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# HALVING THE ERROR IN SECOND ORDER ADAMS-BASHFORTH METHODS VIA A SIMPLE TIME FILTER

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**Abstract:** This paper presents the implementation and rigorous analysis of a simple time filter applied to the second order Adams-Bashforth family of explicit numerical integration schemes. Although the implementation is remarkably straightforward—requiring the modular addition of just a single line of code—the resulting mathematical benefits are substantial, making it highly attractive for legacy scientific codebases. By theoretically modeling the coupled system as a unified linear multistep method, we are able to apply standard stability frameworks to the modified scheme. Specifically, we verify numerical stability using the Jury stability criterion, ensuring that the roots of the characteristic polynomial remain within the unit circle for the desired parameter range. Furthermore, we perform a detailed local truncation error analysis. Our results demonstrate that the filter acts to dampen the parasitic computational mode and effectively halves the leading error coefficient compared to the unfiltered method. This provides a robust enhancement to the original algorithm, yielding superior accuracy with negligible computational cost, as it avoids the expensive function evaluations associated with higher-order or implicit methods.

**Keywords:** Stability analysis, Error analysis, Adam Bashforth, Time filter

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## 1. Introduction

Time filters offer a practical way to generate enhanced numerical methods that meet the needs of both industry and research (McGovern, 2025). By applying a filter after the "solve step," we can transform traditional techniques into new, higher-order methods (Asselin, 1972; Williams, 2009; Guzel and Trenchea, 2018). This approach is highly attractive because it typically requires only a single additional line of code, making it easy to implement in existing software. Despite this simplicity, these modifications yield substantial improvements in accuracy without sacrificing the favorable stability characteristics of the base method. In numerical analysis, explicit methods are widely utilized due to their computational efficiency; however, they often encounter limitations regarding time-step constraints and cumulative truncation errors. Traditional Adams-Bashforth schemes, while providing high-order accuracy through their multistep structure, can produce artificial oscillations or phase lags in the solution. The four-point time filter proposed in this study is specifically designed to systematically dampen these errors and reduce the error magnitude without degrading the method's asymptotic accuracy. This approach is particularly critical for long-term simulations where maintaining stability and adhering to conservation laws are paramount, such as in computational fluid dynamics. Foundational work in this area is presented in (Guzel and Layton, 2018; DeCaria et al., 2022). Explicit methods are common in both modern and legacy codebases. Research has demonstrated that applying filters to these methods can enhance accuracy without compromising stability (Hurl et al., 2014; Guzel and Layton, 2018). This paper investigates the construction of a novel filtered method derived from the second-order Adams-Bashforth (AB2) scheme using a four-point time filter (Guzel and Layton, 2018; Guzel and Trenchea, 2018). These methods form a natural embedded pair, making them particularly well-suited for constant time-stepping. To begin, consider the initial value problem (equation 1)

$$\begin{aligned} u'(t) &= f\big(t, u(t)\big), \text{ for } t > 0 \text{ and} \\ u(0) &= u_0. \end{aligned} \tag{1}$$

Let $t_n$ be the $n$th time step and $u_n$ be the approximation to the exact solution $u(t_n)$. Introducing $n$ as an algorithm parameter, we discretize the equation using the second-order Adams-Bashforth (AB2) method, followed by a simple four-point time filter (equation 2).

$$\begin{aligned} AB2 \ &: \quad w_{n+1} - u_n &&= \tfrac{3h}{2} f(u_n) - \tfrac{h}{2} f(u_{n-1}). \\ Filter \ &: \quad u_{n+1} &&= w_{n+1} + a w_{n+1} + b u_n \\ & && \quad + c u_{n-1} + d u_{n-2} \end{aligned} \tag{2}$$

where $a, b, c$, and $d$ are constants, and $w$ and $u$ represent the unfiltered and once-filtered values, respectively. Filter step is the only four point filter for which the combination of AB2 method with time filter produces much better approximation when $\nu = -2/3$. The combination recovers AB2 method when $\nu = 0$. Proposition 2 establishes that the combined method is zero-stable within the range $-2/3 \le \nu < 2$, but unstable outside this interval.

