DOI: 10.25092/baunfbed.476597 **J. BAUN Inst. Sci. Technol., 20(3) Special Issue, 53-61, (2018)**

Numerical solution and stability analysis of transient MHD duct flow

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Geliş Tarihi (Recived Date): 02.08.2018
Kabul Tarihi (Accepted Date): 24.10.2018

Abstract

This paper simulates the 2D transient magnetohydrodynamic (MHD) flow in a rectangular duct in terms of the velocity of the fluid and the induced magnetic field by using the radial basis function (RBF) approximation. The inhomogeneities in the Poisson's type MHD equations are approximated using the polynomial functions (1+r) and the particular solution is found satisfying both the equations and the boundary conditions (no-slip and insulated walls). The Euler scheme is used for advancing the solution to steady-state with a time increment and a relaxation parameter which are determined for achieving stable solution. It is shown that, as Hartmann number increases, the fluid becomes stagnant at the center of the duct, the flow is flattened and boundary layers are developed on the Hartmann and side walls. These are the well-known characteristics of the MHD duct flow. The stability analysis is also carried in terms of the spectral radius of the coefficient matrix of the discretized coupled system. Stable solutions are obtained with RBF by using quite large time increment and suitable relaxation parameters on the expense of explicit Euler time-integration scheme used.

Keywords: *MHD duct flow, RBF, Euler time-integration, stability.*

Zamana bağımlı MHD kanal akışının nümerik çözümü ve kararlılık analizi

Özet

Bu çalışmada, dikdörtgen kesit içerisindeki iki boyutlu zamana bağlı olan MHD akışı, sıvının hızı ve indüklenen manyetik alan cinsinden radyal baz fonksiyon yaklaştırımı kullanılarak sunulmuştur. Poisson tipinde olan MHD denklemlerindeki homojen olmayan kısımlar, polinom fonksiyonları (1+r) ile yaklaştırılmıştır ve hem denklemleri

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hem de kaymaz ve iletken olmayan sınır koşullarını sağlayan özel bir çözüm bulunmuştur. Euler yöntemi, kararlı çözümü veren zaman aralığı ve yumuşama katsayıları ile kullanılmıştır. Hartmann sayısı artıkça sıvının kanal ortasında durgunlaştığı, akışın düzleştiği, Hartmann ve yan duvarlardaki sınır tabakalarının geliştiği gösterilmiştir. Bunlar MHD kanal akışının en iyi bilinen özellikleridir. Ayrıca, kararlılık analizi, ayrıklaştırılmış birbirine bağlı olan sistemdeki katsayı matrisinin spektral yarıçapı doğrultusunda yapılmıştır. Açık Euler zaman integrasyonu yöntemi kullanılmasına rağmen RBF ile oldukça geniş zaman aralığı ve uygun yumuşama parametreleri kullanılarak kararlı çözümler elde edilmiştir.

Anahtar kelimeler: *MHD kesit akışı, RBF, Euler zaman integrasyonu, kararlılık analizi.*

1. Introduction

The study of magnetohydrodynamic (MHD) flow in channels has many industrial applications such as MHD generators, MHD flowmeters, nuclear reactors and electromagnetic pumps. The MHD equations, governed by the Navier-Stokes equations and Maxwell equations of electromagnetism through Ohm's law have been solved by several numerical methods. Tezer-Sezgin et al. [1] considered the steady MHD flow in a rectangular duct with arbitrarily conducting walls. The numerical results are obtained by using boundary element method (BEM) for moderate values of Hartmann number ($1 \leq M \leq 10$). Dual reciprocity boundary element method (DRBEM) is implemented to solve MHD duct flow with insulating boundary in [2]. The right hand side function is approximated by using osculating radial basis functions (RBF). Tezer-Sezgin [3] proposed the polynomial and Fourier based differential quadrature method (DQM) for solving the steady MHD flow under the effect of a transverse external oblique magnetic field. The numerical results are presented in terms of velocity and induced magnetic field for several values of Hartmann number. In the references [4, 5], BEM is employed to solve MHD flow for large values of Hartmann number ($M \leq 300$). Carabineau et al. [6] developed the pseudospectral collocation method for obtaining numerical solution of MHD flow in the cross-section of square and circular ducts. An exponential higher-order compact (EHOC) difference scheme is applied for solving coupled MHD equations for several values of Hartmann number by Li et al. [7].

