficient matrix of the system is zero. Under this condition, from equations (24) and (25) we can write

$$[-A^2 - CD - iB(AZ + Cd)]q + A^2 + CD - iB(AZ + Cd) = 0 \quad \text{or} \quad q + 1 = 0. \tag{26}$$

Then, the the amplification factor is found as follows

$$q = \frac{-A^2 - CD + iB(AZ + Cd)}{-A^2 - CD - iB(AZ + Cd)} \quad \text{or} \quad q = -1. \tag{27}$$

As a conclusion, since  $|q| = 1$ , the method is unconditionally stable.

## 2.3. Scheme-II: Quintic B-spline collocation method

The second numerical scheme of the problem to be considered in Section 2 is going to be obtained by the quintic B-spline finite element collocation method. The quintic B-spline basis functions  $\phi_m(x)$  for  $m = -2(1)N + 2$  are defined as follows

$$\Phi_m(x) = \frac{1}{h^5} \begin{cases} q_0 = (x - x_{m-3})^5 & , [x_{m-3}, x_{m-2}] \\ q_1 = q_0 - 6(x - x_{m-2})^5 & , [x_{m-2}, x_{m-1}] \\ q_2 = q_1 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & , [x_{m-1}, x_m] \\ q_3 = q_2 - 6(x - x_{m-2})^5 - 20(x - x_m)^5 & , [x_m, x_{m+1}] \\ q_4 = q_3 - 6(x - x_{m-2})^5 + 15(x - x_{m+1})^5 & , [x_{m+1}, x_{m+2}] \\ q_5 = q_4 - 6(x - x_{m-2})^5 - 6(x - x_{m+2})^5 & , [x_{m+2}, x_{m+3}] \\ 0 & , otherwise \end{cases} \quad (28)$$

[13]. Since the set of quintic B-spline basis functions  $\{\Phi_{-2}(x), \Phi_{-1}(x), \dots, \Phi_{N+2}(x)\}$  constitutes a base for the smooth functions defined on the domain  $[x_L, x_R]$ , the approximate solutions  $u_N(x, t)$  and  $v_N(x, t)$  are given as follows

$$u_N(x, t) = \sum_{i=-2}^{N+2} \Phi_i(x) \delta_i(t), \quad v_N(x, t) = \sum_{i=-2}^{N+2} \Phi_i(x) \sigma_i(t). \quad (29)$$

Here  $\delta_i(t)$  and  $\sigma_i(t)$  are time dependent parameters which are going to be found out. Since the quintic B-spline basis functions and their derivatives are zero outside the region  $[x_{m-3}, x_{m+3}]$ , a typical approximate solution can be written as follows over the region  $[x_m, x_{m+1}]$  as follows

$$u_N(x, t) = \sum_{i=m-2}^{m+2} \Phi_i(x) \delta_i(t), \quad v_N(x, t) = \sum_{i=m-2}^{m+2} \Phi_i(x) \sigma_i(t). \quad (30)$$

If we apply the local transformation  $h\xi = x - x_m$ ,  $0 \leq \xi \leq 1$ , on a typical region  $[x_m, x_{m+1}]$  and convert it into the region  $[0,1]$ , the quintic B-spline basis functions over the region  $[0,1]$  are defined as follows in terms of the local variable  $\xi$

$$\begin{aligned} \Phi_{m-2} &= 1 - 5\xi + 10\xi^2 - 10\xi^3 + 5\xi^4 - \xi^5, \\ \Phi_{m-1} &= 26 - 50\xi + 20\xi^2 + 20\xi^3 - 20\xi^4 + 5\xi^5, \\ \Phi_m &= 66 - 60\xi^2 + 30\xi^4 - 10\xi^5, \\ \Phi_{m+1} &= 26 + 50\xi + 20\xi^2 - 20\xi^3 - 20\xi^4 + 10\xi^5, \\ \Phi_{m+2} &= 1 + 5\xi + 10\xi^2 + 10\xi^3 + 5\xi^4 - 5\xi^5, \\ \Phi_{m+3} &= \xi^5. \end{aligned} \quad (31)$$

