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Homotopy methods for fractional linear/nonlinear differential equations with a local derivative operator

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Abstract

In this paper, we consider some linear/nonlinear differential equations (DEs) containing conformable derivative operator. We obtain approximate solutions of these mentioned DEs in the form of infinite series which converges rapidly to their exact values by using and homotopy analysis method (HAM) and modified homotopy perturbation method (MHPM). Using the conformable operator in solutions of different types of DEs makes the solution steps are computable easily. Especially, the conformable operator has been used in modelling DEs and identifying particular problems such as biological, engineering, economic sciences and other some important fields of application. In this context, the aim of this study is to solve some illustrative linear/nonlinear problems as mathematically and to compare the exact solutions with the obtained solutions by considering some plots. Moreover, it is an aim to show the authenticity, applicability, and suitability of the methods constructed with the conformable operator.

Keywords: *Approximate solution, conformable operator, homotopy analysis method, modified homotopy perturbation method, nonlinear differential equations.*

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Lokal türev operatörlü lineer/lineer olmayan diferansiyel denklemler için homotopi metotları

Özet

Bu çalışmada conformable (uyumlu) türev operatörü (CTO) içeren bazı lineer/lineer olmayan diferansiyel denklemler ele alınmıştır. Homotopi analiz metodunu (HAM) ve modifiyeli homotopi pertürbasyon metodunu (MHPM) kullanarak bu bahsi geçen denklemlerin sonsuz seri formunda yaklaşık çözümleri elde edilmiştir. CTO kullanılması farklı türden diferansiyel denklemlerin çözümlerini elde etmede çözüm adımlarının kolay bir şekilde hesaplanmasını sağlamaktadır. Özellikle CTO mühendislik, fiziksel bilimler, ekonomi ve diğer bazı alanlardaki problemleri modellemede kullanılmaktadır. Bu bağlamda, bu çalışmanın amacı bazı lineer/lineer olmayan diferansiyel denklemleri matematiksel olarak çözmek ve çözüm grafiklerini kullanarak elde edilen yaklaşık çözümler ile tam çözümleri karşılaştırmaktır. Ayrıca CTO ile yeniden tanımlanan HAM ve MHPM metotlarının güvenirliğini, uygulanabilirliğini ve elverişliliğini göstermektir.

Anahtar kelimeler: *Yaklaşık çözüm, uyumlu operatör, homotopi analiz metodu, modifiyeli homotopi pertürbasyon metodu, lineer olmayan diferansiyel denklemler.*

1. Introduction

In the last decade, several numerical, approximate and analytical methods have been investigated to get solutions of linear/nonlinear fractional PDEs. Especially, in the physics and engineering areas, numerous applications and theoretical aspects of fractional calculus have been studied. For example, in [1-10] researchers solved some important problems modelled with fractional DEs. Furthermore, conformable derivative operator defined in 2014 [11], is preferred by some researchers [12-19] to apply it to FDEs and to model some special physical, chemical and engineering problems. Moreover, the mentioned approximate methods have been applied extensively to real-life problems by taking these theoretical aspects into consideration. For instance, approximate-analytical methods have included homotopy analysis method (HAM) [20-22], Adomian decomposition method (ADM) [23-25], differential transform method (DTM) [26, 27], homotopy perturbation method (HPM) [28, 29], modified homotopy perturbation method (MHPM) [30], variational iteration method (VIM) [31], sine-Gordon expansion method [32], q-homotopy analysis method (q-HAM), [33], etc.

2. Conformable derivative operator

2.1. Definition

Given a function $\xi:[0,\infty)\to R.$ Then the conformable derivative of ξ order $\alpha\in(0,1]$ is defined for all $t>0$ by [11]

$$T_{*t}^{\alpha}(\xi)(t)=\lim_{\varepsilon\to 0}\frac{\xi\left(t+\varepsilon t^{1-\alpha}\right)-\xi(t)}{\varepsilon}.$$

2.2. Definition

The $\alpha-$fractional integral of ξ is defined by

$$I_{*t}^{\alpha}(\xi)(t)=I_{a}^{1}\left(t^{\alpha-1}\xi\right)=\int_{a}^{t}\frac{\xi(x)}{x^{1-\alpha}}dx,\quad \alpha\in(0,1).$$

2.3. Theorem

Let $\alpha\in(0,1]$ and ξ,ϑ be $\alpha-$ differentiable at a point $t>0.$ Then [11];