The unsteady two-dimensional MHD flows in channels are also studied. Bozkaya et al. [8] solved transient MHD flow problem in a rectangular duct with insulating walls by using DRBEM in space and DQM in time. They found that as Hartmann number increases, the steady-state solutions are reached at a faster rate. In the work [9], the numerical results for the unsteady MHD duct flow with arbitrarily conducting walls are obtained by using a meshless local Petrov-Galerkin (MLPG) method. Dehghan [10] implemented the method of variably scaled radial kernels to solve MHD flow for different geometries of the duct cross-section. The Crank-Nicolson scheme and the method of lines (MOL) are used for the time discretization.

The stability analysis of the BEM solution of the Diffusion equation is studied by Sharp in [11]. It is observed that as the time step decreases, the quality of the approximation deteriorates showing the state of the instability. Ramesh [12] et al. performed the stability analysis of unsteady heat conduction flow by using the eigenvalue decomposition of the coefficient matrix of the multiple reciprocity discretized system.

In this study, the unsteady MHD flow in a rectangular duct with insulating walls is solved by using the RBF approximation in space and Euler scheme in time. The effect of the magnetic field on the velocity and the induced magnetic field is investigated. It is found that, the increase in the Hartmann number develops the boundary layers which is the well-known behavior of the MHD duct flow. The numerical stability analysis is carried for the RBF approximation of the time-dependent convection-diffusion type MHD flow equations when the explicit Euler time integration scheme is used with relaxation parameters. The numerical stability of the velocity and the induced magnetic field is shown in terms of maximum eigenvalues of the discretized coefficient matrices which is the main contribution of this study. The optimal choices of the time increment, relaxation parameter for certain values of the Hartmann number are found numerically to achieve stable solutions. It is observed that the numerical results are stable for the choice of relaxation parameter in the range $0.5 \leq \alpha_{u_1} < 1$ for all $M \leq 100$.

2. The physical problem and mathematical formulation

The unsteady, laminar, fully-developed flow of a viscous, incompressible, electrically conducting fluid is considered in a square duct Ω=[-1,1]×[-1,1]. The fluid is driven by a constant applied pressure gradient in the pipe-axis (*z*-axis) direction and the flow is subjected to a uniform magnetic field in the *y*- direction. The non-dimensional coupled MHD duct flow equations [13] are given in terms of the velocity $V(x,y,t)$ and the induced magnetic field $B(x,y,t)$ as

$$\nabla^2 V + M\frac{\partial B}{\partial y} = -1 + \frac{\partial V}{\partial t} \tag{1}$$

$$\nabla^2 B + M\frac{\partial V}{\partial y} = \frac{\partial B}{\partial t} \tag{2}$$

In $\Omega \times [0, \infty)$ with zero initial, and no-slip insulated walls boundary conditions shown in Figure 1,

$$V(x,y,0) = B(x,y,0) = 0, \qquad (x,y) \in \Omega \tag{3}$$

$$V(x,y,t) = B(x,y,t) = 0, \qquad (x,y) \in \partial\Omega, \quad t \geq 0 \tag{4}$$

where $M = B_0 L\sqrt{\sigma/\nu\rho}$ is the Hartmann number. Here, L, B_0, σ, ν and ρ are the characteristic length, the external magnetic field intensity, electrical conductivity, kinematic viscosity and the density of the fluid, respectively.

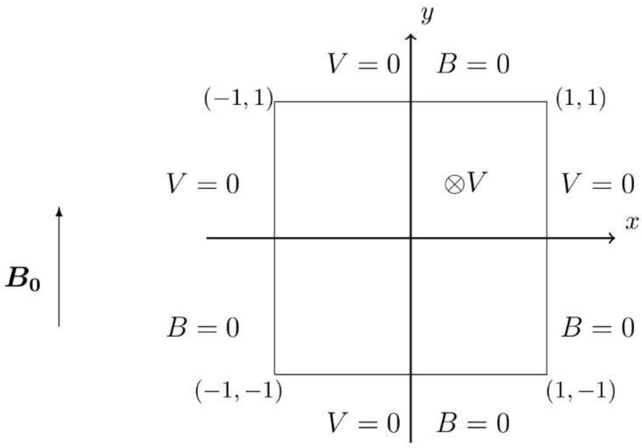

Figure 1. Square duct and boundary conditions.