The pointwise values of  $u_N(x_m, t) = u_m$ ,  $v_N(x_m, t) = v_m$  and their derivatives up to second order for  $m = 0(1)N$  at the point  $(x_m, t)$  in terms of parameters  $\delta_m$  and  $\sigma_m$  are given as follows

$$\begin{aligned} u_m &= \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2}, & v_m &= \sigma_{m-2} + 26\sigma_{m-1} + 66\sigma_m + 26\sigma_{m+1} + \sigma_{m+2}, \\ u'_m &= \frac{5}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}), & v'_m &= \frac{5}{h}(-\sigma_{m-2} - 10\sigma_{m-1} + 10\sigma_{m+1} + \sigma_{m+2}), \\ u''_m &= \frac{20}{h^2}(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}), & v''_m &= \frac{20}{h^2}(\sigma_{m-2} + 2\sigma_{m-1} - 6\sigma_m + 2\sigma_{m+1} + \sigma_{m+2}). \end{aligned} \quad (32)$$

The upper indices in these formulae denote the derivative with respect to  $x$ . If these nodal values are written in their places in Eqs. (12)-(13) and they are arranged accordingly, the following systems of algebraic equations are obtained for  $m = 0(1)N$

$$\begin{aligned}
 & A_1\delta_{m-2}^{n+1} + A_2\delta_{m-1}^{n+1} + A_3\delta_m^{n+1} + A_4\delta_{m+1}^{n+1} + A_5\delta_{m+2}^{n+1} \\
 & + B_1\sigma_{m-2}^{n+1} + B_2\sigma_{m-1}^{n+1} + B_3\sigma_m^{n+1} + B_4\sigma_{m+1}^{n+1} + B_5\sigma_{m+2}^{n+1} = \\
 & \quad C_1\delta_{m-2}^n + C_2\delta_{m-1}^n + C_3\delta_m^n + C_4\delta_{m+1}^n + C_5\delta_{m+2}^n \\
 & \quad + D_1\sigma_{m-2}^n + D_2\sigma_{m-1}^n + D_3\sigma_m^n + D_4\sigma_{m+1}^n + D_5\sigma_{m+2}^n, \quad (33)
 \end{aligned}$$

$$\begin{aligned}
 & E_1\delta_{m-2}^{n+1} + E_2\delta_{m-1}^{n+1} + E_3\delta_m^{n+1} + E_4\delta_{m+1}^{n+1} + E_5\delta_{m+2}^{n+1} \\
 & + F_1\sigma_{m-2}^{n+1} + F_2\sigma_{m-1}^{n+1} + F_3\sigma_m^{n+1} + F_4\sigma_{m+1}^{n+1} + F_5\sigma_{m+2}^{n+1} = \\
 & \quad G_1\delta_{m-2}^n + G_2\delta_{m-1}^n + G_3\delta_m^n + G_4\delta_{m+1}^n + G_5\delta_{m+2}^n \\
 & \quad + H_1\sigma_{m-2}^n + H_2\sigma_{m-1}^n + H_3\sigma_m^n + H_4\sigma_{m+1}^n + H_5\sigma_{m+2}^n. \quad (34)
 \end{aligned}$$

The coefficients  $A_i, B_i, C_i, D_i, E_i, F_i, G_i$  and  $H_i$  for  $i = 1(1)5$  are given in Table 2. This system of equations consists of  $(2N+10)$  unknowns and  $(2N+2)$  equations. If the unknowns  $\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$  corresponding to  $m=0,1,N-1,N$  are eliminated using the boundary conditions  $u_x(x_L, t) = 0, u_{xx}(x_L, t) = u_0''(x_L), u_x(x_R, t) = 0, u_{xx}(x_R, t) = u_0''(x_R)$  and the unknowns  $\sigma_{-2}, \sigma_{-1}, \sigma_{N+1}, \sigma_{N+2}$  are eliminated using the boundary conditions  $v_x(x_L, t) = u_0'''(x_L), v_{xx}(x_L, t) = u_0^{(4)}(x_L), v_x(x_R, t) = u_0'''(x_R), v_{xx}(x_R, t) = u_0^{(4)}(x_R)$ , then a system of  $(2N+2)$  unknowns  $(2N+2)$  equations is obtained. First of all, we write the unknowns of this system of equations in the form of  $\mathbf{d}^T = [\delta_0 \sigma_0 \delta_1 \sigma_1 \dots \delta_N \sigma_N]^T$  and arrange the both sides of the equation in such a way that the coefficients matrices are in agreement with  $\mathbf{d}^T$ . The newly obtained  $(2N+2)$  dimensional square matrices  $\mathbf{A}$  and  $\mathbf{B}$  are used in the system of equations  $\mathbf{A}\mathbf{d}^{n+1} = \mathbf{B}\mathbf{d}^n$  and finally they are solved using an appropriate algorithm. Since  $\mathbf{A}$  and  $\mathbf{B}$  matrices are ten-diagonal matrices, they can be handled easily. In order to compute parameter  $\mathbf{d}^{n+1}$  it is necessary to know the initial parameter  $\mathbf{d}^0$ . Using the initial conditions given in Eq. (8), the following equations