(i) $T_{*t}^{\alpha}(a\xi+b\vartheta)=aT_{*t}^{\alpha}(\xi)+bT_{*t}^{\alpha}(\vartheta)$ for all $a,b\in R,$

(ii) $T_{*t}^{\alpha}\left(t^{k}\right)=kt^{k-\alpha}$ for all $k\in R,$

(iii) $T_{*t}^{\alpha}(\xi(t))=0$ if $\xi(t)=k,$

(iv) $T_{*t}^{\alpha}(\xi\vartheta)=\xi T_{*t}^{\alpha}(\vartheta)+\vartheta T_{*t}^{\alpha}(\xi),$

(v) $T_{*t}^{\alpha}(\xi/\vartheta)=\dfrac{\vartheta T_{*t}^{\alpha}(\xi)-\xi T_{*t}^{\alpha}(\vartheta)}{\vartheta^{2}},$

(vi) If $\xi(t)$ is differentiable, then $T_{*t}^{\alpha}(\xi(t))=t^{1-\alpha}\dfrac{d}{dt}\xi(t).$

2.4. Lemma

Consider ξ as an $n-$times differentiable at $t.$ Then we have $T_{*t}^{\alpha}(\xi(t))=t^{\lceil\alpha\rceil-\alpha}\xi^{\lceil\alpha\rceil}(t),$ for all $t>0,$ $\alpha\in(n,n+1]$ [11].

3. Homotopy analysis method in the conformable sense

This section of the study proposes the solution strategies that are generated by homotopy analysis method in the conformable-type derivative (CHAM). Firstly, we take the following general form of a nonlinear equation:

$$\mathcal{N}\left[\psi(x,t)\right]=0 \tag{1}$$

where $\mathcal{N}(.)$ is a nonlinear operator. Then, the deformation equation is presented as,

$$\begin{aligned}(1-p)\left\{L\left[\phi(x,t;p)-\psi_{0}(x,t)\right]\right\}&\\ =p\hbar H(x,t)\mathcal{N}\left[\phi(x,t;p)\right]&\end{aligned} \tag{2}$$

Let $\psi_{0}(x,t)$ show an initial estimation value of the exact solution of Eq. (1), $p\in[0,1]$ is an embedding parameter, $\hbar\neq 0$ is an supporting parameter, $H(x,t)\neq 0$ is an supporting function, and $L=T_{*t}^{\alpha}$ an supporting linear operator. It is free to choose the supporting parameters by applying the suggested method. Clearly, if $p=0$ and $p=1,$ Eq.(2) turns out to be

$$\phi(x,t;0)=\psi_0(x,t),\ \phi(x,t;1)=\psi(x,t) \tag{3}$$

respectively. Thus, p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial value $\psi_0(x,t)$ to the solution $\psi(x,t)$. Then, we consider the Taylor series expansion of $\phi(x,t;p)$ with respect to p, we get

$$\phi(x,t;p)=\psi_0(x,t)+\sum_{m=1}^{\infty}\psi_m(x,t)\,p^m, \tag{4}$$

where

$$\psi_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;p)}{\partial p^m}\Bigg|_{p=0} \tag{5}$$

If the supporting parameters mentioned above are chosen appropriately, the solution of Eq. (3) exists for $p\in[0,1]$. Then we have

$$\psi(x,t)=\sum_{m=0}^{\infty}\psi_m(x,t). \tag{6}$$

If we take the vector

$$\vec{\psi}_n=\{\psi_0(x,t),\psi_1(x,t),\ldots,\psi_n(x,t)\}, \tag{7}$$

we obtain m th-order altered equation as

$$L[\psi_m(x,t)-\chi_m\psi_{m-1}(x,t)]=\hbar H(x,t)\Re_m(\vec{\psi}_{m-1}(x,t)), \tag{8}$$

where

$$\Re_m(\vec{\psi}_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;p)]}{\partial p^{m-1}}\Bigg|_{p=0}, \tag{9}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{10}$$

Finally, operating the conformable integral operator defined in Definition 2.2. on both side of Eq. (8), we have

$$\psi_m(x,t)=\chi_m\psi_{m-1}(x,t)-\chi_m\sum_{k=0}^{n-1}\psi_{m-1}^{(k)}(x,0^+)\frac{t^k}{k!}+I_{*t}^{\alpha}\hbar H(x,t)\Re_m(\vec{\psi}_{m-1}(x,t)). \tag{11}$$