Remark 1. Time filters centered at $t_n$, rather than $t_{n+1}$, are often employed in geophysical fluid dynamics(Durran, 2010) simulations using the leapfrog integrator to dampen oscillations in the computed solution (Asselin, 1972; Robert, 1966; Williams, 2011). A widely used example is the Robert-Asselin filter, given by equation 3.

$$u_n \leftarrow u_n + \frac{\nu}{2}(u_{n+1} - 2u_n + u_{n-1}), \nu \simeq 0.1. \quad (3)$$

## 2. Materials and Methods

### 2.1. Derivation of Method, Error Analysis and Stability

In this section we use higher-order time filter (Li and Trenchea, 2014) to improve accuracy and of filtered Adams-Bashforth(AB2) (Hairer et al., 1993) by adding four point filter to second order method. The properties of AB2 method given in (Butcher, 2016). Consequently, the integration of a four-point linear time filter with the AB2 method enhances accuracy from first to second-order and effectively halves the error.

Proposition 1. The combination of AB2 and Filter step consistent if and only if $a = -\frac{\nu}{2}, b = \frac{3\nu}{2}, c = -\frac{3\nu}{2}, d = \frac{\nu}{2}$ for some parameter $\nu$. Therefore the Filter step is (equation 4).

$$u_{n+1} = w_{n+1} - \frac{\nu}{2}(w_{n+1} - 3u_n + 3u_{n-1} - u_{n-2}) \quad (4)$$

In this case equivalent 5- step method is (equation 5).

$$\begin{aligned} u_{n+1} - (1+\nu)u_n + \frac{3\nu}{2}u_{n-1} - \frac{\nu}{2}u_{n-2} \\ = \frac{3(2-\nu)}{4}u'_n - \frac{2-\nu}{4}u'_{n-1} \end{aligned} \quad (5)$$

**Proof:** To analyze this system, we transform it into a linear multistep method by eliminating $w_{n+1}$ in (AB2) (equation 6).

$$w_{n+1} = u_n + \frac{3h}{2}f(u_n) - \frac{h}{2}f(u_{n-1}) \quad (6)$$

By substituting $w_{n+1}$ into (Filter) and grouping similar terms, we obtain (equation 7).

$$\begin{aligned} w_{n+1} &= (1+a)[u_n + \frac{3h}{2}u'_n - \frac{h}{2}u'_{n-1}] + \\ &\quad bu_n + cu_{n-1} + du_{n-2} \\ \Rightarrow u_{n+1} &= (1+a+b)u_n + cu_{n-1} + du_{n-2} \\ &\quad + \frac{(1+a)3h}{2}u'_n - \frac{(1+a)h}{2}u'_{n-1} \end{aligned} \quad (7)$$

We must demonstrate that the method is consistent and maintains at least second-order accuracy to preserve the precision achieved by the AB2 method. To this end, we employ a standard Taylor series expansion to determine the local truncation error (LTE) (equation 8),

$$\begin{aligned} LTE &= u(t_{n+1}) - u_{n+1} \\ &= \left(u(t_n) + hu'(t_n) + \frac{h^2}{2}u''(t_n) + \frac{h^3}{6}u'''(t_n) + \frac{h^4}{24}u^{(4)}(t_n)\right) \\ &\quad -(1+a+b)u_n - c\left(u_n - hu'_n + \frac{h^2}{2}u''_n - \frac{h^3}{6}u'''_n + \frac{h^4}{24}u_n^{(4)}\right) \\ &\quad -d\left(y_n - 2hu'_n + \frac{4h^2}{2}u''_n - \frac{8h^3}{6}u'''_n + \frac{16h^4}{24}u_n^{(4)}\right) \\ &\quad -\frac{3(1+a)h}{2}u'_n + \frac{(1+a)h}{2}\left(u'_n - hu''_n + \frac{h^2}{2}u'''_n - \frac{h^3}{6}u_n^{(4)}\right) \end{aligned} \quad (8)$$

