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Horizontal lift in the semi-tensor bundle

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Abstract

The present paper is devoted to some results concerning with the horizontal lift of tensor fields of type (1,0) from manifold B to its semi-tensor (pull-back) bundle tB of type (p,q).

Keywords: *Vector field, horizontal lift, pull-back bundle, semi-tensor bundle.*
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1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $\pi_1 : M_n \to B_m$ the differentiable bundle determined by a submersion π_1. Suppose that $(x^i) = (x^a, x^\alpha)$, $a,b,\ldots = 1,\ldots,n-m; \alpha,\beta,\ldots = n-m+1,\ldots,n; i,j,\ldots = 1,2,\ldots,n$ is a system of local coordinates adapted to the bundle $\pi_1 : M_n \to B_m$, where x^α are coordinates in B_m, and x^a are fiber coordinates of the bundle $\pi_1 : M_n \to B_m$. If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another system of local adapted coordinates in the bundle, then we have [14]

$$\begin{cases} x^{a'} = x^{a'}\left(x^b, x^\beta\right), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has components

$$\left(A^{i'}_j\right) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A^{a'}_b & A^{a'}_\beta \\ 0 & A^{\alpha'}_\beta \end{pmatrix},$$

where

$$A^{\alpha'}_\beta = \frac{\partial x^{\alpha'}}{\partial x^\beta}.$$

Let $\left(T^p_q\right)_x (B_m)(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n)$ be the tensor space at a point $x \in B_m$ with local coordinates $(x^1,\ldots,x^m)$, we have the holonomous frame field

$$\partial_{x^{i_1}} \otimes \partial_{x^{i_2}} \otimes \ldots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \ldots \otimes dx^{j_q},$$

for $i \in \{1,\ldots,m\}^p$, $j \in \{1,\ldots,m\}^q$, over $U \subset B_m$ of this tensor bundle, and for any (p,q)-tensor field t we have [[4], p.163]:

$$t|U = t^{i_1\ldots i_p}_{j_1\ldots j_q} \partial_{x^{i_1}} \otimes \partial_{x^{i_2}} \otimes \ldots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \ldots \otimes dx^{j_q},$$

then by definition the set of all points $(x^I) = (x^a, x^\alpha, x^{\overline{\alpha}})$, $x^{\overline{\alpha}} = t^{i_1\ldots i_p}_{j_1\ldots j_q}$, $\overline{\alpha} = \alpha + m^{p+q}$, $I,J,\ldots = 1,\ldots,n+m^{p+q}$ is a semi-tensor bundle $t^p_q(B_m)$ over the manifold M_n [14]. The semi-tensor bundle $t^p_q(B_m)$ has the natural bundle structure over B_m, its bundle projection $\pi : t^p_q(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$. If we introduce a mapping $\pi_2 : t^p_q(B_m) \to M_n$ by $\pi_2 : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^a, x^\alpha)$, then $t^p_q(B_m)$ has a bundle structure over M_n. It is easily verified that $\pi = \pi_1 \circ \pi_2$ [14].

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On the other hand, let $\varepsilon = \pi : E \to B$ denote a fiber bundle with fiber F. Given a manifold B' and a map $f : B' \to B$, one can construct in a natural way a bundle over B' with the same fiber: Consider the subset

$$f^{*}E = \left\{ \left(b', e\right) \in B' \times E \middle| f\left(b'\right) = \pi\left(e\right) \right\}$$

together with the subspace topology from $B' \times E$, and denote by $\pi_1 : f^{*}E \to B'$, $\pi_2 : f^{*}E \to E$ the projections. $f^{*}\varepsilon = \pi_1 : f^{*}E \to B'$ is a fiber bundle with fiber F, called the pull-back bundle of ε via f [[3], [5], [8], [10], [14]].
From the above definition it follows that the semi-tensor bundle $(t_q^p(B_m), \pi_2)$ is a pull-back bundle of the tensor bundle over B_m by π_1 (see, for example [12], [14]).
In other words, the semi-tensor bundle (induced or pull-back bundle) of the tensor bundle $\left(T_q^p(B_m), \widetilde{\pi}, B_m\right)$ is the bundle $\left(t_q^p(B_m), \pi_2, M_n\right)$ over M_n with a total space $t_q^p(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}}\right) \in M_n \times \left(T_q^p\right)_x(B_m) : \pi_1\left(x^a, x^\alpha\right) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\alpha}}\right) = (x^\alpha) \right\} \subset M_n \times \left(T_q^p\right)_x(B_m)$.To a transformation (1.1) of local coordinates of M_n, there corresponds on $t_q^p(B_m)$ the coordinate transformation