4. Modified homotopy perturbation method in the conformable sense

In this section we illustrate the solution strategies that are generated by modified homotopy perturbation method in conformable-type derivative (CMHPM). Now we introduce a solution algorithm in an effective way for the general nonlinear PDEs. In this regard, firstly, we consider the following nonlinear equation:

$$T_{*t}^{\alpha}u\left(x,t\right)+L\left(u,u_x,u_{xx}\right)+N\left(u,u_x,u_{xx}\right)=f\left(x,t\right),\quad t>0, \tag{12}$$

where L is a linear operator, N is a nonlinear operator, f is a known analytical function and T_{*t}^{α}, $n-1<\alpha\le n,$ shows the conformable derivative of order α. We also have the initial conditions

$$u^k\left(x,0\right)=g_k\left(x\right),\ k=0,1,\ldots,n-1. \tag{13}$$

In view of the homotopy perturbation method (HPM), we can derive the following homotopy:

$$(1-p)T_{*t}^{\alpha}u\left(x,t\right)+p\left[T_{*t}^{\alpha}u\left(x,t\right)+L\left(u,u_x,u_{xx}\right)+N\left(u,u_x,u_{xx}\right)-f\left(x,t\right)\right]=0, \tag{14}$$

or

$$T_{*t}^{\alpha}u\left(x,t\right)+p\left[L\left(u,u_x,u_{xx}\right)+N\left(u,u_x,u_{xx}\right)-f\left(x,t\right)\right]=0. \tag{15}$$

Therefore, we get the solution of Eq. (15) by using the powers of p:

$$u=u_0+pu_1+p^2u_2+\cdots. \tag{16}$$

The modified form of the HPM which was proposed by Odibat [34] can be established based on the assumption that the function $f\left(x,t\right)$ in Eq. (12) can be divided into parts,

$$f\left(x,t\right)=\sum_{n=0}^{\infty}f_n(x,t). \tag{17}$$

Then we have the following homotopy:

$$\left(1-p\right)T_{*t}^{\alpha}u\left(x,t\right)+p\left[T_{*t}^{\alpha}u\left(x,t\right)+L\left(u,u_x,u_{xx}\right)+N\left(u,u_x,u_{xx}\right)\right]=\sum_{n=0}^{\infty}p^n f_n\left(x,t\right), \tag{18}$$

or

$$T_{*t}^{\alpha}u\left(x,t\right)+p\left[L\left(u,u_x,u_{xx}\right)+N\left(u,u_x,u_{xx}\right)\right]=\sum_{n=0}^{\infty}p^n f_n\left(x,t\right), \tag{19}$$

where $p\in\left[0,1\right]$. If we set $f_0\left(x,t\right)=0,\quad f_1\left(x,t\right)=f\left(x,t\right)$ for $n=0$ or $n\ge 2$, then the homotopy Eq. (18) or Eq. (19) reduces to the homotopy Eq. (14) or Eq. (15), respectively. The form of homotopy Eq. (19) allows us to obtain the individual terms $u_0,\ u_1,\ u_2,\ldots$ in Eq. (16). Substituting Eq. (16) in Eq. (15) and collecting the terms with the same powers of $p,$ we get

$$\begin{aligned}
&p^0 : T_{*t}^{\alpha} u_0 = f_0(x,t),\ u_0^{(k)}(x,0) = g_k(x),\\
&p^1 : T_{*t}^{\alpha} u_1 = -L(u_0) - N(u_0) + f_1(x,t),\ u_1^{(k)}(x,0) = 0,\\
&p^2 : T_{*t}^{\alpha} u_2 = -L(u_1) - N(u_0,u_1) + f_2(x,t),\ u_2^{(k)}(x,0) = 0,\\
&\ \vdots
\end{aligned} \tag{20}$$

At this step, by applying the conformable integral operator on both side of Eq. (20), the first few terms of the MHPM solution can be given by

$$\begin{aligned}
u_0 &= \sum_{k=0}^{m-1} u^{(k)}(x,0)\frac{t^k}{k!} + I_{*t}^{\alpha}\left[f_0(x,t)\right],\\
u_1 &= -I_{*t}^{\alpha}\left[L(u_0)\right] - I_{*t}^{\alpha}\left[N(u_0)\right] + I_{*t}^{\alpha}\left[f_1(x,t)\right],\\
u_2 &= -I_{*t}^{\alpha}\left[L(u_1)\right] - I_{*t}^{\alpha}\left[N(u_0,u_1)\right] + I_{*t}^{\alpha}\left[f_2(x,t)\right],\\
u_3 &= -I_{*t}^{\alpha}\left[L(u_2)\right] - I_{*t}^{\alpha}\left[N(u_0,u_1,u_2)\right] + I_{*t}^{\alpha}\left[f_3(x,t)\right],\\
&\ \vdots
\end{aligned} \tag{21}$$