The decoupled form of the MHD flow equations (1)-(2) is obtained

$$\nabla^2 U_1 + M\frac{\partial U_1}{\partial y} = -1 + \frac{\partial U_1}{\partial t} \tag{5}$$

$$\nabla^2 U_2 - M\frac{\partial U_2}{\partial y} = -1 + \frac{\partial U_2}{\partial t} \tag{6}$$

with the initial and boundary conditions

$$U_1(x,y,0) = U_2(x,y,0) = 0, \qquad (x,y) \in \Omega \tag{7}$$

$$U_1(x,y,t) = U_2(x,y,t) = 0, \qquad (x,y) \in \partial\Omega, \quad t \geq 0 \tag{8}$$

by the change of variables

$$U_1 = V + B, \quad U_2 = V - B. \tag{9}$$

3. Radial basis function approximation (RBF)

The decoupled convection-diffusion type MHD flow equations (5) and (6) can be considered as Poisson's type when all the terms except Laplacian are taken as inhomogeneity. Thus, the RBF method is described on the Poisson's type equation $\nabla^2 u = h(x,y) + \frac{\partial u}{\partial t} = f(x,y,t)$ with the boundary condition $Bu = g(x,y,t)$ where $h(x,y) = -1 - \frac{\partial U_1}{\partial y}$ and $h(x,y) = -1 + \frac{\partial U_2}{\partial y}$, respectively for equations (5) and (6). The boundary operator B is given identity *(B=I)* and $g(x,y,t) = 0$. The inhomogeneity function $f(x,y,t)$ and the particular solution $u(x,y,t)$ are approximated by the radial basis functions $\varphi_j(r)$ and $\Psi_j(r)$ as

$$f(x,y,t) = \sum_{j=1}^{n} a_j(t)\varphi_j(r), \quad u(x,y,t) = \sum_{j=1}^{n} a_j(t)\Psi_j(r), \quad (x,y) \in \Omega \tag{10}$$

where $\nabla^2\Psi_j(r) = \varphi_j(r),\ r = ((x - x_j)^2 + (y - y_j)^2)^{1/2}$ being the Euclidean distance and n is the number of unknown coefficients. In this method [14], the approximate particular solution u becomes the solution of the original equation satisfying the boundary condition

$$\sum_{j=1}^{n} a_j(t)B\Psi_j(r) = g(x,y,t), \qquad (x,y) \in \partial\Omega. \tag{11}$$

Discretizing the boundary and the domain by taking N_b boundary and N_i interior points, we get the solution vector $\boldsymbol{u} = \boldsymbol{Ua}$ where $U_{ij} = \Psi_j(r_i),\ 1 \le i,j \le n$ and the unknown vector $\boldsymbol{a}$, depending on time, is the solution of the linear system $\boldsymbol{Ca} = \boldsymbol{d}$ obtained from the collocation of the equations (10)-(11). The nonsingular coefficient matrix $\boldsymbol{C}_{n\times n}$ [15] and the right hand side vector $\boldsymbol{d}_{n\times 1}$ are given as

$$\boldsymbol{C} = \begin{bmatrix} B\Psi_1(r_1) & B\Psi_2(r_1) & \cdots & B\Psi_n(r_1) \\ \vdots & \vdots & \ddots & \vdots \\ B\Psi_1(r_{N_b}) & B\Psi_2(r_{N_b}) & \cdots & B\Psi_n(r_{N_b}) \\ \varphi_1(r_{N_b+1}) & \varphi_2(r_{N_b+1}) & \cdots & \varphi_n(r_{N_b+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(r_n) & \varphi_2(r_n) & \cdots & \varphi_n(r_n) \end{bmatrix}, \qquad \boldsymbol{d} = \begin{bmatrix} g(x_1, y_1, t) \\ \vdots \\ g(x_{N_b}, y_{N_b}, t) \\ f(x_{N_{b+1}}, y_{N_b+1}, t) \\ \vdots \\ f(x_n, y_n, t) \end{bmatrix}.$$

The solution $\boldsymbol{u} = \boldsymbol{UC^{-1}d}$ can be rearranged as partitioning the contributions coming from the boundary condition and interior solution [16]

$$\boldsymbol{u} = \boldsymbol{g_u} + \boldsymbol{Kf} \tag{12}$$

where $\boldsymbol{f} = \{f_i\} = \{h_i\} + \left\{\frac{\partial u_i}{\partial t}\right\}, 1 \le i \le N_b + N_i,\ \boldsymbol{g_u} = \boldsymbol{R_1 u_{bc}}$ and $\boldsymbol{K} = [\underset{n\times N_b}{\boldsymbol{0}}\ \underset{n\times N_i}{\boldsymbol{R_2}}]$. Here, $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ are the submatrices of $\boldsymbol{UC}^{-1}$ as $\boldsymbol{UC^{-1}} = [\underset{n\times N_b}{\mathbf{R_1}}\ \underset{n\times N_i}{\boldsymbol{R_2}}]$, and $\boldsymbol{u_{bc}}$ is the vector containing boundary values of the solution $\boldsymbol{u}$.