$$\begin{cases} x^{a'} = x^{a'}\left(x^b, x^\beta\right), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'} = t_{\alpha_1'\ldots\alpha_q'}^{\beta_1'\ldots\beta_p'} = A_{\alpha_1\ldots\alpha_p}^{\beta_1'\ldots\beta_p'} A_{\alpha_1'\ldots\alpha_q'}^{\beta_1\ldots\beta_q} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} = A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} x^{\overline{\beta}}. \end{cases} \tag{1.2}$$

The Jacobian of (1.2) is given by [14]:

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_b^{a'} & 0 & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & t_{(\sigma)}^{(\alpha)} \partial_\beta A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\sigma)} & A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} \end{pmatrix}, \tag{1.3}$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $I, J ... {=} 1, ..., n + m^{p+q}$, $t_{(\sigma)}^{(\alpha)} = t_{\sigma_1\ldots\sigma_q}^{\alpha_1\ldots\alpha_p}$, $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$.
It is easily verified that the condition $Det\bar{A} \neq 0$ is equivalent to the condition:

$$Det(A_b^{a'}) \neq 0, Det(A_\beta^{\alpha'}) \neq 0, Det(A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)}) \neq 0.$$

Also, $\dim t_q^p(B_m) {=} n + m^{p+q}$. In the special case $n{=}m$, $t_q^p(B_m)$ is a tensor bundle $T_q^p(B_m)$ [[6], p.118]. In the special case, the semi-tensor bundles $t_0^1(B_m)$ $(p = 1,\ q = 0)$ and $t_1^0(B_m)$ $(p = 0,\ q = 1)$ are semi-tangent and semi-cotangent bundles, respectively. We note that semi-tangent and semi-cotangent bundle were examined in [[1], [7], [9]] and [[11], [13], [15], [16]], respectively. Also, Fattaev studied the special class of semi-tensor bundle [2]. We denote by $\Im_q^p(t_q^p(B_m))$ and $\Im_q^p(B_m)$ the modules over $F\left(t_q^p(B_m)\right)$ and $F\left(B_m\right)$ of all tensor fields of type (p, q) on $t_q^p(B_m)$ and B_m respectively, where $F\left(t_q^p(B_m)\right)$ and $F\left(B_m\right)$ denote the rings of real-valued C^∞ $-$functions on $t_q^p(B_m)$ and B_m, respectively.

2. Horizontal lifts of vector fields and $\gamma-$Operator

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [9] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. If we take account of (1.3), we can prove that ${}^{HH}\widetilde{X}' = \bar{A}\left({}^{HH}\widetilde{X}\right)$, where ${}^{HH}\widetilde{X}$ is a vector field defined by

$${}^{HH}\widetilde{X} = \begin{pmatrix} \widetilde{X}^b \\ X^\beta \\ X^l\left(\sum_{\mu=1}^{q} \Gamma_{l\beta_\mu}^{\varepsilon} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} - \sum_{\lambda=1}^{p} \Gamma_{l\varepsilon}^{\beta_\lambda} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\right) \end{pmatrix}, \tag{2.1}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t_q^p(B_m)$. We call ${}^{HH}\widetilde{X}$ the horizontal lift of the vector field of the vector field $\widetilde{X}$ to $t_q^p(B_m)$ [14].
Now, consider $A \in \Im_q^p(B_m)$ and $\varphi \in \Im_1^1(B_m)$, then ${}^{vv}A \in \Im_0^1(t_q^p(B_m))$ (vertical lift), $\gamma\varphi \in \Im_0^1(t_q^p(B_m))$ and $\widetilde{\gamma}\varphi \in \Im_0^1(t_q^p(B_m))$ have respectively, components on the semi-tensor bundle $t_q^p(B_m)$ [14]

$${}^{vv}A = \begin{pmatrix} 0 \\ 0 \\ A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \end{pmatrix}, \gamma\varphi = \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} \varphi_\varepsilon^{\alpha_\lambda} \end{pmatrix}, \widetilde{\gamma}\varphi = \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \varphi_{\beta_\mu}^{\varepsilon} \end{pmatrix} \tag{2.2}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t_q^p(B_m)$, where $A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}$, $\varphi_\varepsilon^{\alpha_\lambda}$ and $\varphi_{\beta_\mu}^{\varepsilon}$ are local components of A and φ.
On the other hand, ${}^{vv}f$ the vertical lift of function $f \in \Im_0^0(B_m)$ on $t_q^p(B_m)$ is defined by [14]:

$${}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. \tag{2.3}$$

Theorem 2.1. *For any vector fields $\widetilde{X}$, $\widetilde{Y}$ on M_n and $f \in \Im_0^0(B_m)$, we have*

$${}^{HH}\widetilde{X}\,{}^{vv}f = {}^{vv}(Xf).$$

Proof. Let $\widetilde{X} \in \Im_0^1(M_n)$. Then we get by (2.1) and (2.3):

