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# GENERALIZED SCREW TRANSFORMATION AND ITS APPLICATIONS IN ROBOTICS

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**Abstract:** The workspace of the robots can be expressed in terms of the Clifford algebra of the dual quaternions. In this paper, after a review of some basic properties of the generalized dual quaternions we shall use them to kinematical modeling of the robotics in a generalized space**.**

**Keywords:** Dual Generalized Quaternion, Robotic, Screw Motion

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## 1 INTRODUCTION

Dual quaternions are powerful mathematical tools for the spatial analysis of rigid body motions. The dual quaternions introduced by Clifford in his seminal paper "Preliminary sketch of bi-quaternions" [4] and later on the work of Study [26] who utilized the dual numbers to represent the position of two skew lines in space. The use of dual numbers, dual numbers matrix and dual quaternions in instantaneous spatial kinematics is considered in [28,29]. In [1] the algebra of dual quaternions is developed by using the Hamilton operators. Properties of these operators are used to find some mathematical expressions for the screw motion. The screw motions in Minkowski 3-space $\mathbb{R}_1^3$ by using the dual split quaternions are studied in [13]. The dual number transformations in the area of robotics for the treatment of the manipulator kinematics are studied in [8] and by employing the dual matrices the closed-form solutions for the various types of robot manipulators are considered in [23]. The dual form of the Jacobian of a manipulator by using the dual orthogonal matrices is computed in [21]. Funda and Paul [7] have carried out a computational analysis of screw transformations in robotics. They have showed that the dual quaternions represent simultaneously the rotation and translation transformations for dealing with the kinematics of robot chains more efficiently than any other approach. A closed-form solution of the inverse kinematics of a 6 degree of freedom robot manipulator in terms of line transformations using the dual quaternions is computed in [12]. The 3D rigid body motion transformation of point, lines and planes useful for computer vision and robotics is investigated by using the algebra of motors in [3]. Also, the kinematic control laws for the free rigid bodies, by using the dual quaternions are given in [10]. A brief introduction of the generalized quaternions is provided in [5, 24]. Recently, we have studied the generalized quaternions, and have given some of their algebraic properties [15,16]. A matrix corresponding to Hamilton operators that is defined for generalized quaternions has determined a Homothetic motion in [17]. Furthermore, in [14] the authors have showed how that these operators can be used to described rotation in. In [15], it is demonstrated that how a unit generalized quaternion can be used to described rotation in 4-dimensional space. In [22], the dual generalized quaternions are defined and some of their algebraic properties are provided. Also, the De-Moivre's and Euler's formulas for these quaternions are studied.

In this paper, after reviewing some of the algebraic properties of the dual generalized quaternion, it is shown that this quaternion is a screw transformation in the generalized space $\mathbb{E}_{\alpha\beta}^3$. Therefore, we use the screw displacement formulated in terms of the dual generalized quaternions to represent the kinematical equations of the robot.

## 2 PRELIMINARIES

In this section, we give a brief summary of the generalized inner product, dual numbers and generalized quaternions. For detailed information about these quaternions, we refer the reader to [15] and [16].

**Definition 2.1.** For the vectors $\vec{x}=(x_1,x_2,x_3)$ and $\vec{y}=(y_1,y_2,y_3)$, the generalized inner product on $\mathbb{R}^3$ is given by

$$g(\vec{x},\vec{y})=\alpha x_1 y_1+\beta x_2 y_2+\alpha\beta x_3 y_3, \qquad (1)$$

where $\alpha$ and $\beta$ are positive numbers.