The application of the RBF approximation to unsteady MHD duct flow equations (5)-(6) gives

$$\boldsymbol{U_1} = \boldsymbol{g_{u_1}} - M\boldsymbol{K}\frac{\partial \boldsymbol{F}}{\partial y}\boldsymbol{F}^{-1}\boldsymbol{U_1} - \boldsymbol{Kl} + \boldsymbol{K}\frac{\partial \boldsymbol{U_1}}{\partial t} \tag{13}$$

$$\boldsymbol{U_2} = \boldsymbol{g_{u_2}} + M\boldsymbol{K}\frac{\partial \boldsymbol{F}}{\partial y}\boldsymbol{F}^{-1}\boldsymbol{U_2} - \boldsymbol{Kl} + \boldsymbol{K}\frac{\partial \boldsymbol{U_2}}{\partial t} \tag{14}$$

where $\boldsymbol{F}$ is the coordinate matrix constructed from $F_{ij} = \varphi_j(r_i)$ and $l_{ij} = 1,\ 1 \le i,j \le n$.

The explicit Euler method is used for the time derivatives in the equations (13)-(14) with the relaxation parameters α_{u_1} and α_{u_2}, we obtain the final discretized system for the unsteady MHD duct flow equations

$$\boldsymbol{C}_{\boldsymbol{u}_1}\boldsymbol{U}_{\boldsymbol{1}}^{(m+1)} = \boldsymbol{b}_{\boldsymbol{u}_1}^{(m)} - \boldsymbol{L}_{\boldsymbol{u}_1}\boldsymbol{U}_{\boldsymbol{1}}^{(m)} \tag{15}$$

$$\boldsymbol{C}_{\boldsymbol{u}_2}\boldsymbol{U}_{\boldsymbol{2}}^{(m+1)} = \boldsymbol{b}_{\boldsymbol{u}_2}^{(m)} - \boldsymbol{L}_{\boldsymbol{u}_2}\boldsymbol{U}_{\boldsymbol{2}}^{(m)} \tag{16}$$

where

$$\boldsymbol{C}_{\boldsymbol{u}_1} = \alpha_{u_1}\boldsymbol{S}_{\boldsymbol{u}_1} - \frac{1}{\Delta t}\boldsymbol{K}, \qquad \boldsymbol{L}_{\boldsymbol{u}_1} = (1-\alpha_{u_1})\boldsymbol{S}_{\boldsymbol{u}_1} - \frac{1}{\Delta t}\boldsymbol{K},$$

$$\boldsymbol{C}_{\boldsymbol{u}_2} = \alpha_{u_2}\boldsymbol{S}_{\boldsymbol{u}_2} - \frac{1}{\Delta t}\boldsymbol{K}, \qquad \boldsymbol{L}_{\boldsymbol{u}_2} = (1-\alpha_{u_2})\boldsymbol{S}_{\boldsymbol{u}_2} - \frac{1}{\Delta t}\boldsymbol{K},$$

$$\boldsymbol{S}_{\boldsymbol{u}_1} = \boldsymbol{I} + M\boldsymbol{K}\frac{\partial \boldsymbol{F}}{\partial y}\boldsymbol{F}^{-1}, \qquad \boldsymbol{S}_{\boldsymbol{u}_2} = \boldsymbol{I} - M\boldsymbol{K}\frac{\partial \boldsymbol{F}}{\partial y}\boldsymbol{F}^{-1},$$

$$\boldsymbol{b}_{\boldsymbol{u}_1} = \boldsymbol{g}_{\boldsymbol{u}_1} - \boldsymbol{K}\boldsymbol{l}, \qquad \boldsymbol{b}_{\boldsymbol{u}_2} = \boldsymbol{g}_{\boldsymbol{u}_2} - \boldsymbol{K}\boldsymbol{l}.$$