We invoke the localization assumption (Griffiths and Higham, 2010)—specifically, that the numerical solution matches the exact solution at previous time steps $(u_n = u(t_n) \ldots, u(t_1) = u_1)$. Consequently, the coefficients of the local truncation error (LTE) expansion yield (equation 9).

$$\begin{aligned} LTE &= u_n(1 - 1 - a - b - c - d) \\ &\quad + hu'_n\left(1 + c + 2d - \frac{3(1+a)}{2} + \frac{1+a}{2}\right) \\ &\quad + \frac{h^2u''_n}{2}\left(1 - c - 4d - (1+a)\right) \\ &\quad + \frac{h^3u'''_n}{6}(1 + c + 8d + \frac{6(1+a)}{4}) + O(h^4) \end{aligned} \quad (9)$$

Thus, second-order accuracy is achieved when the coefficients of terms up to $h^2$ vanish, yielding (equation 10):

$$\begin{aligned} a + b + c + d &= 0 \\ 1 + c + 2d - (1+a) &= 0 \Rightarrow a = c + 2d \\ 1 - c - 4d - (1+a) &= 0 \Rightarrow -a - c - 4d = 0 \end{aligned} \quad (10)$$

Solving this system yields (equation 11).

$$a = -d, b = 3d, c = -3d. \quad (11)$$

Setting $d = \frac{\nu}{2}$ results in the following second-order accurate system (equation 12);

$$\begin{aligned} w_{n+1} &= u_n + \frac{3h}{2}u'_n - \frac{h}{2}u'_{n-1} \\ u_{n+1} &= w_{n+1} - \frac{\nu}{2}(w_{n+1} - 3u_n + 3u_{n-1} - u_{n-2}) \end{aligned} \quad (12)$$

Combining (AB2) and (Filter) results in the following linear multistep method (equation 13):

$$\begin{aligned} u_{n+1} = \quad & (1 - \frac{\nu}{2})[u_n + \frac{3h}{2}u'_n - \frac{h}{2}u'_{n-1})] + \\ & \frac{3\nu}{2}u_n - \frac{3\nu}{2}u_{n-1} + \frac{\nu}{2}u_{n-2} \end{aligned} \quad (13)$$

By rearranging the terms, we arrive at (equation 14).

$$\begin{aligned} u_{n+1} - \quad & (1+\nu)u_n + \frac{3\nu}{2}u_{n-1} - \frac{\nu}{2}u_{n-2} \\ & = \frac{3(2-\nu)}{4}u'_n - \frac{2-\nu}{4}u'_{n-1} \end{aligned} \quad (14)$$

Remark 2. The local truncation error (LTE) for the linear multistep method (1) is given by (equation 15).

$$LTE = \frac{10 + 7\nu}{24}h^3u'''_n \quad (15)$$

This result is obtained by substituting $d=\frac{\nu}{2}$ into equation 9, which yields (equation 16).

$$\begin{aligned} LTE &= \frac{h^3 u_n'''}{6}\left(1+c+8d+\frac{6(1+a)}{4}\right)+O(h^4) \\ &= \left[1-\frac{3\nu}{2}+4\nu+\frac{3}{2}-\frac{3\nu}{4}\right]\frac{h^3u_n'''}{6}+O(h^4) \\ &= \frac{10+7\nu}{24}h^3u_n'''+O(h^4) \end{aligned} \tag{16}$$

Proposition 2. The linear multi step method (LMM) (equation 5) is 0- stable for $-2/3 \le \nu < 2$.

Proof. To analyze the stability, we consider the characteristic polynomials of the linear multistep method (LMM) given in (1). Recall that for a general LMM, the first characteristic polynomial $\rho(\zeta)$ is determined by the coefficients of the time steps $u_n$, while the second characteristic polynomial $\sigma(\zeta)$ is determined by the coefficients of the function evaluations $f_n$. We begin by examining the first characteristic polynomial of (equation 17):

$$\begin{aligned} &r^3-(1+\nu)r^2+\frac{3\nu}{2}r-\frac{\nu}{2}=0 \\ &\Rightarrow (r-1)(r^2-\nu r+\frac{\nu}{2})=0 \end{aligned} \tag{17}$$