If $\alpha > 0$ and $\beta < 0$ then $g(\vec{x}, \vec{y})$ is called the generalized Lorentzian inner product. The vector space on $R^3$ equipped with the generalized inner product is called 3-dimensional generalized space and denoted by $E^3_{\alpha\beta}$. The cross product in $E^3_{\alpha\beta}$ is defined by

$$\vec{x} \times \vec{y} = \beta(x_2 y_3 - x_3 y_2)\vec{i} + \alpha(x_3 y_1 - x_1 y_3)\vec{j} + (x_1 y_2 - x_2 y_1)\vec{k}. \tag{2}$$

**Special cases:**

1. If $\alpha = \beta = 1$, then $E^3_{\alpha\beta}$ is an Euclidean 3-space $E^3$.
2. If $\alpha = 1, \beta = -1$, then $E^3_{\alpha\beta}$ is a non-Euclidean 3-space $E^3_2$.

**Definition 2.2.** A generalized quaternion $q$ is defined as

$$q = a_0 1 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \tag{3}$$

where $a_0, a_1, a_2$ and $a_3$ are real numbers and $1, \vec{i}, \vec{j}, \vec{k}$ of $q$ may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\begin{aligned} &\vec{i}^2 = -\alpha, \quad \vec{j}^2 = -\beta, \quad \vec{k}^2 = -\alpha\beta, \\ &\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j}, \end{aligned} \tag{4}$$

and

$$\vec{k}\vec{i} = \alpha\vec{j} = -\vec{i}\vec{k}, \quad \alpha, \beta \in R. \tag{5}$$

The set of all generalized quaternions is denoted by $H_{\alpha\beta}$. A generalized quaternion $q$ is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. Therefore $H_{\alpha\beta}$ is form a 4-dimensional real space which contains the real axis and a 3-dimensional real linear space $E^3_{\alpha\beta}$, so that, $H_{\alpha\beta} = R \oplus E^3_{\alpha\beta}$.

**Definition 2.3.** Each element of the set

$$D = \{A = a + \varepsilon a^* : a, a^* \in R \;\;\&\;\; \varepsilon \neq 0\} \tag{6}$$

is called a dual number. Summation and multiplication of two dual numbers are defined as similar to the complex numbers but it is must be forgotten that $\varepsilon^2 = 0$. Thus, $D$ is a commutative ring with a unit element [18].

## 3 GENERALIZED DUAL QUATERNION

**Definition 3.1.** A generalized dual quaternion $Q$ is an expression of form

$$Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k} \tag{7}$$

where $A_0, A_1, A_2$ and $A_3$ are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units which satisfy in the above equalities. As a consequence of this definition, a generalized dual quaternion $Q$ can also be written as;

$$Q = q + \varepsilon q^*, \tag{8}$$

where $q, q^* \in H_{\alpha\beta}$ are real and pure generalized dual quaternion components, respectively. It is useful, therefore, to define the following terms:

The scalar part of $Q$ is $S_Q = A_0$.

The dual vector part of $Q$ is $\vec{V}_Q = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$.

If $S_Q = 0$, then $Q$ is called pure generalized dual quaternion, we may be called its generalized dual vector. The set of all generalized dual vectors denoted by $D^3_{\alpha\beta}$.

The Hamilton conjugate of $Q$ is

$$\begin{aligned} \bar{Q} &= S_Q - \vec{V}_Q, \\ &= \bar{q} + \varepsilon \bar{q}^*. \end{aligned} \tag{9}$$

The norm of $Q$ is

$$\begin{aligned} N_Q &= \bar{Q}Q = Q\bar{Q} = A_\circ^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \\ &= (a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2) + \\ &\quad 2\varepsilon(a_0 a_0^* + \alpha a_1 a_1^* + \beta a_2 a_2^* + \alpha\beta a_3 a_3^*) \\ &= q\bar{q} + \varepsilon(q\bar{q}^* + q^*\bar{q}). \end{aligned} \tag{10}$$

The norm of a generalized dual quaternion, in general, is not a real number but a dual number.

The reciprocal of $Q$ is $Q^{-1} = \dfrac{\bar{Q}}{N_Q}$. The reciprocal of a generalized dual quarternion $Q$ exists if and only if $N_Q \neq 0$.