The iteration continues until the stopping criteria $\left\|\boldsymbol{z}^{(m+1)} - \boldsymbol{z}^{(m)}\right\|_\infty < 10^{-6}$ is satisfied for reaching steady-state where $\boldsymbol{z}$ denotes $\boldsymbol{U_1}$ and $\boldsymbol{U_2}$ for the solutions of (15) and (16), respectively. The solutions, the velocity and the induced magnetic field are obtained from the back transformation $\boldsymbol{V = (U_1 + U_2)}/2$ and $\boldsymbol{B = (U_1 - U_2)}/2$. In the equations (15)-(16), the vectors $\boldsymbol{b}_{\boldsymbol{u}_1}^{(m)}$ and $\boldsymbol{b}_{\boldsymbol{u}_2}^{(m)}$ are known and do not contribute to the stability analysis. Thus, the stability conditions for the RBF space – Euler time discretized system of MHD duct flow equations are [12]

$$\rho(\boldsymbol{C}_{\boldsymbol{u}_1}^{-1}\boldsymbol{L}_{\boldsymbol{u}_1}) < 1 \tag{17}$$

$$\rho(\boldsymbol{C}_{\boldsymbol{u}_2}^{-1}\boldsymbol{L}_{\boldsymbol{u}_2}) < 1 \tag{18}$$

where $\rho\left(\boldsymbol{C}_{\boldsymbol{u}_1}^{-1}\boldsymbol{L}_{\boldsymbol{u}_1}\right)$ and $\rho\left(\boldsymbol{C}_{\boldsymbol{u}_2}^{-1}\boldsymbol{L}_{\boldsymbol{u}_2}\right)$ are the spectral radius of the matrices $\boldsymbol{C}_{\boldsymbol{u}_1}^{-1}\boldsymbol{L}_{\boldsymbol{u}_1}$ and $\boldsymbol{C}_{\boldsymbol{u}_2}^{-1}\boldsymbol{L}_{\boldsymbol{u}_2}$, respectively. These matrices differ only in the ± sign of the matrices $\boldsymbol{S}_{\boldsymbol{u}_1}$ and $\boldsymbol{S}_{\boldsymbol{u}_2}$.

4. Numerical results

In the RBF space discretization for the MHD duct flow equations (15)-(16), we use the polynomial function $\varphi = 1 + r$. In order to obtain smooth numerical results the boundary is discretized by taking N_b=*100, 156, 236* and *336* points for the Hartmann number values *M=10, 30, 50* and *100*, respectively, with the fixed time increment $\Delta t = 0.1$ and the relaxation parameters $\alpha_{u_1} = \alpha_{u_2} = 0.6$. Steady-state solutions of the velocity and the induced magnetic field are shown in terms of equivelocity and current lines in Figure 2. It is observed that velocity contours are symmetric with respect to center lines *x=0* and *y=0.* The flow attains its maximum value through the center of the duct making one vortex at the center. As the Hartmann number increases, fluid shows flattening tendency which is an expected behavior of the MHD duct flow. The increase in the magnetic field intensity develops the boundary layers for the flow near the walls with the thickness *O(1/M)* on the Hartmann walls and $O(1/\sqrt{M})$ on the side walls.

Induced magnetic loops are anti-symmetric with respect to the centerline *y=0.* As M increases, boundary layers are also developed for induced magnetic field on the walls parallel to the applied magnetic field. The magnitude of the induced magnetic field decreases due to the convection dominance in the equation (2) as M increases.

In order to show that numerical results of MHD duct flow obtained from the RBF space-Euler time discretized systems (15)-(16) are stable, the spectral radius (maximum eigenvalue in magnitude) of the coefficient matrices $\boldsymbol{C}_{\boldsymbol{u_1}}^{-1}\boldsymbol{L}_{\boldsymbol{u_1}}$ and $\boldsymbol{C}_{\boldsymbol{u_2}}^{-1}\boldsymbol{L}_{\boldsymbol{u_2}}$ are computed for several values of the time increment Δt, relaxation parameters $\alpha_{u_1} = \alpha_{u_2}$ and the Hartmann number *M.* The maximum eigenvalues in magnitude ($\rho = \max_{1 \le j \le n} |\lambda_j|$) are presented in Tables 1-2.

