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Research Article

GAMMA Renewal Function in Censored Data

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Abstract

In this study, the renewal process whose times between the intervals are gamma-diffused has been examined. In the situation where the sampling is Randomly Right Censored, a Parametric Estimator is recommended, which depends on the Maximum Likelihood Estimators of the unknown parameters of the gamma diffusion; and the statistical characteristics of these estimators have been investigated.

Keywords: Gamma renewal process, Renewal function, Random right censoring

1. Introduction

Let $\{X_n, n=1,2,...\}$ be the series of random variables which has independent non-negative same F diffusion function; and let $F(0)=P(X_n=0)<1$, i.e. X_n not be equal to zero with one probability.

After X_n: $(n-1)$ renewals, let n denote the time until the renewal. The $S_0=0$, $S_n=X_1+...+X_n$, $n\geq 1$, and S_n random variable and n is the time elapsed until renewal. For each $t\geq 0$, $N(t)$ let the definition be:

$$N(t)=\sup\{n:S_n\leq t\}$$

$N(t)$ is the number of the renewals only until t time; i.e. $[0,t]$ is the number of the renewals between the intervals. The $N(t)$ which is defined such is called as Renewal Random Variable and $\{N(t),\ t\geq 0\}$ as stochastic process renewal process (Ross 1983).

Let the $\{N(t),\ t\geq 0\}$ be a renewal process,

The average value function M given with $M(t)=E(N(t))$, $t\geq 0$

is called as the renewal function (Karlin and Taylor, 1975). Here, $M(t)$, $[0,t]$ is the average number of the renewals made in time intervals.

Let $I_k=\begin{cases}1, & S_k\leq t\\ 0, & S_k>t\end{cases}$;

$$N(t)=\sum_{k=1}^{\infty} I_k$$

is obtained as follows:

$$E(N(t))=\sum_{k=1}^{\infty} F^{k*}(t)$$

Then,

$$M(t)=\sum_{k=1}^{\infty} F^{k*}(t)\ , \qquad t\geq 0 \tag{1.1}$$

The * Stieltjes Convulution operation is shown here.

By using the (1.1) expression, an integral equation may be obtained for M renewal function. By taking the integral $\int_0^t F^{k*}(t-x)dF(x)$, which is mathematical instead of $F^{(k+1)*}(t)$, the following equation is found:

$$\begin{aligned} M(t) &= F(t)+\int_0^t M(t-x)dF(x)\\ &= F(t)+F*M(t), \quad t\geq 0 \end{aligned} \tag{1.2}$$

This integral equation is called as the renewal equation. The (1.2) equation may also be written as follows:

$$M(t)=F(t)+\int_o^t F(t-x)dM(x), \qquad t\geq 0 \tag{1.3}$$

Generally, the M renewal function cannot be obtained analytically except for two-parameter exponential, smooth, hyper exponential and gamma diffusion. In this case, M can be expressed in numerical manner. There are various methods in the literature that are based on Laplace and Reverse-Laplace conversion calculations, on force series expansion, on cubic spline approach, and numerical calculation of the renewal integral equation (Baxter et al. 1982; Xie, 1989). A method which is easily programmed, and which gives good results with its properties like being easy and convergent nearly in all situations is the RS (Rieman-Stieltjes) of Xie, which is described below (Xie 1989).

In the light of the Riemann-Stieltjes integral definition, the $\int_a^b g(x)dh(x)$ integral being the defragment of the $[a,b]$ interval and $\Delta=\{x_0,x_1,...,x_n\}$, the following can be obtained:

$$\int_a^b g(x)dh(x)\approx\sum_{i=1}^{n} g((x_i+x_{i-1})/2)(h(x_i)-h(x_{i-1})) \tag{1.4}$$

In this formula which can be used to calculate the Riemann-Stieltjes integral in numerical manner, it is clear that as the defragmentation norm decreases, the approach will be better.

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Now, let us deal with the (1.3) equation, i.e.

$$M(t) = F(t) + \int_0^t F(t-x)dM(x), \quad t \geq 0$$

integral equation. Let t a given value and $\{t_0, t_1, ..., t_n\}$ be a defragmentation which ensures that $0 = t_0 < t_1 < ... < t_n = t$. In this case;

$$M(t_i) = F(t_i) + \int_0^{t_i} F(t_i - x)dM(x) \quad ;$$

And by using the (1.4) expression;, the following equation is found;

$$M(t_i) \approx F(t_i) + \sum_{j=1}^{i} F(t_i - (t_j + t_{j-1})/2)(M(t_j) - M(t_{j-1}))$$

$$T_i = \sum_{j=1}^{i-1} F(t_i - (t_j + t_{j-1})/2)(M(t_j) - M(t_{j-1}))$$

Thus, when $M(t_i)$ successively, and $\widehat{M}(t_0) = 0$; it can be calculated with the following formula:

$$\widehat{M}(t_i) = \frac{F(t_i) + T_i - F(t_i - (t_i + t_{i-1})/2)\widehat{M}(t_{i-1})}{1 - F(t_i - (t_i + t_{i-1})/2)}, \; i = 1,2,...,n$$

(Xie 1989).