$$\begin{aligned}
 (u_N)_{xx}(x_L, 0) &= \frac{20}{h^2}(\delta_{-2}^0 + 2\delta_{-1}^0 - 6\delta_0^0 + 2\delta_1^0 + \delta_2^0) = u_0''(x_L), \\
 (u_N)_x(x_L, 0) &= \frac{5}{h}(-\delta_{-2}^0 - 10\delta_{-1}^0 + 10\delta_1^0 + \delta_2^0) = u_0'(x_L), \\
 u_N(x_m, 0) &= \delta_{m-2}^0 + 26\delta_{m-1}^0 + 66\delta_m^0 + 26\delta_{m+1}^0 + \delta_{m+2}^0 = u_0(x_m), \quad m = 0(1)N \quad (35) \\
 (u_N)_x(x_R, 0) &= \frac{5}{h}(-\delta_{N-2}^0 - 10\delta_{N-1}^0 + 10\delta_{N+1}^0 + \delta_{N+2}^0) = u_0'(x_R), \\
 (u_N)_{xx}(x_R, 0) &= \frac{20}{h^2}(\delta_{N-2}^0 + 2\delta_{N-1}^0 - 6\delta_N^0 + 2\delta_{N+1}^0 + \delta_{N+2}^0) = u_0''(x_R)
 \end{aligned}$$

and

$$\begin{aligned}
 (v_N)_{xx}(x_L, 0) &= \frac{20}{h^2}(\sigma_{-2}^0 + 2\sigma_{-1}^0 - 6\sigma_0^0 + 2\sigma_1^0 + \sigma_2^0) = u_0^{(4)}(x_L), \\
 (v_N)_x(x_L, 0) &= \frac{5}{h}(-\sigma_{-2}^0 - 10\sigma_{-1}^0 + 10\sigma_1^0 + \sigma_2^0) = u_0'''(x_L), \\
 v_N(x_m, 0) &= \sigma_{m-2}^0 + 26\sigma_{m-1}^0 + 66\sigma_m^0 + 26\sigma_{m+1}^0 + \sigma_{m+2}^0 = u_0''(x_m), m = 0(1)N \quad (36) \\
 (v_N)_x(x_R, 0) &= \frac{5}{h}(-\sigma_{N-2}^0 - 10\sigma_{N-1}^0 + 10\sigma_{N+1}^0 + \sigma_{N+2}^0) = u_0'''(x_R), \\
 (v_N)_{xx}(x_R, 0) &= \frac{20}{h^2}(\sigma_{N-2}^0 + 2\sigma_{N-1}^0 - 6\sigma_N^0 + 2\sigma_{N+1}^0 + \sigma_{N+2}^0) = u_0^{(4)}(x_R)
 \end{aligned}$$

Tablo 2. The values of the coefficients of the equation systems given by Eqs.(33)-(34).