$^{HH}\widetilde{X}^{vv}f = ^{HH}\widetilde{X}^I \partial_I(^{vv}f)$

$$\begin{aligned} ^{HH}\widetilde{X}^{vv}f &= {}^{HH}\widetilde{X}^a \underbrace{\partial_a(^{vv}f)}_{0} + {}^{HH}\widetilde{X}^\alpha \partial_\alpha(^{vv}f) + {}^{HH}\widetilde{X}^{\overline{\alpha}} \underbrace{\partial_{\overline{\alpha}}(^{vv}f)}_{0} \\ &= X^\alpha \partial_\alpha(^{vv}f) \\ &= {}^{vv}(Xf), \end{aligned}$$

which gives Theorem 2.1. □

Theorem 2.2. *Let $\widetilde{X}$ be a projectable vector field on M_n. For the Lie product, we have*

$[^{HH}\widetilde{X}, ^{vv}A] = ^{vv}(\nabla_X A)$

for any $A \in \Im_q^p(B_m)$.

Proof. If $A, B \in \Im_q^p(B_m)$ and $\begin{pmatrix} [^{HH}\widetilde{X}, ^{vv}A]^b \\ [^{HH}\widetilde{X}, ^{vv}A]^\beta \\ [^{HH}\widetilde{X}, ^{vv}A]^{\overline{\beta}} \end{pmatrix}$ are components of $[^{HH}\widetilde{X}, ^{vv}A]^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t_q^p(B_m)$, then we have

$$\begin{aligned} [^{HH}\widetilde{X}, ^{vv}A]^J &= (^{HH}\widetilde{X})^I \partial_I(^{vv}A)^J - (^{vv}A)^I \partial_I(^{HH}\widetilde{X})^J \\ &= \left(^{HH}\widetilde{X}\right)^a \partial_a(^{vv}A)^J + \left(^{HH}\widetilde{X}\right)^\alpha \partial_\alpha(^{vv}A)^J + \left(^{HH}\widetilde{X}\right)^{\overline{\alpha}} \partial_{\overline{\alpha}}(^{vv}A)^J \\ &\quad - \underbrace{(^{vv}A)^a}_{0} \partial_a(^{HH}\widetilde{X})^J - \underbrace{(^{vv}A)^\alpha}_{0} \partial_\alpha(^{HH}\widetilde{X})^J - (^{vv}A)^{\overline{\alpha}} \partial_{\overline{\alpha}}(^{HH}\widetilde{X})^J \\ &= \left(^{HH}\widetilde{X}\right)^a \partial_a(^{vv}A)^J + \left(^{HH}\widetilde{X}\right)^\alpha \partial_\alpha(^{vv}A)^J \\ &\quad + \left(^{HH}\widetilde{X}\right)^{\overline{\alpha}} \partial_{\overline{\alpha}}(^{vv}A)^J - (^{vv}A)^{\overline{\alpha}} \partial_{\overline{\alpha}}(^{HH}\widetilde{X})^J. \end{aligned}$$

Firstly, if $J = b$, we have

$$\begin{aligned} [^{HH}\widetilde{X}, ^{vv}A]^b &= \left(^{HH}\widetilde{X}\right)^a \partial_a \underbrace{(^{vv}A)^b}_{0} + \left(^{HH}\widetilde{X}\right)^\alpha \partial_\alpha \underbrace{(^{vv}A)^b}_{0} \\ &\quad + \left(^{HH}\widetilde{X}\right)^{\overline{\alpha}} \partial_{\overline{\alpha}} \underbrace{(^{vv}A)^b}_{0} - \underbrace{(^{vv}A)^{\overline{\alpha}}}_{A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}} \partial_{\overline{\alpha}} \underbrace{(^{HH}\widetilde{X})^b}_{\widetilde{X}^b} \\ &= A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \underbrace{\partial_{\overline{\alpha}}\widetilde{X}^b}_{0} \\ &= 0, \end{aligned}$$