Then we get the solution in the series form as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

5. Numerical examples

5.1. Example

We consider the one-dimensional linear Klein-Gordon equation [35]

$$T_{*t}^{\alpha} u(x,t) - u_{xx}(x,t) + u(x,t) = 6x^3\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \left(x^3 - 6x\right)t^3,\ t>0,\ \ x\in R,\ 1<\alpha\le 2, \tag{22}$$

with the initial conditions

$$u(x,0) = 0,\ u_t(x,0) = 0. \tag{23}$$

Firstly, we will solve this problem by using the mentioned HAM. Choosing the operator

$$L\big(\phi(x,t;p)\big) = T_{*t}^{\alpha}\phi(x,t;p)$$

with the property that $L[k]=0,\ k$ is a constant. We use the initial approximation $u(x,0)=0.$ Choosing $H(x,t)=1,$ we can construct the $m.$ order modified equation as

$$L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar\Re_m\left(\vec{u}_{m-1}(x,t)\right) \tag{24}$$

where

$$\Re_m\left(\vec{u}_{m-1}(x,t)\right)=T_{*t}^{\alpha}u_{m-1}-\left(u_{m-1}\right)_{xx}+u_{m-1}-\left(1-\chi_m\right)\left[6x^3\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}+\left(x^3-6x\right)t^3\right] \tag{25}$$

Therefore, the solution of Eq. (24) for $m\geq 1$ becomes

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar I_{*t}^{\alpha}\Re_m\left(\vec{u}_{m-1}(x,t)\right). \tag{26}$$

From Eqs. (23), (25) and (26), we obtain

$$u_0(x,t)=0,$$

$$u_1(x,t)=-\hbar\frac{x^3t^3}{\Gamma(4-\alpha)}-\hbar\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)},$$

$$u_2(x,t)=-\hbar\frac{x^3t^3}{\Gamma(4-\alpha)}-\hbar\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)}-\hbar^2\frac{x^3t^3}{\Gamma(4-\alpha)}-\hbar^2\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)}$$
$$-\hbar^2\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)\Gamma(4-\alpha)}-\hbar^2\frac{\left(x^3-12x\right)t^{2\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)},$$
$$\vdots$$

Then the approximate solution of Eq. (22) is presented by

$$\begin{aligned}u(x,t)&=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots\\&=-\hbar\frac{x^3t^3}{\Gamma(4-\alpha)}-\hbar\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)}-\hbar\frac{x^3t^3}{\Gamma(4-\alpha)}-\hbar\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)}\\&\quad-\hbar^2\frac{x^3t^3}{\Gamma(4-\alpha)}-\hbar^2\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)}-\hbar^2\frac{\left(x^3-6x\right)t^{\alpha+3}}{(\alpha+2)(\alpha+3)\Gamma(4-\alpha)}\\&\quad-\hbar^2\frac{\left(x^3-12x\right)t^{2\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)}+\cdots\end{aligned}$$

Then the exact solution of the Eq. (22) subject to the initial conditions Eq. (23) for $\alpha=2,$ is obtained with HAM as $u(x,t)=x^3t^3.$