$i$	$A_i$	$B_i$	$C_i$	$D_i$	$E_i$	$F_i$	$G_i$	$H_i$
1	$1 - \frac{5kZ_m}{2h}$	$-c - \frac{5kd}{2h} + \frac{20e}{h^2}$	$A_{6-i}$	$B_{6-i}$	$\frac{20}{h^2}$	-1	$\frac{-20}{h^2}$	1
2	$26 - \frac{50kZ_m}{2h}$	$-26c - \frac{50kd}{2h} + \frac{40e}{h^2}$	$A_{6-i}$	$B_{6-i}$	$\frac{40}{h^2}$	-26	$\frac{-40}{h^2}$	26
3	66	$-66c - \frac{120e}{h^2}$	$A_{6-i}$	$B_{6-i}$	$-\frac{120}{h^2}$	-66	$\frac{120}{h^2}$	66
4	$26 + \frac{50kZ_m}{2h}$	$-26c + \frac{50kd}{2h} + \frac{40e}{h^2}$	$A_{6-i}$	$B_{6-i}$	$\frac{40}{h^2}$	-26	$\frac{-40}{h^2}$	26
5	$1 + \frac{5kZ_m}{2h}$	$-c + \frac{5kd}{2h} + \frac{20e}{h^2}$	$A_{6-i}$	$B_{6-i}$	$\frac{20}{h^2}$	-1	$\frac{-20}{h^2}$	1

are obtained. The initial parameter  $\mathbf{d}^0$  is found by the solution of these systems of equations. The internal iteration given by equation (23) is applied 3 or 5 times at each time step to the nonlinear terms of the equation system, thus the approximate solution is improved.

**2.4. Stability analysis**

The stability analysis of the difference equations (33) - (34) obtained by applying the quintic B-spline finite element collocation method is going to be done by von Neumann method. In place of  $u$  in the nonlinear term  $u^{p-1}u_x$  in Eq. (6), a local constant  $\hat{Z}$  is going to be taken. In that condition, the term  $Z_m$  in the difference equation given by Eq. (33) is going to be constant in the form of  $a + pb\hat{Z}^{p-1}$ ,  $i$  is the imaginary unit,  $\varphi$  is an arbitrary real number, the amplification factor  $q = q(\varphi)$  is a complex number, special solutions  $\delta_m^n = Pq^n e^{im\varphi}$ ,  $\sigma_m^n = Wq^n e^{im\varphi}$  are written in Eqs. (33) - (34) and the Euler formula  $e^{i\varphi} = \cos\varphi + i\sin\varphi$  is used and after some arrangements the following homogenous equations system is obtained

$$[(A + iZB)q - (A - iZB)]P + [(D + iBd)q - (D - iBd)]W = 0 \tag{37}$$

$$(q + 1)(CP - AW) = 0 \tag{38}$$

where  $A = 2(26\cos\varphi + \cos2\varphi + 33)$ ,  $B = \frac{5k}{h}i(10\sin\varphi + \sin2\varphi)$ ,  $C = \frac{40}{h^2}(2\cos\varphi + \cos2\varphi - 3)$ ,  $D = -Ac + Ce$ ,  $Z = \max Z_m$ . It is well known that this homogenous equation system has at least one nonzero solution when the determinant of the coefficient matrix of the system is zero. Therefore from Eqs. (37)-(38) we can write

$$[-A^2 - CD - iB(AZ + Cd)]q + A^2 + CD - iB(AZ + Cd) = 0 \text{ or } q + 1 = 0. \tag{39}$$

Then, the amplification factor is found as follows

$$q = \frac{-A^2 - CD + iB(AZ + Cd)}{-A^2 - CD - iB(AZ + Cd)} \text{ or } q = -1 \tag{40}$$

As a conclusion, since  $|q| = 1$ , the method is unconditionally stable.

### 3. Application of the methods and comparisons

In this section, the methods proposed in Section 2 for the Rosenau-KdV-RLW equation system have been applied to one example and numerical results were obtained. To demonstrate the efficiency and effectiveness of the proposed methods, we have used the fundamental conservation characteristics of the Rosenau-KdV-RLW equation defined as follows

$$Q(t) = \int_{x_L}^{x_R} u(x, t) dx = \int_{x_L}^{x_R} u(x, 0) dx = Q(0) \quad (41)$$

$$E(t) = \int_{x_L}^{x_R} (u^2(x, t) + cu_x^2(x, t) + u_{xx}^2(x, t)) dx = E(0) \quad (42)$$

and known as mass and energy invariants, besides, we have used the error norms defined as follows

$$L_2 = \sqrt{h \sum_{i=1}^N |u_i^{analitik} - u_i^{numerik}|^2}, \quad L_\infty = \max_{1 \leq i \leq N} |u_i^{analitik} - u_i^{numerik}| \quad (43)$$

If  $L$  represents one of these error norms, then the following formula

$$\begin{aligned} & \text{Rate} \\ &= \frac{\ln(L(t, h_1)/L(t, h_2))}{\ln(h_1/h_2)} \end{aligned} \quad (44)$$

is going to be used as a convergence rate.