by virtue of (2.1) and (2.2). Secondly, if $J = \beta$, we have

$$\begin{aligned} [^{HH}\widetilde{X}, ^{vv}A]^\beta &= \left(^{HH}\widetilde{X}\right)^a \partial_a \underbrace{(^{vv}A)^\beta}_{0} + \left(^{HH}\widetilde{X}\right)^\alpha \partial_\alpha \underbrace{(^{vv}A)^\beta}_{0} \\ &\quad + \left(^{HH}\widetilde{X}\right)^{\overline{\alpha}} \partial_{\overline{\alpha}} \underbrace{(^{vv}A)^\beta}_{0} - \underbrace{(^{vv}A)^{\overline{\alpha}}}_{A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}} \partial_{\overline{\alpha}} \underbrace{(^{HH}\widetilde{X})^\beta}_{\widetilde{X}^b} \\ &= A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \underbrace{\partial_{\overline{\alpha}}X^\beta}_{0} \\ &= 0, \end{aligned}$$

by virtue of (2.1) and (2.2). Thirdly, if $J=\overline{\beta}$, then we have

$$\begin{aligned}[][^{HH}\widetilde{X},^{vv}A]^{\overline{\beta}} &= \left(^{HH}\widetilde{X}\right)^{a}\partial_{a}\underbrace{\underbrace{(^{vv}A)^{\overline{\beta}}}_{A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}}}_{0}+\left(^{HH}\widetilde{X}\right)^{\alpha}\partial_{\alpha}\underbrace{(^{vv}A)^{\overline{\beta}}}_{A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}} \\ &\quad+\left(^{HH}\widetilde{X}\right)^{\overline{\alpha}}\partial_{\overline{\alpha}}\underbrace{\underbrace{(^{vv}A)^{\overline{\beta}}}_{A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}}}_{0}-\underbrace{(^{vv}A)^{\overline{\alpha}}}_{A^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_q}}\partial_{\overline{\alpha}}(^{HH}\widetilde{X})^{\overline{\beta}} \\ &= \left(^{HH}\widetilde{X}\right)^{\alpha}\partial_{\alpha}\underbrace{(^{vv}A)^{\overline{\beta}}}_{A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}}-\underbrace{(^{vv}A)^{\overline{\alpha}}}_{A^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_q}}\partial_{\overline{\alpha}}(^{HH}\widetilde{X})^{\overline{\beta}} \\ &= X^{\alpha}\partial_{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}-X^{\alpha}A^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_q}\underbrace{\partial_{\overline{\alpha}}t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}}_{\delta^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_p}\delta^{\alpha_1\ldots\alpha_q}_{\beta_1\ldots\beta_q}}\sum_{\lambda=1}^{p}\Gamma^{\beta_\lambda}_{\varepsilon\ \alpha} \\ &\quad+X^{\alpha}A^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_q}\underbrace{\partial_{\overline{\alpha}}t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\sigma\ldots\beta_q}}_{\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p}\delta^{\alpha_1\ldots\alpha_q}_{\beta_1\ldots\sigma\ldots\beta_q}}\sum_{\mu=1}^{q}\Gamma^{\sigma}_{\alpha\beta_\mu} \\ &= X^{\alpha}\partial_{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}+X^{\alpha}A^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\sum_{\lambda=1}^{p}\Gamma^{\beta_\lambda}_{\varepsilon\ \alpha}-X^{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\sigma\ldots\beta_q}\sum_{\mu=1}^{q}\Gamma^{\sigma}_{\alpha\beta_\mu} \\ &= X^{\alpha}(\partial_{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}+\sum_{\lambda=1}^{p}\Gamma^{\beta_\lambda}_{\varepsilon\ \alpha}A^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}-\sum_{\mu=1}^{q}\Gamma^{\sigma}_{\alpha\beta_\mu}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\sigma\ldots\beta_q}) \\ &= (\nabla_X A)^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{aligned}$$

by virtue of (2.1) and (2.2). On the other hand, we know that $^{vv}(\nabla_X A)$ have components

$$^{vv}(\nabla_X A)=\begin{pmatrix}0\\0\\(\nabla_X A)^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix}$$

with respect to the coordinates $(x^b,x^{\beta},x^{\overline{\beta}})$ on $t^p_q(B_m)$. Thus Theorem 2.2 is proved. □

We denote the curvature tensor of ∇ by $R\in\Im^1_3(B_m)$. Then $R(X,Y)$ is an element of $\Im^1_1(B_m)$ such that,

$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z$

for any $X,Y,Z\in\Im^1_0(B_m)$.
From (2.1) we have:

Theorem 2.3. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projections X and Y on B_m, respectively. For the Lie product, we have* $[^{HH}\widetilde{X},^{HH}\widetilde{Y}]=^{HH}\widetilde{[X,Y]}+(\widetilde{\gamma}-\gamma)R(X,Y)$.