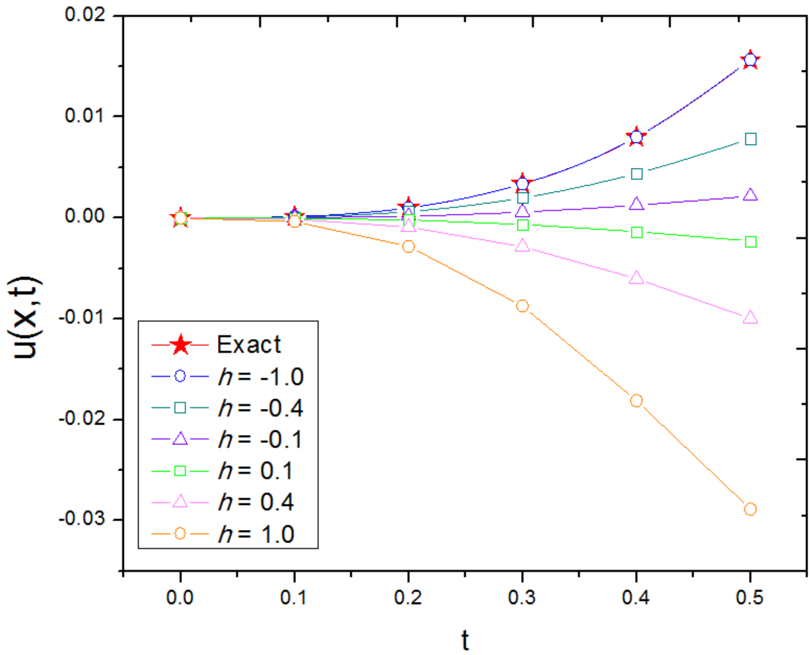


Figure 1. Comparison the HAM and the exact solutions at $\alpha = 2,\ x = 0.5.$

Secondly, we solve the Eq. (22) by using the MHPM. Let us take the initial conditions in Eq. (23) into consideration and use the homotopy in Eq. (19) and finally set $f_0(x,t) = 6x^3 \dfrac{t^{3-\alpha}}{\Gamma(4-\alpha)},\ f_1(x,t) = (x^3 - 6x)t^3,\ \cdots,\ f_n(x,t) = 0,\ n \geq 2.$ The form of homotopy in Eq. (19) allows us to obtain the individual terms $u_0, u_1, u_2, \ldots$ in Eq. (16). Substituting Eq. (16) in Eq. (19) and collecting the terms with the same powers of p, we obtain

$$
\begin{aligned}
&p^0 : T_{*t}^{\alpha} u_0 = f_0(x,t), && u_0(x,0) = 0,\ (u_0)_t(x,0) = 0,\\
&p^1 : T_{*t}^{\alpha} u_1 = (u_0)_{xx} - u_0 + f_1(x,t), && u_1(x,0) = 0,\ (u_1)_t(x,0) = 0,\\
&p^2 : T_{*t}^{\alpha} u_2 = (u_1)_{xx} - u_1,\ u_2^{(k)}(x,0) = 0, && u_2(x,0) = 0,\ (u_2)_t(x,0) = 0,\\
&p^3 : T_{*t}^{\alpha} u_3 = (u_2)_{xx} - u_2,\ u_3^{(k)}(x,0) = 0, && u_3(x,0) = 0,\ (u_3)_t(x,0) = 0,\\
&\vdots
\end{aligned}
\tag{27}
$$

Now, by applying the operator I_{*t}^{α} on both side of Eq. (27), the first few terms of the MHPM solution can be given by

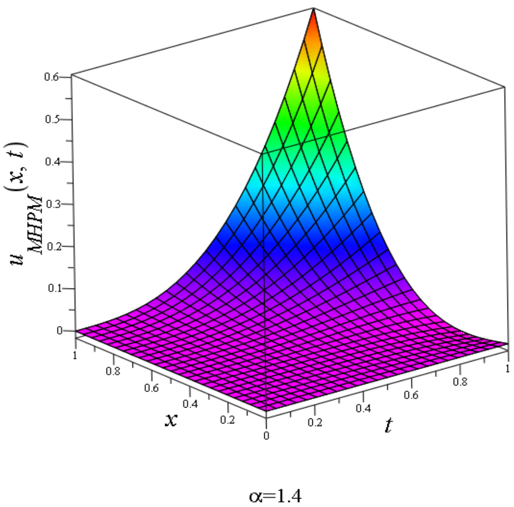
$$
\begin{aligned}
u_0(x,t) &= \frac{x^3 t^3}{\Gamma(4-\alpha)},\\
u_1(x,t) &= -\frac{(x^3 - 6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)\Gamma(4-\alpha)} + \frac{(x^3 - 6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)},
\end{aligned}
$$

$$u_2(x,t)=\frac{6xt^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)}-\frac{6xt^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)\Gamma(4-\alpha)}$$
$$-\frac{(x^3-6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)}+\frac{(x^3-6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)\Gamma(4-\alpha)},$$
$$\vdots$$

In this way, the rest terms of the series can be calculated. The approximate solution of Eq. (22) is given by

$$u(x,t)=\frac{x^3t^3}{\Gamma(4-\alpha)}-\frac{(x^3-6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)\Gamma(4-\alpha)}+\frac{(x^3-6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)}$$
$$+\frac{6xt^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)}-\frac{6xt^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)\Gamma(4-\alpha)}$$
$$-\frac{(x^3-6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)}+\frac{(x^3-6x)t^{\alpha+3}}{(\alpha+2)(\alpha+3)(2\alpha+2)(2\alpha+3)\Gamma(4-\alpha)}+\cdots$$

Then the exact solution of the Eq. (22) subject to the initial conditions Eq. (23) for special case of $\alpha=2,$ is obtained with MHPM as $u(x,t)=x^3t^3$.



