

Solving fractional difference equations by discrete Adomian decomposition method

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Abstract

In this paper, we propose the discrete Adomian decomposition method(DADM) to solve linear as well as nonlinear fractional partial difference equations and provide few examples to illustrate the applicability of proposed method. The results show that DADM is efficient, accurate and can be applied to other fractional difference equations.

Keywords: Discrete Adomian decomposition method, fractional order, partial difference equations.

Ayrık Adomian ayrışım metodu ile kesirli mertebe fark denklemlerinin çözümü

Özet

Bu makalede, hem lineer hem de lineer olmayan kesirli mertebe kısmi fark denklemlerini çözmek için ayrık Adomian ayrışım metodunu(DADM) önerdik ve önerilen metodun uygulanabilirliğini göstermek için birkaç örnek verdik. Sonuçlar, DADM'nin etkili, doğru ve diğer kesirli mertebe fark denklemlerine uygulanabileceğini gösterdi.

Anahtar kelimeler: Ayrık Adomian ayrışım metodu, kesirli mertebe, kısmi fark denklemleri.

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1. Introduction and preliminaries

Fractional calculus has received increasing attention as one of the most important interdisciplinary subjects in mathematical physics, chemistry, mechanical and electrical properties of real phenomena [24, 27]. On the other hand recently, discrete fractional calculus gain much attention [1, 8-11].

Most fractional differential equation do not have precise analytic solution. Therefore various techniques have been developed to solve these equations [4, 18-20]. There are many studies with these techniques [5, 6, 21, 23, 25]. Adomian decomposition method(ADM) is widely used to provide on analytical approximation to linear/nonlinear problems. ADM was first introduced by Adomian[4, 5]. Wazwaz has applied ADM to solve various differential equations [26-20]. Later on the discrete ADM(DADM) was used to obtain numerical solution of partial differential equations [12, 15].

In this paper, we propose the discrete Adomian decomposition method to solve fractional partial difference equations. To the best of our knowledge, this is the first time the DADM have been used to solve linear or nonlinear fractional order difference equations. This method can be efficiently used to lead to accurate solutions for standard fractional partial difference equations.

Definition 2.1:[9, 10]Let $f: \mathbb{N}_a \rightarrow \mathbb{R}$ and $\alpha > 0$, the α th –order fractional sum of f is defined by

$${}_a\Delta_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{\alpha-1} f(s), \quad t \in \mathbb{N}_{a+\alpha},$$

where $\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$, $\sigma(s) = s + 1$.

The trivial sum is

$${}_a\Delta_t^{-0} f(t) = f(t), \quad t \in \mathbb{N}_a$$

and the falling function is

$$t^{(\alpha)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \alpha)}.$$

This definition is analogous to Riemann-Liouville fractional integral.

Throughout, we assume that if $t + 1 - \alpha \in \{0, -1, \dots, -k, \dots\}$, then $t^{(\alpha)} = 0$.

$$\Delta^{-\alpha} t^{(\gamma)} = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} t^{(\alpha+\gamma)}, \quad \gamma \in \mathbb{R}^+$$

is known power rule.

Definition 2.2:[1, 3] Let $f: \mathbb{N}_a \rightarrow \mathbb{R}$ and $\alpha > 0$. Let $m \in \mathbb{N}_0$, such that $m - 1 < \alpha \leq m$. The α th –order Caputo-like delta difference is given by

$$\begin{aligned} {}_a^C \Delta_t^\alpha f(t) &= {}_a^C \Delta_t^{-(m-\alpha)} \Delta^m f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-(m-\alpha)} (t-\sigma(s))^{(m-\alpha-1)} \Delta^m f(s), \quad t \in \mathbb{N}_{a+m-\alpha}. \end{aligned}$$

${}_a^C \Delta_t^\alpha$ maps functions defined on \mathbb{N}_a to functions defined on $\mathbb{N}_{a+m-\alpha}$.

For special case $0 < \alpha \leq 1$,

$${}_a^C \Delta_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=a}^{t-(1-\alpha)} (t-\sigma(s))^{(-\alpha)} \Delta f(s),$$

where $\Delta f(s) = f(s+1) - f(s)$.

For the initial point $a \in \mathbb{R}$, the discrete Leibnitz sum law holds

$${}_{a+(1-\alpha)} \Delta_t^{-\alpha} {}_a^C \Delta_t^\alpha f(t) = f(t) - f(a), \quad t \in \mathbb{N}_a,$$

where $0 < \alpha \leq 1$.