Here, r denotes the complex variable associated with the time-shift operator, assuming a solution of the form $u_n=r^n$. Thus the method is zero stable if and only if magnitude of all roots of first characteristic polynomial are strictly less than or roots on boundary must not be repeated. Since $r_1=1$ then all other two roots of (equation 18).

$$r^2-\nu r+\frac{\nu}{2}=0 \tag{18}$$

must strictly less than 1 or equal to 1. To achieve this we use the properties of Jury test criteria(Griffiths and Higham, 2010; Jury, 1964) for stability thus the roots lie strictly inside the unit circle if the following inequalities hold (equation 19); Let $q(r)=r^2-\nu r+\frac{\nu}{2}$

$$\begin{aligned} &q(1)>0 \quad \Rightarrow 1>\frac{\nu}{2}\Rightarrow \nu<2 \\ &q(-1)>0 \quad \Rightarrow 1+\frac{3\nu}{2}>0\Rightarrow \nu>-\frac{2}{3} \\ &\frac{\nu}{2}<1 \quad \Rightarrow \nu<2 \end{aligned} \tag{19}$$

Thus, the roots have absolute values strictly less than 1 when (equation 20):

$$-\frac{2}{3}<\nu<2. \tag{20}$$

We also check the case where one of the roots is exactly equal to $-1$ (equation 21):

$$(-1)^2-\nu(-1)+\frac{\nu}{2}=0 \Rightarrow 1+\frac{3\nu}{2}=0\Rightarrow \nu = -\frac{2}{3} \tag{21}$$

and the roots are 1 and 1/3. Therefore combining all these intervals yields 0- stability condition for the interval (equation 22).

$$-\frac{2}{3}\le\nu<2. \tag{22}$$

To evaluate the stability (Hurl et al., 2014) of method, we employ the boundary locus curve method. We choose the minimum admissible value, $\nu=-2/3$, to minimize the error. The stability region is defined as the set of all $z=h\lambda$ for which the roots $r_i$ of the stability polynomial equation satisfy the root condition (equation 23).:

$$\begin{aligned} &\rho(r)-z\sigma(r)=0\Rightarrow z=\frac{r^3-\frac{1}{3}r^2-r+\frac{1}{3}}{2r^2-\frac{2}{3}r} \\ &\Rightarrow z=\frac{r^2-1}{2r} \end{aligned} \tag{23}$$

where any roots on the boundary must not be repeated and all others must be strictly less than 1. The boundary of this region is mapped by substituting $r=e^{i\theta}$ for $\theta\in[0,2\pi]$ into the characteristic equation 24.

$$z(\theta)=\frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \tag{24}$$

Application to the method by substituting the specific characteristic polynomials derived from the, we obtain the following expression for the boundary locus curve (equation 25).

$$z(\theta)=\frac{e^{2i\theta}-1}{2e^{i\theta}}\Rightarrow z(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2} \tag{25}$$

set $e^{i\theta}=\cos\theta+i\sin\theta$ and

$e^{-i\theta}=\cos\theta-i\sin\theta$ then we obtain

$z(\theta) = i\sin\theta$

The corresponding root locus curve for $\theta\in[0,2\pi]$ compared to standart AB2 method is presented in Figure 1.