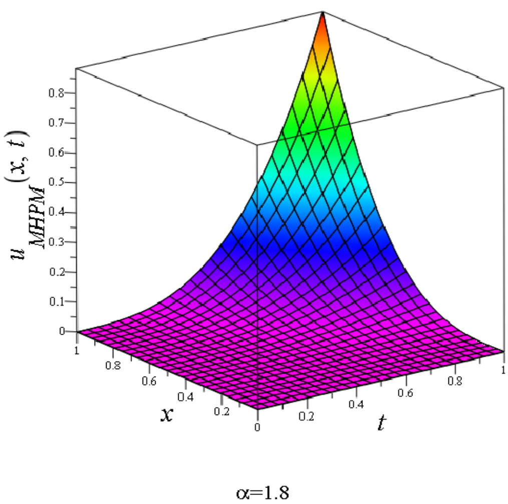


Figure 2. HAM sol. with $\alpha=1.4$ and $\alpha=1.8$ for Example 1

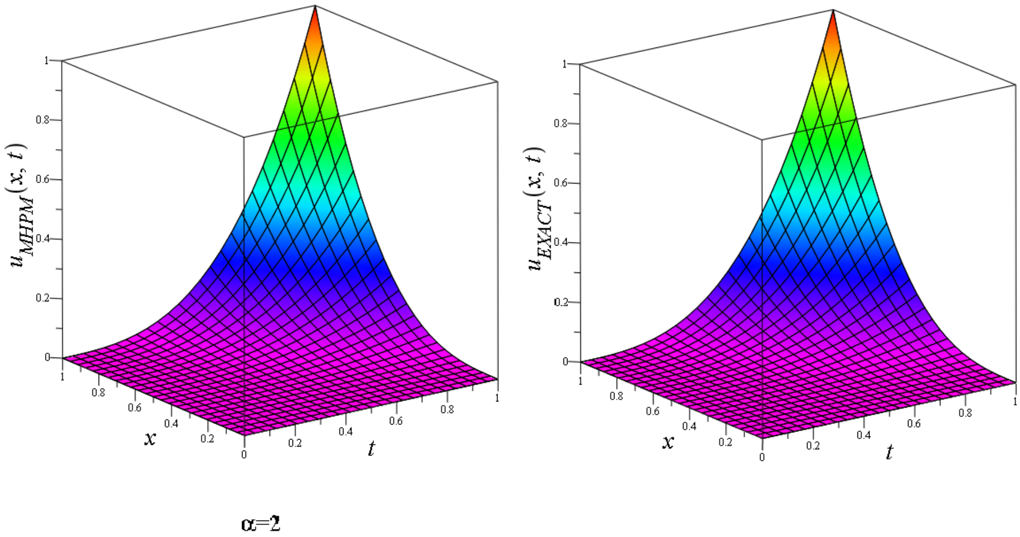


Figure 3. HAM solution and exact solution with $\alpha = 2$ for Example 1.

5.2. Example

Now let us consider the following nonlinear differential equation [36]

$$T_{*t}^{\alpha} u(x,t) + u^2(x,t) = 2x \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + x^2 t^4, \quad t > 0, \quad 0 \le x \le 1, \quad 0 < \alpha \le 1, \tag{28}$$

with the initial condition

$$u(x,0) = 0, \tag{29}$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = t^2. \tag{30}$$

Firstly, we will apply the HAM to the problem. Choosing $H(x,t) = 1,$ we can construct the m. order modified equation as

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar \Re_m\left(\vec{u}_{m-1}(x,t)\right) \tag{31}$$

where

$$\Re_m\left(\vec{u}_{m-1}(x,t)\right) = T_{*t}^{\alpha} u_{m-1} + u_{m-1}^2 - (1-\chi_m)\left[2x \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + x^2 t^4\right]. \tag{32}$$

Now the solution of Eq. (28) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar I_{*t}^{\alpha} \Re_m\left(\vec{u}_{m-1}(x,t)\right). \tag{33}$$