Unit generalized dual quaternion:

$$N_Q = 1. \tag{11}$$

The generalized quaternion multiplication is, in general, not commutative. If $Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ and $P = B_0 + B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$ are the two generalized dual quarternions and let $R = QP$ then $R$ is given by

$$\begin{aligned} R = {} & S_Q S_P - g(\vec{V}_Q, \vec{V}_P) + S_Q\vec{V}_P + S_P\vec{V}_Q + \vec{V}_Q \times \vec{V}_P \\ &= (A_0B_0 - \alpha A_1B_1 - \beta A_2B_2 - \alpha\beta A_3B_3) + (A_0B_1 + A_1B_0 - \beta A_2B_3 + \beta A_3B_2)\vec{i} \\ &+ (A_0B_1 + A_1B_0 + \beta A_2B_3 - \beta A_3B_2)\vec{j} + (A_0B_3 + A_1B_2 - A_2B_1 + A_3B_0)\vec{k} \\ &= qp + \varepsilon(qp^* + q^*p). \end{aligned} \tag{12}$$

The generalized quaternion product can be described by a matrix-vector product as

$$QP = \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}. \tag{13}$$

**Theorem 3.1.** (E. Study map) There exists a one-to-one correspondence between directed lines of $\mathrm{E}^3_{\alpha\beta}$ and ordered pair of vectors $A = (\vec{a}, \vec{a}^*)$ such that $g(\vec{a}, \vec{a}) = 1$ and $g(\vec{a}, \vec{a}^*) = 0$.

*Proof:* The proof can be found in [19, 20].

**Special case:**

1) If $\alpha = \beta = 1$, then we have E. Study mapping in Euclidean 3-space $\mathrm{E}^3$ [9].

2) If $\alpha = 1, \beta = -1,$ then we have E. Study mapping for a space which is isomorphic to Minkowski 3-space $E_1^3$ [27].

**Theorem 3.2.** Let $A, B$ be two unit generalized dual vectors in $D_{\alpha\beta}^3$ and $\alpha, \beta \in \mathrm{R}^+$. Then we have $g(A,B) = \cos\Phi$ where $\cos\Phi = \cos\varphi - \varepsilon\varphi^* \sin\varphi.$

*Proof:* The proof can be found in [19].

## 4. ROBOTIC MOTIONS

Chasles's theorem states that every rigid body transformation can be composed of a rotation about an axis and a translation along that axis. This is the transformation executed by a turning screw and is referred to as screw transformation [11]. There are at least four methods used commonly to represent a general spatial screw transformation, including:

(1) Dual orthogonal 3×3 matrix, (2) dual unit quaternion, (3) dual special unitary 2×2 matrix, and (4) dual Pauli spin matrices [6].

The first and most common method in the robotics community is based on homogenous matrix transformation. In robotics, this matrix is used to describe one coordinate system with respect to another one [25]. In robotic motions, both the rotation and the translation moves of screws or axis are simultaneously done in a transformation in unit dual quaternions. The representation of the robotic motion in terms of dual quaternions is studied by several authors [7,10,25].

## 5. GENERALIZED SCREW TRANSFORMATION

In paper [2], authors described the unit dual quaternion is a screw transformation in Euclidean 3-sapce. In this section, we show that a unit dual generalized quaternion is a generalized screw transformation in the generalized space $E_{\alpha\beta}^3$.

First let $\alpha, \beta \in \mathrm{R}^+$. We suppose that given two lines $L_1$ and $L_2$ represented with unit generalized dual vectors $A$ and $B$, respectively. Dual quaternionic multiplication of $A$ and $B$ is determined by

$$AB = -g(A,B) + A \times B, \tag{14}$$

where $g(A,B) = \cos\Phi$ and $\|A \times B\| = \sin\Phi$. $\Phi = \varphi - \varepsilon\varphi^*$ is the dual angle between $A$ and $B$. Then we find

$$AB = -\cos\Phi + S\sin\Phi, \tag{15}$$

where $S = s + \varepsilon s^* = \dfrac{A \times B}{\|A \times B\|}$ is the unit generalized dual vector which is orthogonal to both $A$ and $B$. In addition, the conjugate $AB$ is

$$\overline{AB} = BA = -(-\cos\Phi + S\sin\Phi) = -Q. \tag{16}$$

The inverses $A^{-1}$ and $B^{-1}$ respectively of $A$ and $B$ are

$$A^{-1} = \frac{\overline{A}}{N_A} = -A, \; B^{-1} = \frac{\overline{B}}{N_B} = -B. \tag{17}$$