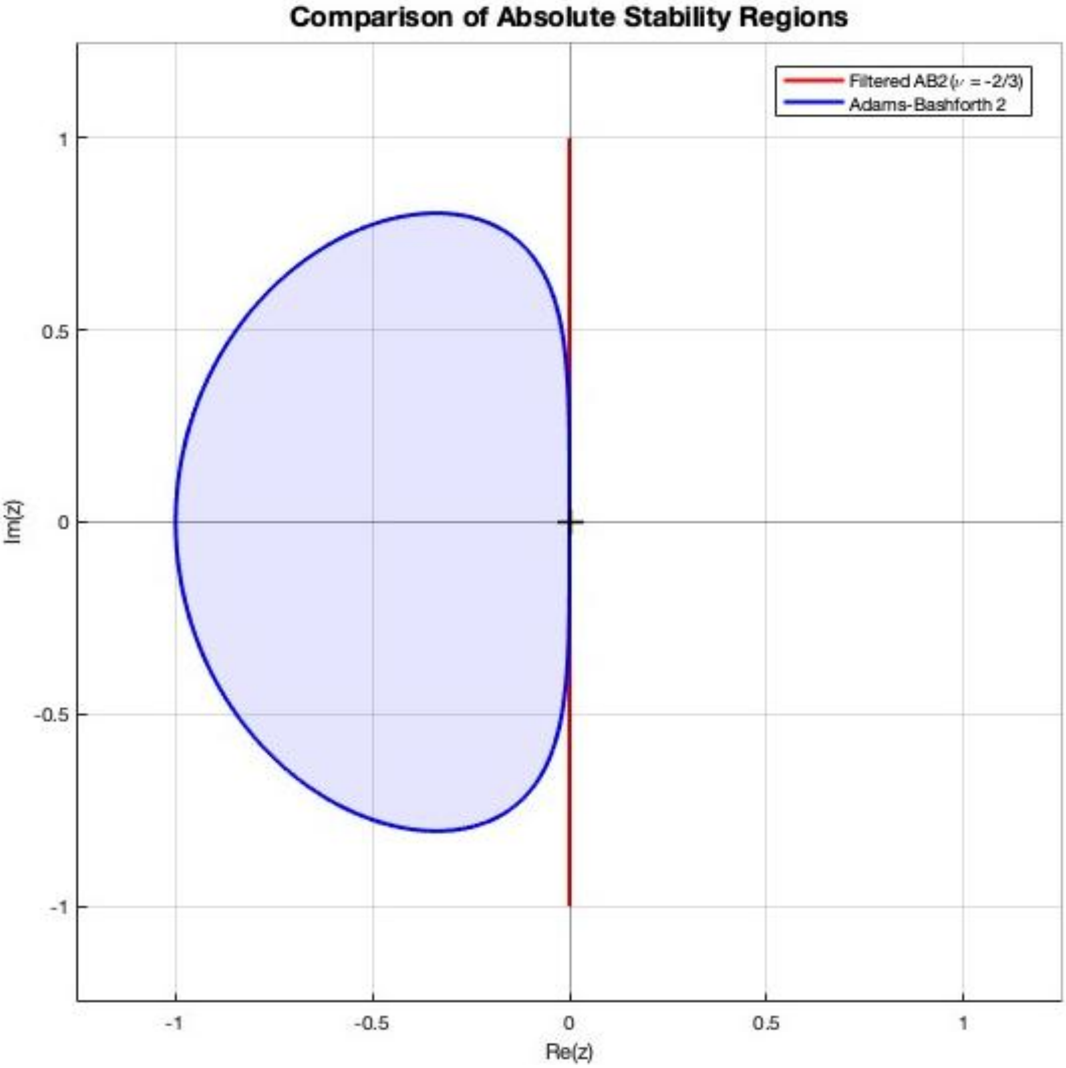


**Figure 1.** Boundary locus curve of filtered AB2 compared to AB2 method.

## 3. Results

In this section, we present two numerical experiments designed to compare the performance of the standard second-order Adams-Bashforth (AB2) method against the filtered AB2 scheme, utilizing a constant step size.

### 3.1. Dahlquist's Test Equation

We consider the classical linear scalar test problem defined by the first-order initial value problem (IVP) (equation 26):

$$u'(t) = \lambda u(t), \quad t > 0, \qquad u(0) = 1. \tag{26}$$

For the specific case where $\lambda = -1$, this system yields the exact analytical solution $u(t) = e^{-t}$.

The numerical results summarizing the performance of the standard AB2 method and the proposed filtered scheme (with $\nu = -2/3$) are presented in Table 1 and Figure 2. First, we observe the convergence behavior of both schemes. As the step size $h$ is successively halved, the computed convergence rates for both the AB2 and the proposed method asymptotically approach 2. This confirms that the proposed filtering technique preserves the classical second-order accuracy $O(h^2)$ of the underlying Adams-Bashforth method.