**Example:** The exact solution of the Rosenau-KdV-RLW equation given by Eq. (1) for parametric values of  $a=1$ ,  $b=0.5$ ,  $c=1$ ,  $d=1$ ,  $e=1$ ,  $p=2$  is

$$u(x, t) = k_1 \operatorname{sech}^4[k_2(x - k_3 t)] \quad (45)$$

where  $k_1 = -5(25 - 13\sqrt{457})/456$ ,  $k_2 = \sqrt{-13 + \sqrt{457}}/\sqrt{288}$ ,  $k_3 = (241 + 13\sqrt{457})/266$  [1, 4, 14]. By taking  $t=0$  in the exact solution, the initial condition of the problem can be obtained. In Tables 3-4, the error norms  $L_2$  and  $L_\infty$  of the numerical solutions obtained by cubic B-spline collocation methods and quintic B-spline collocation methods at time  $T = 30$  for  $x_L = -40$ ,  $x_R = 100$  and are compared with those of some published ones in the literature. As it is seen from Tables 3-4, the errors of the numerical solutions obtained by the present methods are smaller than the errors of the compared ones.

Tablo 3. A comparison of numerical results error using  $L_2$  for various  $h=k$  and  $x \in [-40,100]$  at time  $T=30$ .

$h=k$	Scheme-I	Scheme-II	[1]	[4]
0.25	2.55603E-1	2.37668E-1	2.94337E-0	1.86617E-0
0.125	6.46775E-2	6.00345E-2	8.05629E-1	5.18662E-1
0.0625	1.62185E-2	1.50476E-2	2.05276E-1	1.33174E-1
0.03125	4.05772E-3	3.76437E-3	5.15696E-2	3.35296E-2

Tablo 4. A comparison of numerical results error using  $L_\infty$  for various  $h=k$  and  $x \in [-40,100]$  at time  $T=0$ .

$h=k$	Scheme-I	Scheme-II	[1]	[4]
0.25	9.85936E-2	9.10323E-2	9.86753E-1	6.99597E-1
0.125	2.49932E-2	2.30177E-2	2.14488E-1	1.97127E-1
0.0625	6.26876E-3	5.76980E-3	5.19201E-2	5.06954E-2
0.03125	1.56845E-3	1.44360E-3	1.28858E-2	1.27669E-2

In Table 5, the convergence rates for the error norms  $L_2$  and  $L_\infty$  of the numerical solutions obtained by cubic B-spline collocation method and quintic B-spline collocation method at time  $T=30$  for  $x_L = -40$ ,  $x_R = 100$  are calculated and displayed. The convergence rates obtained by the present method have been compared with those obtained by Wang and Dai [4]. It is seen that the convergence rates of the proposed methods are larger. It is also clearly seen that the greatest convergence rate has been obtained by the quintic B-spline collocation method.

In Table 6, for values of  $x_L = -40$ ,  $x_R = 160$ ,  $k=h=0.25$  at times  $T=0, 15, 30, 45, 60$  mass and energy invariants have been calculated and compared with those given in [1]. According to the results obtained, the fundamental conservation properties of the Rosenau-KdV-RLW equation are preserved with those obtained by the proposed numerical schemes in the range  $[0,60]$ .

Tablo 5. A comparison of convergence rates for  $L_2$  and  $L_\infty$  at  $T=30$

$h=k$	Scheme-I		Scheme-II		[4]	
	Rate $L_2$	Rate $L_\infty$	Rate $L_2$	Rate $L_\infty$	Rate $L_2$	Rate $L_\infty$
0.5	-----	-----	-----	-----	-----	-----
0.25	1.93025	1.92143	1.93992	1.92951	1.84721	1.82740
0.125	1.98257	1.97996	1.98508	1.98363	1.96149	1.95920
0.0625	1.99562	1.99528	1.99625	1.99838	1.98980	1.98945

Tablo 6. Acomparison of mass and energy invariants for  $h=k=0.25$ ,  $x \in [-40,100]$  at time  $T=30$