This method is beneficial especially when f probability density function is not known and has opposite points.

When instead of F diffusion function, f probability density function is given, and if F does not return in a closed form, F diffusion function may be calculated easily with the following formula:

$$F(t_i) = F(t_{i-1}) + f((t_i + t_{i-1})/2)\frac{t}{n}, \qquad i = 1,2,...,n$$

The (1.2) and (1.3) equations are theoretically equal. Although the (1.2) equation is more common in the literature, the RS Method has been limited to the solution of (1.3) equation. If the non-equal step lengths are used, the (1.3) equation seems simpler for the successive solution. Meanwhile, instead of (1.2), the use of (1.3) has a basic advantage, which $F(t)$ is nearly constant for big t 's; and thus, the rounding error in $F(t_i) - F(t_{i-1})$ will be relatively bigger.

In a $\{N(t), \; t \geq 0\}$ renewal process, if the times between the renewals have the $\alpha \; (>0)$ and $\beta \; (>0)$ -scale parameter gamma diffusion, $\{N(t), \; t \geq 0\}$ process is called as the gamma renewal process, and the average value function of this process is called as gamma renewal function.

In gamma renewal process, the density and diffusion function of each renewal time are as follows, respectively;

$$f(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha} t^{\alpha-1} e^{-t/\beta}, \; t \geq 0$$

and

$$F(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^t x^{\alpha-1} e^{-x/\beta} dx, \; t \geq 0$$

In case the α is a natural number of the shape parameter, it is known that for F, a closed form is

$$F(t) = 1 - \sum_{k=0}^{\alpha-1} \frac{(t/\beta)^k}{k!} e^{-t/\beta}, \; t \geq 0$$

and

$$F^{n*}(t) = 1 - \sum_{k=0}^{n\alpha-1} \frac{(t/\beta)^k}{k!} e^{-t/\beta}, \quad t \geq 0$$

In case the α is a natural number of the shape parameter, it is known that for M gamma renewal function is $i^2 = -1$ and $c = 2\pi i/\alpha$; the following is obtained in closed form:

$$M(t) = \frac{t}{\alpha\beta} + \frac{1}{\alpha}\sum_{r=1}^{\alpha-1} \frac{c^r}{1-c^r}(1 - e^{-t(1-c^r)/\beta}), \quad t \geq 0 \qquad (1.5)$$

(Aydoğdu 1997).

For $\alpha = 1,2,3,4$, respectively, from (1.5), the following analytical expressions are obtained:

$$M(t) = \frac{t}{\beta}, \quad t \geq 0$$

$$M(t) = \frac{t}{2\beta} - \frac{1}{4} + \frac{1}{4}e^{-2t/\beta}, \quad t \geq 0$$

$$M(t) = \frac{t}{3\beta} - \frac{1}{3} + \frac{1}{3}e^{-3t/2\beta}(\cos(\frac{\sqrt{3}t}{2\beta}) + \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}t}{2\beta})), \quad t \geq 0$$

$$M(t) = \frac{t}{4\beta} - \frac{3}{8} + \frac{1}{8}e^{-2t/\beta} + \frac{1}{4}e^{-t/\beta}(\cos(\frac{t}{\beta}) + \sin(\frac{t}{\beta})), \quad t \geq 0$$

In case α is not a natural number of the shape parameter, it is known that for M, the gamma renewal function can be calculated with the RS method of Xie (1989) in a numerical manner.

2. Parameter Estimation in Case of a Random Right Censored Sampling

Censoring is the ignoring of the observations obtained from the units obtained from sampling due to some limitations like time and costs or the ignoring of the data that could not be obtained. Various types of censorship are given in the literature (Lawless 2003). We are only dealing with the random right censoring, which is one of the censorship types. In this part, while the F diffusion function is known as functional, obtaining the maximum likelihood estimators will be dealt with based on the random right sampling censorship of the unknown $\theta_1, \theta_2, ..., \theta_r$ parameters.

Let us accept that, for $i = 1,2,...,n$ the life of the Y_i i. part and the T_i i. part shows the censorship random variable. Here, both Y_i ($i = 1,2,...,n$)'s and T_i ($i = 1,2,...,n$)'s are independent and same-diffused random variables. In addition, Y_i ($i = 1,2,...,n$) and T_i ($i = 1,2,...,n$) random variables are independent.

Let $i = 1,2,...,n$, under the censorship of i. component T_i with random variable our observation for i. component is $X_i = \min(Y_i, T_i)$. The random variables $X_1, ..., X_n$ that rare formed like this are called as 'Ransom Right Censored n -unit sampling".

In application about renewal, these types are observations are very common. For example, if a new product is sold at different times and the producer observes the expiry duration of these products for 12 months, the life spans of the sold products appear as Random Right Censored data.