However, a direct comparison of the absolute errors reveals a distinct advantage for the proposed method. The final column in Table 1. Representing the ratio of the proposed method's error to the standard AB2 error, stabilizes at approximately 0.40 as $h \to 0$. This indicates that the proposed method consistently reduces the global error by roughly 60% compared to the standard AB2 scheme. Consequently, while both methods share the same asymptotic order, the proposed algorithm achieves significantly higher precision for a given time step $h$.

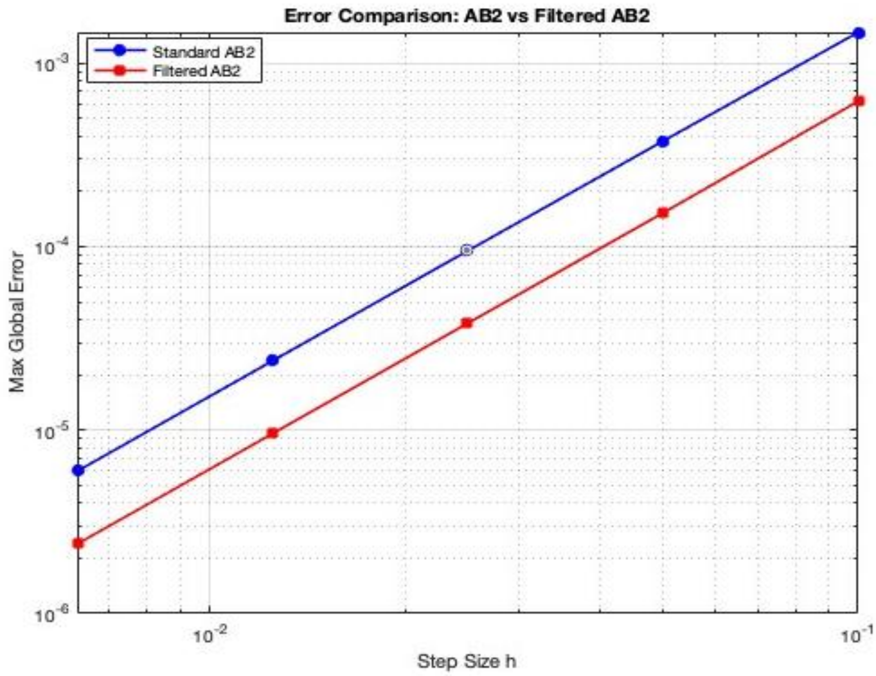


**Figure 2.** Error comparison of filtered AB2 compared to AB2 method.

**Table 1.** Comparison of errors and convergence rates between AB2 and the Filtered AB2 Method

| $h$ | Standard AB2 | | Filtered AB2 | | Error Rate |
|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Filtered/Standard |
| 0.10000 | 1.4714e-3 | ----- | 6.1947e-4 | | 0.4210 |
| 0.05000 | 3.7553e-4 | 1.9702 | 1.5247e-4 | 2.0225 | 0.4060 |
| 0.02500 | 9.4843e-5 | 1.9853 | 3.8187e-5 | 1.9973 | 0.4026 |
| 0.01250 | 2.3831e-5 | 1.9927 | 9.5625e-6 | 1.9976 | 0.4013 |
| 0.00625 | 5.9726e-6 | 1.9964 | 2.3928e-7 | 1.9987 | 0.4006 |

### 3.2. The Simple Harmonic Oscillator

We next investigate the performance of the filtered scheme on the linear oscillatory problem defined by (equation 27):

$$u'' + \omega^2 u = 0$$
$$u(0) = 1, \quad (27)$$
$$u'(0) = 0$$

The exact solution $u(t) = \cos(\omega t)$ represents a non-decaying oscillation, making it an ideal test case for detecting artificial damping or phase drift introduced by the numerical scheme. The system was integrated using both the standard AB2 method and the proposed filtered AB2 method with $\omega = 4$ over the interval $t \in [0,10]$. Figure 3 presents the trajectory of the computed solutions compared to the exact solution $u(t) = \cos(4t)$. While both methods capture the fundamental frequency of the oscillation, the standard AB2 method exhibits a noticeable phase error that grows linearly with time. In contrast, the filtered AB2 scheme significantly reduces this phase lag, maintaining a trajectory that adheres more closely to the exact solution peaks. The global error analysis confirms that the proposed method yields a smaller maximum error for the same fixed step size $h = 0.05$.