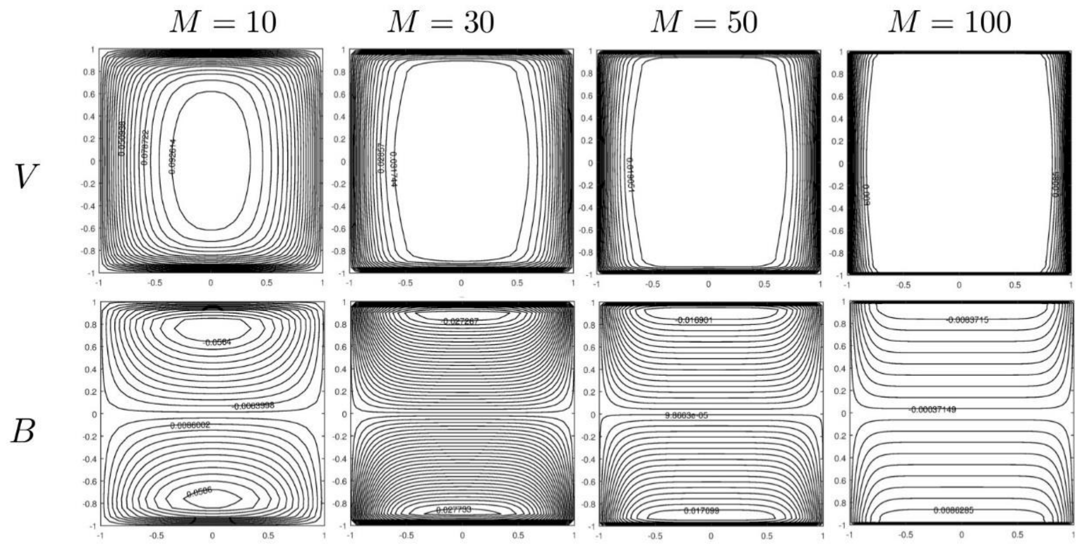

Figure 2. Velocity and induced magnetic field contours for $\Delta t = 0.1$.

Table 1 shows the effects of the relaxation parameters and the time increment on the spectral radius of the coefficient matrices $\boldsymbol{C}_{\boldsymbol{u_1}}^{-1}\boldsymbol{L}_{\boldsymbol{u_1}}$ or $\boldsymbol{C}_{\boldsymbol{u_2}}^{-1}\boldsymbol{L}_{\boldsymbol{u_2}}$ for a fixed Hartmann number $M=1$ and $N_b=100$. It is found that as relaxation parameter decreases, the maximum eigenvalue increases and for the choice of $\alpha_{u_1} \le 0.5$ the method becomes unstable. This is an expected result since Euler method with a relaxation parameter tends to be explicit scheme with the small value of α_{u_1} ($u^{(m+1)} = (1-\alpha_{u_1})u^{(m)} - \alpha_{u_1}u^{(m)}$, u representing $\boldsymbol{U_1}$ and $\boldsymbol{U_2}$ in (15) and (16), respectively). As Δt decreases, spectral radius increases but still does not exceed 1 for $0.6 \le \alpha_{u_1} < 1$.

Table 1. Spectral radius ρ for *N_b=100, M=1.*

$\alpha \backslash \Delta t$	0.9	0.8	0.5	0.1	0.01
0.9	0.111	0.124	0.222	0.646	0.950
0.8	0.250	0.250	0.250	0.633	0.950
0.7	0.429	0.429	0.429	0.620	0.950
0.6	0.667	0.667	0.667	0.667	0.950

In Table 2, the maximum eigenvalues are obtained for different values of M and Δt with a fixed $\alpha_{u_1} = 0.9$. An increase in the Hartmann number decreases the spectral radius of the coefficient matrix. The variation of $M \geq 5$ gives always the same eigenvalue which is less than one for the choice of $\Delta t \leq 0.5$. This shows that for large values of Hartmann number one does not require smaller time increment to achieve stable solution.

Table 2. Spectral radius ρ for $\alpha_{u_1} = 0.9$.

Δt	*M=1*	*M=5*	*M=10*	*M=50*	*M=100*
0.01	0.950	0.898	0.771	0.266	0.111
0.1	0.646	0.443	0.197	0.111	0.111
0.5	0.222	0.111	0.111	0.111	0.111
0.8	0.124	0.111	0.111	0.111	0.111
0.9	0.111	0.111	0.111	0.111	0.111

5. Conclusion

In this study, the RBF approximation is developed for solving the equations of unsteady MHD flow in a rectangular duct with insulating walls. The Euler scheme with a relaxation parameter is used for the time integration in the MHD equations. The impact of the external magnetic field is analyzed on the behaviors of the flow and the induced magnetic field by simulating equivelocity and equal current lines. Numerical results show that as the Hartmann number increases, boundary layers are formed and the flow is flattened. Numerical stability analysis of RBF space - Euler time approximation is also performed in terms of spectral radius of related coefficient matrices of the discretized system. It is found that quite large time increment can be used for achieving stable numerical results when suitable relaxation parameters are used for a certain Hartmann number.

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