Proof. If $\widetilde{X}$ and $\widetilde{Y}$ are projectable vector fields on M_n with projection $X,Y\in\Im^1_0(B_m)$ and $\begin{pmatrix}[^{HH}\widetilde{X},^{HH}\widetilde{Y}]^b\\ [^{HH}\widetilde{X},^{HH}\widetilde{Y}]^{\beta}\\ [^{HH}\widetilde{X},^{HH}\widetilde{Y}]^{\overline{\beta}}\end{pmatrix}$ are components of $[^{HH}\widetilde{X},^{HH}\widetilde{Y}]^J$ with respect to the coordinates $(x^b,x^{\beta},x^{\overline{\beta}})$ on $t(B_m)$, then we have
$[^{HH}\widetilde{X},^{HH}\widetilde{Y}]^J=(^{HH}\widetilde{X})^I\partial_I(^{HH}\widetilde{Y})^J-(^{HH}\widetilde{Y})^I\partial_I(^{HH}\widetilde{X})^J$.

Firstly, if $J=b$, we have

$$\begin{aligned}[][^{HH}\widetilde{X},^{HH}\widetilde{Y}]^b &= (^{HH}\widetilde{X})^I\partial_I(^{HH}\widetilde{Y})^b-(^{HH}\widetilde{Y})^I\partial_I(^{HH}\widetilde{X})^b \\ &= (^{HH}\widetilde{X})^a\partial_a\underbrace{\underbrace{(^{HH}\widetilde{Y})^b}_{\widetilde{Y}^b}}_{0}+(^{HH}\widetilde{X})^{\alpha}\partial_{\alpha}(^{HH}\widetilde{Y})^b+(^{HH}\widetilde{X})^{\overline{\alpha}}\partial_{\overline{\alpha}}\underbrace{\underbrace{(^{HH}\widetilde{Y})^b}_{\widetilde{Y}^b}}_{0} \\ &\quad-(^{HH}\widetilde{Y})^a\partial_a\underbrace{\underbrace{(^{HH}\widetilde{X})^b}_{\widetilde{X}^b}}_{0}-(^{HH}\widetilde{Y})^{\alpha}\partial_{\alpha}(^{HH}\widetilde{X})^b-(^{HH}\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}\underbrace{\underbrace{(^{HH}\widetilde{X})^b}_{\widetilde{X}^b}}_{0} \\ &= (^{HH}\widetilde{X})^{\alpha}\partial_{\alpha}(^{HH}\widetilde{Y})^b-(^{HH}\widetilde{Y})^{\alpha}\partial_{\alpha}(^{HH}\widetilde{X})^b \\ &= X^{\alpha}\partial_{\alpha}\widetilde{Y}^b-Y^{\alpha}\partial_{\alpha}\widetilde{X}^b \\ &= \widetilde{[X,Y]}^b\end{aligned}$$

by virtue of (2.1). Secondly, if $J=\beta$, we have

$$
\begin{aligned}
[{}^{HH}\widetilde{X},{}^{HH}\widetilde{Y}]^{\beta} &= ({}^{HH}\widetilde{X})^{I}\partial_{I}({}^{HH}\widetilde{Y})^{\beta}-({}^{HH}\widetilde{Y})^{I}\partial_{I}({}^{HH}\widetilde{X})^{\beta}\\
&= ({}^{HH}\widetilde{X})^{a}\underbrace{\partial_{a}\underbrace{({}^{HH}\widetilde{Y})^{\beta}}_{\widetilde{Y}^{\beta}}}_{0}+({}^{HH}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{HH}\widetilde{Y})^{\beta}+({}^{HH}\widetilde{X})^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}\underbrace{({}^{HH}\widetilde{Y})^{\beta}}_{\widetilde{Y}^{\beta}}}_{0}\\
&\quad -({}^{HH}\widetilde{Y})^{a}\underbrace{\partial_{a}\underbrace{({}^{HH}\widetilde{X})^{\beta}}_{\widetilde{X}^{\beta}}}_{0}-({}^{HH}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{HH}\widetilde{X})^{\beta}-({}^{HH}\widetilde{Y})^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}\underbrace{({}^{HH}\widetilde{X})^{\beta}}_{\widetilde{X}^{\beta}}}_{0}\\
&= ({}^{HH}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{HH}\widetilde{Y})^{\beta}-({}^{HH}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{HH}\widetilde{X})^{\beta}\\
&= X^{\alpha}\partial_{\alpha}\widetilde{Y}^{\beta}-Y^{\alpha}\partial_{\alpha}\widetilde{X}^{\beta}\\
&= [X,Y]^{\beta}
\end{aligned}
$$