## 4. Discussion and Conclusion

While the standard Adams-Bashforth method is satisfactory, the addition of a time filter offers a superior alternative. This approach is conceptually straightforward and can be easily integrated into existing codebases by adding just a single line. By analyzing the coupled system as a linear multistep method, we proved that this filtering step preserves zero-stability while effectively halving the truncation error. These theoretical findings are fully consistent with the numerical results.

This paper has demonstrated that the integration of a simple four-point time filter into the standard second-order Adams-Bashforth (AB2) method provides a robust and highly efficient enhancement for numerical integration. Through a comprehensive linear multistep analysis, we have established that the filtered method remains 0-stable for the parameter range $-2/3 \leq \upsilon < 2$. Specifically, by selecting the optimal value of $\upsilon = -2/3$, the local truncation error is effectively halved compared to the original AB2 scheme. Numerical experiments validate these theoretical findings. For the linear scalar test problem, the filtered method reduced the global error by approximately 60% while maintaining the expected second-order convergence rate. Furthermore, application to the simple harmonic oscillator revealed that the filter significantly mitigates phase lag and artificial damping, maintaining a trajectory much closer to the exact solution than the standard AB2 method. Given that the filter requires negligible computational overhead, often just a single line of code, it represents a superior alternative for industry and research applications requiring high-fidelity time integration.

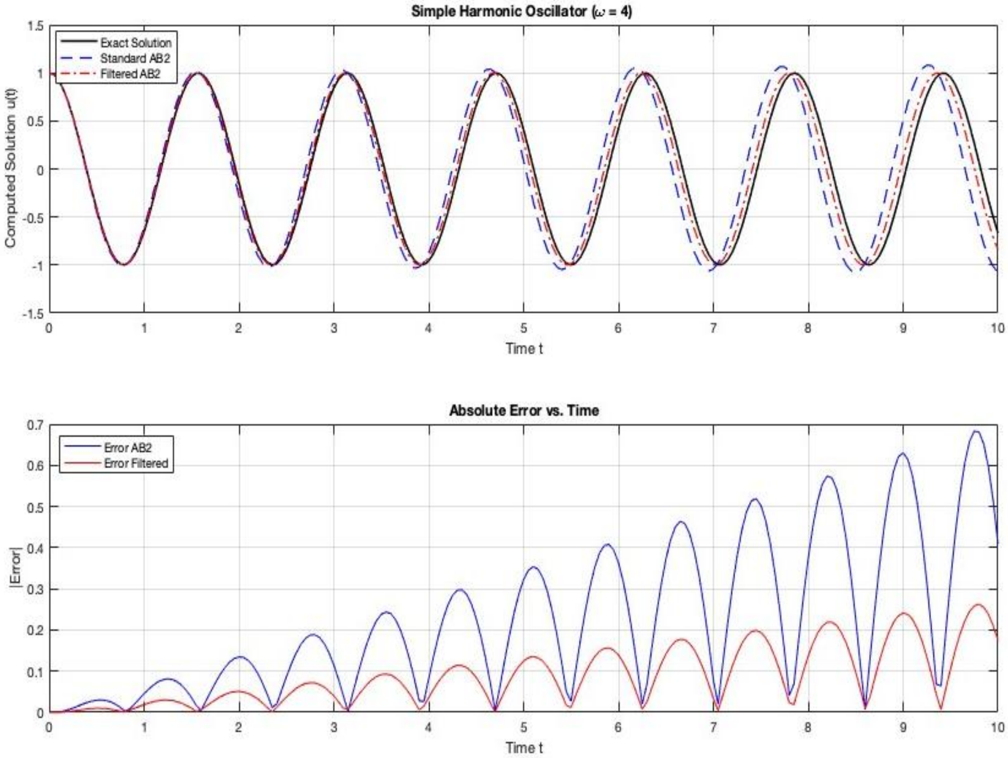


**Figure 3.** Error comparison of filtered AB2 compared to AB2 method.

**Author Contributions**

The percentages of the author' contributions are presented below. The author reviewed and approved the final version of the manuscript.