2. Discrete Adomian decomposition method in fractional difference equations

To illustrate the methodology to this method, we consider the nonlinear fractional difference equation in the following general form

$${}_a^C \Delta_t^\alpha U_{k,t} + LU_{k,t} + NU_{k,t} = g_{k,t}, \quad t \in \mathbb{N}_{a+1-\alpha}, \quad k \in \mathbb{N}_0 \tag{1}$$

with initial condition

$$U_{k,a} = f_k \tag{2}$$

where $0 < \alpha \leq 1$, $U_{k,t}$ is the unknown function, $g_{k,t}$ is the source term, L linear difference and N is the nonlinear difference operator.

Applying the fractional sum ${}_{a+(1-\alpha)} \Delta_t^{-\alpha}$ to both sides of Eq(1) and using the initial conditions, we get

$$U_{k,t} = G_{k,t} - {}_{a+(1-\alpha)} \Delta_t^{-\alpha} \{LU_{k,t} + NU_{k,t}\}, \tag{3}$$

where $G_{k,t}$ represents the term arising from the source term and from using the given initial conditions.

The DADM decomposes the solution into a series

$$U_{k,t} = \sum_{n=0}^{\infty} U_{k_n,t} \quad (4)$$

and decomposes the nonlinear term $NU_{k,t}$ into a series

$$NU_{k,t} = \sum_{n=0}^{\infty} A_n, \quad (5)$$

where A_n , depending on $U_{k_0,t}, U_{k_1,t}, \dots, U_{k_n,t}$ are called the Adomian polynomials. For the nonlinearity $NU_{k,t} = M(U_{k,t})$, we determine the Adomian polynomials by using definitional formula

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[M \left(\sum_{j=0}^{\infty} U_{k_j,t} \lambda^j \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (6)$$

where λ is s grouping parameter of convenience.

If the zeros component $U_{k_n,0}$ is given then the remaining components where $n > 1$ can be determined by using recurrence relations as follows

$$\begin{aligned} U_{k_0,t} &= G_{k,t}, \\ U_{k_{n+1},t} &= -{}_{a+(1-\alpha)}\Delta_t^{-\alpha} \{LU_{k_n,t} + A_n\}, \quad n \geq 0. \end{aligned} \quad (7)$$

Therefore we obtain the solution from (4).

3. Applications to fractional difference equations

Example 4.1: Consider the fractional order discrete diffusion equation

$${}^C_0\Delta_t^\alpha U_{k,t} = \Delta_k^2 U_{k,t} + k\Delta_k U_{k,t} + U_{k,t}, \quad k \in \mathbb{N}_0, \quad t \in \mathbb{N}_{1-\alpha}, \quad 0 < \alpha \leq 1 \quad (8)$$

with initial condition

$$U_{k,0} = k, \quad (9)$$

where Δ_k is the forward partial difference which is defined as usual, i.e., $\Delta_k U_{k,t} = U_{k+1,t} - U_{k,t}$.

Applying ${}_{1-\alpha}\Delta_t^{-\alpha}$ on both sides of Eq(8) and using the initial condition, we get

$$U_{k,t} = k + {}_{1-\alpha}\Delta_t^{-\alpha} (U_{k+2,t} + (k-2)U_{k+1,t} + (2-k)U_{k,t}). \quad (10)$$

Substituting Eq(4) into Eq(10) we have

$$\sum_{n=0}^{\infty} U_{k_n,t} = k + {}_{1-\alpha}\Delta_t^{-\alpha} \left(\sum_{n=0}^{\infty} U_{k_n+2,t} + \sum_{n=0}^{\infty} (k-2)U_{k_n+1,t} + \sum_{n=0}^{\infty} (2-k)U_{k_n,t} \right)$$

Thus we have following recurrence relations

$$U_{k_0,t} = U_{k,0}$$

$$U_{k_{n+1},t} = {}_{1-\alpha}\Delta_t^{-\alpha} (U_{k_n+2,t} + (k-2)U_{k_n+1,t} + (2-k)U_{k_n,t}), \quad n = 0,1,2 \dots$$

Thus

$$U_{k_0,t} = k$$

and

$$U_{k_1,t} = k \frac{2t^{(\alpha)}}{\Gamma(\alpha+1)}$$

$$U_{k_2,t} = k \frac{2^2(t+(\alpha-1))^{(2\alpha)}}{\Gamma(2\alpha+1)}$$

$$\vdots$$

$$U_{k_n,t} = k \frac{2^n(t+(n-1)(\alpha-1))^{(n\alpha)}}{\Gamma(n\alpha+1)}.$$

Therefore from (4) we obtain the solution

$$U_{k,t} = \sum_{n=0}^{\infty} k \frac{2^n(t+(n-1)(\alpha-1))^{(n\alpha)}}{\Gamma(n\alpha+1)} = kE_{(\alpha)}(2, t),$$

where $E_{(\alpha)}$ is the discrete Mittag-Leffler function.