by virtue of (2.1). Thirdly, if $J=\overline{\beta}$, then we have

$$
\begin{aligned}
[{}^{HH}X,{}^{HH}Y]^{\overline{\beta}} &= {}^{HH}X^{I}\partial_{I}({}^{HH}Y)^{\overline{\beta}}-{}^{HH}Y^{I}\partial_{I}({}^{HH}X)^{\overline{\beta}}\\
&= {}^{HH}X^{a}\underbrace{\partial_{a}{}^{HH}Y^{\overline{\beta}}}_{0}+{}^{HH}X^{\alpha}\partial_{\alpha}{}^{HH}Y^{\overline{\beta}}+{}^{HH}X^{\overline{\alpha}}\partial_{\overline{\alpha}}{}^{HH}Y^{\overline{\beta}}\\
&\quad -{}^{HH}Y^{a}\underbrace{\partial_{a}{}^{HH}X^{\overline{\beta}}}_{0}-{}^{HH}Y^{\alpha}\partial_{\alpha}{}^{HH}X^{\overline{\beta}}-{}^{HH}Y^{\overline{\alpha}}\partial_{\overline{\alpha}}{}^{HH}X^{\overline{\beta}}\\
&= X^{\alpha}\partial_{\alpha}{}^{HH}Y^{\overline{\beta}}-\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\alpha}_{\varepsilon\beta_{\lambda}}X^{\beta_{\lambda}}\partial_{\overline{\alpha}}{}^{HH}Y^{\overline{\beta}}\\
&\quad +\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}X^{\beta_{\mu}}\Gamma^{\sigma}_{\beta_{\mu}\alpha}\partial_{\overline{\alpha}}{}^{HH}Y^{\overline{\beta}}-Y^{\alpha}\partial_{\alpha}{}^{HH}X^{\overline{\beta}}\\
&\quad +\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\alpha}_{\varepsilon\beta_{\lambda}}Y^{\beta_{\lambda}}\partial_{\overline{\alpha}}{}^{HH}X^{\overline{\beta}}-\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}Y^{\beta_{\mu}}\Gamma^{\sigma}_{\beta_{\mu}\alpha}\partial_{\overline{\alpha}}{}^{HH}X^{\overline{\beta}}\\
&= X^{\alpha}\partial_{\alpha}\left(\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}Y^{\alpha}\Gamma^{\sigma}_{\alpha\beta_{\mu}}\right)-X^{\alpha}\partial_{\alpha}\left(\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\beta_{\lambda}}_{\varepsilon\alpha}Y^{\alpha}\right)\\
&\quad +\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\alpha}_{\varepsilon\beta_{\lambda}}X^{\beta_{\lambda}}\underbrace{\partial_{\overline{\alpha}}\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}}_{\delta^{\varepsilon}_{\alpha}}\Gamma^{\beta_{\lambda}}_{\varepsilon\alpha}Y^{\alpha}\\
&\quad +\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}X^{\beta_{\mu}}\Gamma^{\sigma}_{\beta_{\mu}\alpha}\underbrace{\partial_{\overline{\alpha}}\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}}_{\delta^{\alpha}_{\sigma}}Y^{\alpha}\Gamma^{\sigma}_{\alpha\beta_{\mu}}\\
&\quad -Y^{\alpha}\partial_{\alpha}\left(\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}X^{\alpha}\Gamma^{\sigma}_{\alpha\beta_{\mu}}\right)+Y^{\alpha}\partial_{\alpha}\left(\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\beta_{\lambda}}_{\varepsilon\alpha}X^{\alpha}\right)\\
&\quad -\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\alpha}_{\varepsilon\beta_{\lambda}}Y^{\beta_{\lambda}}\underbrace{\partial_{\overline{\alpha}}\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}}_{\delta^{\varepsilon}_{\alpha}}\Gamma^{\beta_{\lambda}}_{\varepsilon\alpha}X^{\alpha}\\
&\quad -\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}Y^{\beta_{\mu}}\Gamma^{\sigma}_{\beta_{\mu}\alpha}\underbrace{\partial_{\overline{\alpha}}\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}}_{\delta^{\alpha}_{\sigma}}X^{\alpha}\Gamma^{\sigma}_{\alpha\beta_{\mu}}\\
&= -X^{\alpha}\partial_{\alpha}\left(\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\beta_{\lambda}}_{\varepsilon\alpha}Y^{\alpha}\right)+\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}X^{\beta_{\lambda}}\Gamma^{\alpha}_{\varepsilon\beta}Y^{\theta}\Gamma^{\beta_{\lambda}}_{\alpha\theta}\\
&\quad +X^{\alpha}\partial_{\alpha}\Big(\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}Y^{\alpha}\Gamma^{\sigma}_{\alpha\beta_{\mu}}\Big)+\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}X^{\beta_{\mu}}\Gamma^{\sigma}_{\beta_{\mu}\alpha}Y^{\theta}\Gamma^{\alpha}_{\theta\beta_{\mu}}\\
&\quad +Y^{\alpha}\partial_{\alpha}\left(\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}\Gamma^{\beta_{\lambda}}_{\varepsilon\alpha}X^{\alpha}\right)-\sum_{\lambda=1}^{p}t^{\alpha_{1}\ldots\varepsilon\ldots\alpha_{p}}_{\beta_{1}\ldots\beta_{q}}Y^{\beta_{\lambda}}\Gamma^{\alpha}_{\varepsilon\beta_{\lambda}}X^{\theta}\Gamma^{\beta_{\lambda}}_{\alpha\theta}\\
&\quad -Y^{\alpha}\partial_{\alpha}\Big(\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}X^{\alpha}\Gamma^{\sigma}_{\alpha\beta_{\mu}}\Big)-\sum_{\mu=1}^{q}t^{\alpha_{1}\ldots\alpha_{p}}_{\beta_{1}\ldots\sigma\ldots\beta_{q}}Y^{\beta_{\mu}}\Gamma^{\sigma}_{\beta_{\mu}\alpha}X^{\theta}\Gamma^{\alpha}_{\theta\beta_{\mu}}
\end{aligned}
$$