From Eqs. (29), (32) and (33), we get

$$
\begin{aligned}
&u_0(x,t)=0,\\
&u_1(x,t)=-\hbar\frac{xt^2}{\Gamma(3-\alpha)}-\hbar\frac{x^2t^{\alpha+4}}{(\alpha+4)},\\
&u_2(x,t)=-\hbar\frac{xt^2}{\Gamma(3-\alpha)}-\hbar\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\hbar^2\frac{xt^2}{\Gamma(3-\alpha)}-\hbar^2\frac{x^2t^{\alpha+4}}{(\alpha+4)},\\
&u_3(x,t)=-\hbar\frac{xt^2}{\Gamma(3-\alpha)}-2\hbar^2\frac{xt^2}{\Gamma(3-\alpha)}-\hbar^3\frac{xt^2}{\Gamma(3-\alpha)}-\hbar\frac{x^2t^{\alpha+4}}{(\alpha+4)}\\
&\qquad-2\hbar^2\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\hbar^3\frac{x^2t^{\alpha+4}}{(\alpha+4)}+\hbar^3\frac{x^2t^{\alpha+4}}{(\alpha+4)\Gamma(3-\alpha)\Gamma(3-\alpha)}\\
&\qquad+2\hbar^3\frac{x^3t^{2\alpha+6}}{(\alpha+4)(2\alpha+6)\Gamma(3-\alpha)}+\hbar^3\frac{x^4t^{3\alpha+8}}{(\alpha+4)(\alpha+4)(3\alpha+8)},\\
&\vdots
\end{aligned}
$$

These steps give that the approximate solution of Eq. (28) as

$$
\begin{aligned}
u(x,t)=&-\hbar\frac{xt^2}{\Gamma(3-\alpha)}-\hbar\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\hbar\frac{xt^2}{\Gamma(3-\alpha)}-\hbar\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\hbar^2\frac{xt^2}{\Gamma(3-\alpha)}\\
&-\hbar^2\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\hbar\frac{xt^2}{\Gamma(3-\alpha)}-2\hbar^2\frac{xt^2}{\Gamma(3-\alpha)}-\hbar^3\frac{xt^2}{\Gamma(3-\alpha)}-\hbar\frac{x^2t^{\alpha+4}}{(\alpha+4)}\\
&-2\hbar^2\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\hbar^3\frac{x^2t^{\alpha+4}}{(\alpha+4)}+\hbar^3\frac{x^2t^{\alpha+4}}{(\alpha+4)\Gamma(3-\alpha)\Gamma(3-\alpha)}\\
&+2\hbar^3\frac{x^3t^{2\alpha+6}}{(\alpha+4)(2\alpha+6)\Gamma(3-\alpha)}+\hbar^3\frac{x^4t^{3\alpha+8}}{(\alpha+4)(\alpha+4)(3\alpha+8)}+\cdots
\end{aligned}
$$

Then the exact solution of Eq. (28) for $\alpha=1$ is obtained with the HAM as $u(x,t)=xt^2$.
Secondly, we solve the mentioned problem by applying the MHPM to it. Considering the initial condition in Eq. (29) and the homotopy in Eq. (19), we set $f_0(x,t)=2x\frac{t^{2-\alpha}}{\Gamma(3-\alpha)},\ f_1(x,t)=x^2t^4,\cdots,\ f_n(x,t)=0,\ n\geq 2.$ Then we obtain

$$
\begin{aligned}
&p^0: T_{*t}^{\alpha}u_0=f_0(x,t), && u_0(x,0)=0,\\
&p^1: T_{*t}^{\alpha}u_1=-u_0^2+f_1(x,t), && u_1(x,0)=0,\\
&p^2: T_{*t}^{\alpha}u_2=-2u_0u_1,\ u_2^{(k)}(x,0)=0, && u_2(x,0)=0,\\
&p^3: T_{*t}^{\alpha}u_3=-\left(u_1^2+2u_0u_1\right),\ u_3^{(k)}(x,0)=0, && u_3(x,0)=0,\\
&\vdots
\end{aligned}
\tag{34}
$$