Example 4.2: Consider the fractional order difference Schrödinger equation

$$i {}_0^C\Delta_t^\alpha U_{k,t} + \Delta_k^2 U_{k,t} + q|U_{k,t}|^2 U_{k,t} = 0 \quad k \in \mathbb{N}_0, \quad t \in \mathbb{N}_{1-\alpha}, \quad 0 < \alpha \leq 1 \quad (11)$$

with the initial condition

$$U_{k,0} = e^{ilk}. \quad (12)$$

Applying ${}_{1-\alpha}\Delta_t^{-\alpha}$ on both sides of Eq(11) and using the initial condition, we obtain

$$U_{k,t} = e^{ilk} + i {}_{1-\alpha}\Delta_t^{-\alpha} \left(\Delta_k^2 U_{k,t} + q|U_{k,t}|^2 U_{k,t} \right). \quad (13)$$

The nonlinear term is $(U_{k,t}) = |U_{k,t}|^2 U_{k,t} = U_{k,t}^2 \overline{U_{k,t}}$, which decomposed as an Adomian polynomials.

Substituting Eq(4) and Eq(5) into Eq(13) we get

$$\sum_{n=0}^{\infty} U_{k_n,t} = e^{ilk} + i_{1-\alpha} \Delta_t^{-\alpha} \left(\sum_{n=0}^{\infty} \Delta_k^2 U_{k_n,t} + q \sum_{n=0}^{\infty} A_n \right).$$

Since initial condition $U_{k_0,t}$ is given we get following recurrence relations.

$$\begin{aligned} U_{k_0,t} &= U_{k,0} \\ U_{k_{n+1},t} &= i_{1-\alpha} \Delta_t^{-\alpha} (U_{k_{n+2},t} - 2U_{k_{n+1},t} + U_{k_n,t} + qA_n) \quad n = 0,1,2 \dots \end{aligned}$$

According to Eq(6) we can compute first few components of Adomian polynomials as follows:

$$\begin{aligned} A_0 &= U_{k_0,t}^2 \overline{U_{k_0,t}} \\ A_1 &= 2U_{k_0,t} U_{k_1,t} \overline{U_{k_0,t}} + U_{k_0,t}^2 \overline{U_{k_1,t}} \\ A_2 &= 2U_{k_0,t} U_{k_2,t} \overline{U_{k_0,t}} + U_{k_1,t}^2 \overline{U_{k_0,t}} + 2U_{k_0,t} U_{k_1,t} \overline{U_{k_1,t}} + U_{k_0,t}^2 \overline{U_{k_2,t}} \\ A_3 &= 2(U_{k_0,t} U_{k_3,t} \overline{U_{k_0,t}} + U_{k_1,t} U_{k_2,t} \overline{U_{k_0,t}} + U_{k_0,t} U_{k_2,t} \overline{U_{k_1,t}} + U_{k_0,t} U_{k_1,t} \overline{U_{k_2,t}}) \\ &\quad + U_{k_1,t}^2 \overline{U_{k_1,t}} + U_{k_0,t}^2 \overline{U_{k_3,t}} \\ &\vdots \\ A_n &= \sum_{i=0}^n \sum_{j=0}^{n-i} U_{k_i,t} U_{k_j,t} \overline{U_{k_{n-i-j},t}}. \end{aligned}$$

Thus we have the following solution:

$$\begin{aligned} U_{k_0,t} &= e^{ilk} \\ U_{k_1,t} &= i \frac{e^{ilk} \omega t^{(\alpha)}}{\Gamma(\alpha + 1)} \\ U_{k_2,t} &= -\frac{\omega^2 e^{ilk} (t + (\alpha - 1))^{(2\alpha)}}{\Gamma(2\alpha + 1)} \\ U_{k_3,t} &= -i \frac{\omega^3 e^{ilk} (t + 2(\alpha - 1))^{(3\alpha)}}{\Gamma(3\alpha + 1)} \\ &\vdots \\ U_{k_n,t} &= i^n \frac{\omega^n e^{ilk} (t + (n - 1)(\alpha - 1))^{(n\alpha)}}{\Gamma(n\alpha + 1)}, \end{aligned}$$

where $\omega = (e^{il} - 1)^2 + q$.

Thus from (4) we get the solution

$$U_{k,t} = \sum_{n=0}^{\infty} i^n \frac{\omega^n e^{ilk} (t + (n - 1)(\alpha - 1))^{(n\alpha)}}{\Gamma(n\alpha + 1)} = e^{ilk} E_{(\alpha)}(i\omega, t).$$

4. Conclusions

Discrete Adomian decomposition method is successfully applied to fractional partial difference equations with discrete time derivative $\alpha(0 < \alpha \leq 1)$. From the obtained results, we conclude that DADM can provide highly accurate solutions for fractional

partial difference equations. It can be promising method to solve other nonlinear partial difference equations.

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