$$
\begin{aligned}
= & -\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\Gamma_{\varepsilon}^{\beta_\lambda}{}_{\alpha}X^{\alpha}(\partial_\alpha Y^{\alpha})-\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}X^{\alpha}Y^{\theta}\partial_\alpha\Gamma_{\theta}^{\beta_\lambda}{}_{\varepsilon} \\
& -\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}X^{\alpha}Y^{\theta}\Gamma_{\alpha}^{\beta_\lambda}{}_{\gamma}\Gamma_{\theta\varepsilon}^{\gamma}+\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}X^{\alpha}(\partial_\alpha Y^{\alpha})\Gamma_{\alpha\beta_\mu}^{\sigma} \\
& +\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}X^{\alpha}Y^{\theta}\partial_\alpha\Gamma_{\theta\beta_\mu}^{\sigma}+\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}X^{\alpha}Y^{\theta}\Gamma_{\alpha\gamma}^{\sigma}\Gamma_{\theta\beta_\mu}^{\gamma} \\
& +\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\Gamma_{\varepsilon}^{\beta_\lambda}{}_{\alpha}Y^{\alpha}(\partial_\alpha X^{\alpha})+\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}X^{\alpha}Y^{\theta}\partial_\theta\Gamma_{\alpha}^{\beta_\lambda}{}_{\varepsilon} \\
& +\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}X^{\alpha}Y^{\theta}\Gamma_{\theta}^{\beta_\lambda}{}_{\gamma}\Gamma_{\alpha\varepsilon}^{\gamma}-\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}Y^{\alpha}(\partial_\alpha X^{\alpha})\Gamma_{\alpha\beta_\mu}^{\sigma} \\
& -\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}Y^{\alpha}X^{\theta}\partial_\alpha\Gamma_{\theta\beta_\mu}^{\sigma}-\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}X^{\alpha}Y^{\theta}\Gamma_{\theta\gamma}^{\sigma}\Gamma_{\alpha\beta_\mu}^{\gamma} \\
= & -\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\Gamma_{\varepsilon}^{\beta_\lambda}{}_{\alpha}\underbrace{(X^{\alpha}(\partial_\alpha Y^{\alpha})-Y^{\alpha}(\partial_\alpha X^{\alpha}))}_{[X,Y]^{\alpha}} \\
& +[\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\underbrace{(X^{\alpha}(\partial_\alpha Y^{\alpha})-Y^{\alpha}(\partial_\alpha X^{\alpha}))}_{[X,Y]^{\alpha}}\Gamma_{\alpha\beta_\mu}^{\sigma}] \\
& -\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\underbrace{X^{\alpha}Y^{\theta}\left(\partial_\alpha\Gamma_{\theta}^{\beta_\lambda}{}_{\varepsilon}-\partial_\theta\Gamma_{\alpha}^{\beta_\lambda}{}_{\varepsilon}+\Gamma_{\alpha}^{\beta_\lambda}{}_{\gamma}\Gamma_{\theta\varepsilon}^{\gamma}-\Gamma_{\theta}^{\beta_\lambda}{}_{\gamma}\Gamma_{\alpha\varepsilon}^{\gamma}\right)}_{(R(X,Y))_{\varepsilon}^{\beta_\lambda}} \\
& +\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\underbrace{X^{\alpha}Y^{\theta}(\partial_\alpha\Gamma_{\theta\beta_\mu}^{\sigma}-\partial_\theta\Gamma_{\alpha\beta_\mu}^{\sigma}+\Gamma_{\alpha\gamma}^{\sigma}\Gamma_{\theta\beta_\mu}^{\gamma}-\Gamma_{\theta\gamma}^{\sigma}\Gamma_{\alpha\beta_\mu}^{\gamma})}_{(R(X,Y))_{\beta_\mu}^{\varepsilon}} \\
= & -\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\Gamma_{\varepsilon}^{\beta_\lambda}{}_{\alpha}[X,Y]^{\alpha}+\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}[X,Y]^{\alpha}\Gamma_{\alpha\beta_\mu}^{\sigma} \\
& -\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}(R(X,Y))_{\varepsilon}^{\beta_\lambda}+\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(R(X,Y))_{\beta_\mu}^{\sigma} \\
= & [X,Y]^{\alpha}\left(-\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\Gamma_{\varepsilon}^{\beta_\lambda}{}_{\alpha}+\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\Gamma_{\alpha\beta_\mu}^{\sigma}\right) \\
& -\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}(R(X,Y))_{\varepsilon}^{\beta_\lambda}+\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(R(X,Y))_{\beta_\mu}^{\sigma}
\end{aligned}
$$