Following the same solution steps in the Example 1, the first few terms of the MHPM solution can be obtained as

$$u_0(x,t)=\frac{xt^2}{\Gamma(3-\alpha)},$$

$$u_1(x,t)=\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\frac{x^2t^{\alpha+4}}{(\alpha+4)\Gamma(3-\alpha)\Gamma(3-\alpha)},$$

$$u_2(x,t)=-\frac{2x^3t^{2\alpha+6}}{(\alpha+4)(2\alpha+6)\Gamma(3-\alpha)}+\frac{2x^3t^{2\alpha+6}}{(\alpha+4)(2\alpha+6)\Gamma(3-\alpha)\Gamma(3-\alpha)\Gamma(3-\alpha)},$$

$$\vdots$$

The rest parts of the series can be given as the same way. Then the approximate solution of Eq. (33) is given by

$$u(x,t)=\frac{xt^2}{\Gamma(3-\alpha)}+\frac{x^2t^{\alpha+4}}{(\alpha+4)}-\frac{x^2t^{\alpha+4}}{(\alpha+4)\Gamma(3-\alpha)\Gamma(3-\alpha)}-\frac{2x^3t^{2\alpha+6}}{(\alpha+4)(2\alpha+6)\Gamma(3-\alpha)}$$
$$+\frac{2x^3t^{2\alpha+6}}{(\alpha+4)(2\alpha+6)\Gamma(3-\alpha)\Gamma(3-\alpha)\Gamma(3-\alpha)}+\cdots$$

The last equation means that the exact solution of the Eq. (28) for $\alpha=1$ is obtained with the proposed MHPM as $u(x,t)=xt^2$.

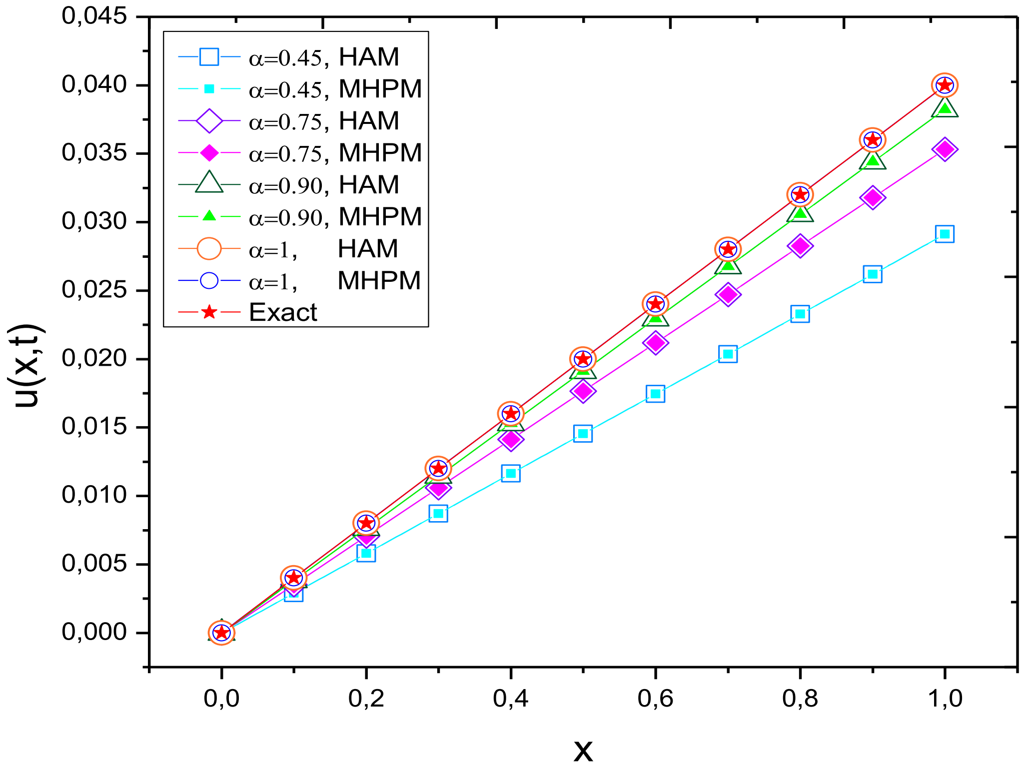


Figure 4. Comparison the numerical solutions with the exact solution at $\hbar=-1, t=0.2$.

6. Conclusion

In this work, approximate-analytical solutions of some linear/nonlinear PDEs are obtained by using the HAM and MHPM methods considering the conformable derivative operator. The fundamental solutions for non-homogeneous Klein-Gordon equation and a nonlinear PDE have been investigated by applying these suggested methods. The results of numerical computations have been illustrated by the figures under the variation of order α, time value t, distance term x and the auxiliary parameter $\hbar$. The results of this study find out that the HAM and MHPM in the conformable derivative mean are applicable and suitable methods that can evaluate the components of infinite series smoothly and with ease in short notice even in nonlinear PDEs and the results have proven the accuracy and influence of these mentioned methods.

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