$T$	Scheme-I		Scheme-II		[1] $\theta = -1$	
	Q	E	Q	E	Q	E
0	21.67925844	43.70855146	21.67925844	43.70855146	21.67925844	43.70855146
15	21.67922349	43.70931861	21.67922326	43.71412237	21.68257703	43.72652015
30	21.67919030	43.70919982	21.67919310	43.71401660	21.68264127	43.72664228
45	21.67879169	43.70910121	21.67891685	43.71391006	21.68342617	43.72664409
60	21.68231910	43.70900042	21.68069226	43.71380341	21.67462536	43.72664408

In Figs. 1-2, the graphics of the exact and numerical solutions obtained using cubic and quintic B-spline collocation method on the region  $[-40,100]$  for values of  $h=k=0.25$  at times  $T=10, T=20, T=30$  are illustrated. In those graphics, numerical solutions and exact ones overlap in such a way that they are indistinguishable.

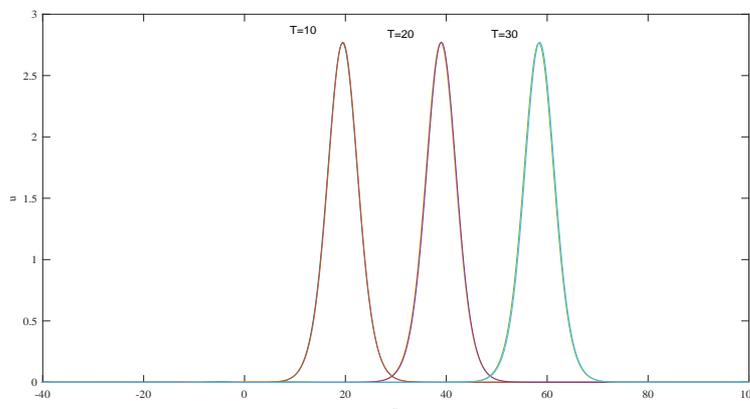


Figure 1. The graphics of exact and numerical solutions obtained using Scheme-I on the region  $[-40,100]$  for values of  $h=k=0.25$  at times  $T=10, T=20$  and  $T=30$ .

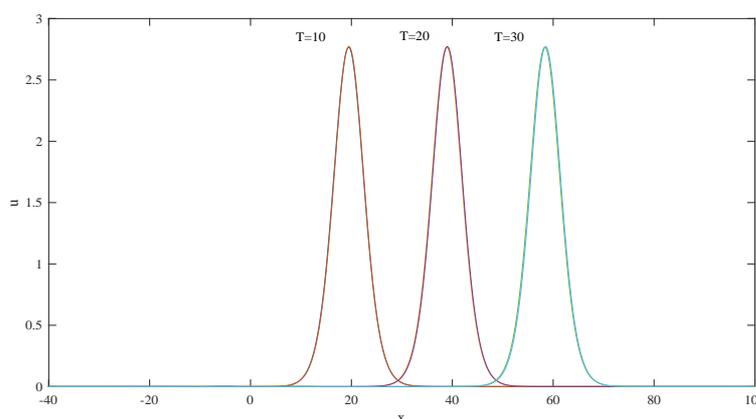


Figure 2. The graphics of exact and numerical solutions obtained using Scheme-II on the region  $[-40,100]$  for values of  $h=k=0.25$  at times  $T=10, T=20$  and  $T=30$ .

#### 4. Conclusion

In this study, the Rosenau-KdV-RLW equation is firstly converted into the partial differential equation system given by Eqs.(6)-(7). Then, the resulting system has been solved separately by both the cubic and quintic B-spline finite element collocation methods. The methods have been examined on an example of simulation of solitary waves. The error norms and invariants have been computed to determine the accuracy of the proposed methods. The error norms have been compared by Refs.[1-4]. It is seen that the error norms are smaller than the other ones. The mass and energy invariants for solved example are sufficiently constant during the simulation time for given both methods. The stability analysis of the methods has been made by the von

Neumann method. It is found that they are unconditionally stable. The convergence rates of the methods are found as nearly about two. It is obviously seen that the obtained results are in very good agreement with the exact ones. Although both methods produce close results, the quintic B-spline finite element collocation method gives better results than the other one. Consequently, the presented methods can also be applied to many partial differential equations including higher-order derivatives widely encountered in engineering and science.

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