by virtue of (2.1). On the other hand, we know that ${}^{HH}\widetilde{[X,Y]}+(\widetilde{\gamma}-\gamma)R(X,Y)$ have components

$$
\begin{aligned}
= & {}^{HH}\widetilde{[X,Y]}+(\widetilde{\gamma}-\gamma)R(X,Y) \\
= & {}^{HH}\widetilde{[X,Y]}+\widetilde{\gamma}R(X,Y)-\gamma R(X,Y) \\
= & \begin{pmatrix} \widetilde{[X,Y]}^{b} \\ [X,Y]^{\beta} \\ [X,Y]^{\alpha}\left(\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\Gamma_{\alpha\beta_\mu}^{\sigma}-\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\Gamma_{\varepsilon}^{\beta_\lambda}{}_{\alpha}\right) \end{pmatrix} \\
& +\begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(R(X,Y))_{\beta_\mu}^{\sigma} \end{pmatrix}-\begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}(R(X,Y))_{\varepsilon}^{\beta_\lambda} \end{pmatrix} \\
= & \begin{pmatrix} \widetilde{[X,Y]}^{b} \\ [X,Y]^{\beta} \\ [X,Y]^{\alpha}\left(\sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\Gamma_{\alpha\beta_\mu}^{\sigma}-\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\Gamma_{\varepsilon}^{\beta_\lambda}{}_{\alpha}\right) \end{pmatrix} \\
& +\begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\sigma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(R(X,Y))_{\beta_\mu}^{\sigma}-\sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}(R(X,Y))_{\varepsilon}^{\beta_\lambda} \end{pmatrix}
\end{aligned}
$$

with respect to the coordinates $(x^{b},x^{\beta},x^{\overline{\beta}})$ on $t_q^p(B_m)$. Thus Theorem 2.3 is proved. □

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