| | A.G. |
|---|---|
| C | 100 |
| D | 100 |
| S | 100 |
| DCP | 100 |
| DAI | 100 |
| L | 100 |
| W | 100 |
| CR | 100 |
| SR | 100 |
| PM | 100 |
| FA | 100 |

C= concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

**Conflict of Interest**

The author declared that there is no conflict of interest.

**Ethical Consideration**

Ethics committee approval was not required for this study because of there was no study on animals or humans.

## References


Asselin, R. (1972). Frequency Filter for Time Integrations. *Monthly Weather Review*, *100*(6), 487-490. https://doi.org/10.1175/15200493(1972)100<0487:FFFTI>2.3.CO;2

Butcher, J. C. (2016). *Numerical methods for ordinary differential equations* (3. bs.). John Wiley and Sons, Ltd. https://doi.org/10.1002/9781119121534

DeCaria, V., Gottlieb, S., Grant, Z. J., & Layton, W. J. (2022). A general linear method approach to the design and optimization of efficient, accurate, and easily implemented time-stepping methods in CFD. *Journal of Computational Physics*, *455*, Article 110927. https://doi.org/10.1016/j.jcp.2021.110927

Durran, D. R. (2010). *Numerical methods for fluid dynamics* (2. bs., Cilt 32). Springer. https://doi.org/10.1007/978-1-4419-6412-0

Griffiths, D. F., & Higham, D. J. (2010). *Numerical methods for ordinary differential equations*. Springer-Verlag London, Ltd. https://doi.org/10.1007/978-0-85729-148-6

Guzel, A., & Layton, W. (2018). Time filters increase accuracy of the fully implicit method. *BIT Numerical Mathematics*, *58*(2), 301–315. https://doi.org/10.1007/s10543-018-0695-z

Guzel, A., & Trenchea, C. (2018). The Williams step increases the stability and accuracy of the hoRA time filter. *Applied Numerical Mathematics*, *131*, 158–173. https://doi.org/10.1016/j.apnum.2018.05.003

Hairer, E., Nørsett, S. P., & Wanner, G. (1993). Solving Ordinary Differential Equations I. In *Springer Series in Computational Mathematics*. Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-540-78862-1

Hurl, N., Layton, W., Li, Y., & Trenchea, C. (2014). Stability analysis of the Crank–Nicolson-Leapfrog method with the Robert–Asselin–Williams time filter. *BIT Numerical Mathematics*, *54*(4), 1009–1021. https://doi.org/10.1007/s10543-014-0493-1

Jury, E. (1964). *Theory and Application of the Z-transform Method*. John Wiley and Sons.

Li, Y., & Trenchea, C. (2014). A higher-order Robert–Asselin type time filter. *Journal of Computational Physics*, *259*, 23–32. https://doi.org/10.1016/j.jcp.2013.11.022

McGovern, S. M. (2025). Adaptive step selection for a filtered implicit method. *Journal of Scientific Computing*, *103*(2). https://doi.org/10.1007/s10915-025-02861-w

Robert, A. J. (1966). The integration of a low order spectral form of the primitive meteorological equations. *Journal of the Meteorological Society of Japan. Ser. II*, *44*(5), 237–245. https://doi.org/10.2151/jmsj1965.44.5_237

Williams, P. D. (2009). A proposed modification to the Robert–Asselin time filter. *Monthly Weather Review*, *137*(8), 2538–2546. https://doi.org/10.1175/2009MWR2724.1

Williams, P. D. (2011). The RAW filter: An improvement to the Robert–Asselin filter in semi-implicit integrations. *Monthly Weather Review*, *139*(6), 1996–2007. https://doi.org/10.1175/2010MWR3601.1