Thus we can write

$$Q = -BA = B^{-1}A. \tag{18}$$

The screw transformation on $B$ is a unit generalized dual quaternion $Q$, *i.e.*

$$Q=\cos\Phi+S\sin\Phi. \tag{19}$$

**Corollary 5.1.** $Q=\cos\Phi+S\sin\Phi$ is called a generalized screw transformation. Hence, we can say that the expression $B=QA$ which is found by left side multiplication of $A$ by $Q$, rotate $A$ around the screw axis $S$, with a dual elliptical angle $\Phi$. Since $\Phi=\varphi-\varepsilon\varphi^*$, a rotation of angle $\varphi$, a translation of $\varphi^*$ occurs and $\varphi^*/\varphi$ is the step. In this statement, we can give Fig. 1.

**Example 5.1.** Find the generalized screw transformation which transforms the

$$l_1=\{x=t,\, y=z=0\} \tag{20}$$

line to the $l_2=\{\frac{x}{1/\sqrt{2}}=\frac{y}{1/\sqrt{2}}, z=2\}$ line.

First, according to the Theorem 3.1., we obtain the unit dual vectors corresponding to $l_1$ and $l_2$ lines in 3-space $D^3_{\alpha\beta}$. For the line $l_1$, we have $M=(0,0,0), \vec{a}=(1,0,0)$ and $\vec{a}^*=(1,0,0)$, and for the line $l_2$, $N=(0,0,2)$, $\vec{b}=(1/\sqrt{2},1/\sqrt{2},0)$ and $\vec{b}^*=2/\sqrt{2}(-\beta,\alpha,0)$.

So $A_0$ and $B_0$ are the unit dual vectors corresponding to the $l_1$ and $l_2$,

$$A_0=\frac{A}{N_A}=\frac{\vec{a}+\varepsilon\vec{a}^*}{N_A}=\frac{1}{\sqrt{2}}(1,0,0), \tag{21}$$

and

$$B_0=\frac{B}{N_B}=\frac{\vec{b}+\varepsilon\vec{b}^*}{N_B}=\frac{\sqrt{2}}{\sqrt{\alpha+\beta}}\left[(1/\sqrt{2},1/\sqrt{2},0)+\varepsilon\sqrt{2}(-\beta,\alpha,0)\right]. \tag{22}$$

Now, we find the screw transformation $Q=Q_0+Q_1\vec{i}+Q_2\vec{j}+Q_3\vec{k}$ ,

$$Q=B_0(A_0)^{-1}=g(B_0,A_0)-B_0\times A_0. \tag{23}$$

We obtain

$$\begin{aligned} &Q_0=\frac{\sqrt{\alpha}}{\sqrt{\alpha+\beta}}(1-2\beta\varepsilon),\ Q_1=0,\ Q_2=0 \\ &Q_3=-\frac{\sqrt{\alpha}}{\sqrt{\alpha+\beta}}(\frac{1}{\alpha}+2\varepsilon). \end{aligned} \tag{24}$$

The generalized screw transformation $Q$ depends on $\alpha,\beta$ and for different of the positive values $\alpha,\beta$ we have different screw transformation.

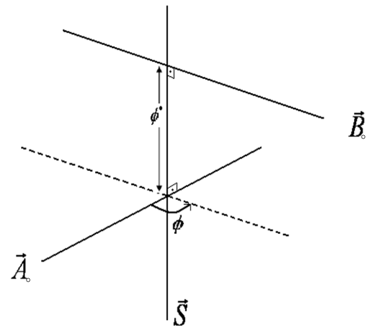


**Figure 1.** Screw motion

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