Let us show Y_i random variable diffusion function with F and probability density function with f for $i = 1,2,...,n$. While F is formally known, let some parameters be unknown. Let $\theta_1, \theta_2, \ldots, \theta_r$ show the unknown parameters of F. In this case, we can make the estimation at a point with the help of the estimations of the $\theta_1, \theta_2, \ldots, \theta_r$ parameters based on the $X_1, \ldots, X_n$ Random Right Censored Sampling.

Let us show the T_i ($i = 1, 2, \ldots, n$) random variable diffusion function with G and the probability density function with g. For $i = 1, 2, \ldots, n$,

$$I_i = \begin{cases} 1, & Y_i \le T_i \\ 0, & Y_i > T_i \end{cases}$$

and

$$X_i = \min(Y_i, T_i) = Y_i I_i + T_i(1 - I_i)$$

can be written. The event ($I_i = 1$) shows that censoring is not performed for i. Component, while the event ($I_i = 0$) shows that censoring is performed. Now, first of all, let us find the common diffusions of the random variables X_i and I_i ($i = 1, 2, \ldots, n$) in order to form the probability function in $X_1, \ldots, X_n$ sampling which is random right censored.

Since

$$\begin{aligned} X_i = x_i, I_i = 1 &\Leftrightarrow X_i = x_i \quad , Y_i \le T_i \\ &\Leftrightarrow X_i = Y_i = x_i , \ X_i \le T_i \\ &\Leftrightarrow Y_i = x_i \quad , x_i \le T_i , \end{aligned}$$

$$f_{x_i, I_i}(x_i, 1) = f(x_i)(1 - G(x_i^-))$$

and since

$$\begin{aligned} X_i = x_i, I_i = 0 &\Leftrightarrow X_i = x_i \quad , Y_i > T_i \\ &\Leftrightarrow X_i = x_i \quad , Y_i > T_i \quad , X_i = T_i \\ &\Leftrightarrow Y_i = x_i \quad , Y_i > x_i , \end{aligned}$$

$$f_{x_i, I_i}(x_i, 0) = g(x_i)(1 - F(x_i))$$

Therefore, for $\delta_i \in \{0, 1\}$, the following equation may be written:

$$f_{X_i, I_i}(x_i, \delta_i) = \left[f(x_i)\right]^{\delta_i}\left[1 - F(x_i)\right]^{1-\delta_i}\left[f_{T_i}(x_i)\right]^{1-\delta_i}\left[1 - F_{T_i}(x_i)\right]^{\delta_i}$$

In this case for the Random Right Censored $X_1, \ldots, X_n$ sampling, the $L(\theta_1, \theta_2, \ldots \theta_r)$ probability function can be defined as follows (Lawless 2003).

$$L(\theta_1, \theta_2, \ldots \theta_r) = \prod_{i=1}^{n}\left[f(x_i)\right]^{\delta_i}\left[1 - F(X_i)\right]^{1-\delta_i}\prod_{i=1}^{n}\left[f_T(x_i)\right]^{1-\delta_i}\left[1 - F(x_i^-)\right]^{\delta_i}$$

$$= \prod_{i=1}^{n}\left[f(x_i)\right]^{\delta_i}\left[1 - F(X_i)\right]^{1-\delta_i}\prod_{i=1}^{n}\left[g(x_i)\right]^{1-\delta_i}\left[1 - G(x_i)\right]^{\delta_i}$$

The $\theta_1, \theta_2, \ldots, \theta_r$ values which make this L function maximum are called as the maximum likelihood estimations of the parameters, and these estimations are shown with $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r$, respectively. If the G diffusion does not include the $\theta_1, \theta_2, \ldots, \theta_r$ parameters, the second multiplication in the probability function will not influence the finding process of the maximum likelihood method estimators.

In order to work properly with the maximum likelihood method, let us separate the sets of the observations into sub-sets like D and C. Let D show the decomposition times observed, and let C show the set of the indices of the censored observations. Under the assumption of the D set being empty, the probability function may be written as follows with the help of the C and D sets:

$$L(\theta_1, \ldots, \theta_r) = \prod_{i \in D} f(x_i)\prod_{i \in C}(1 - F(x_i))\prod_{i \in C} g(x_i)\prod_{i \in D}(1 - G(x_i)) \qquad (2.1)$$

I many cases, the maximum likelihood estimators $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r$ are obtained with the following equation:

$$\frac{\partial \log L(\theta)}{\partial \theta_i} = 0, \qquad i = 1, 2, \ldots, r$$

Except for some F diffusion that are used commonly, there is no analytical solution of this equation system. In this case, the need appears for a solution with a numerical method. In this study, the Newton-Raphson Method, which is one of the most-commonly used solution methods, has been used.

2.1.Estimation of the Gamma Distribution Parameters

Let there be a sampling with n units randomly right censored from a Gamma Distribution with $X_1, \ldots, X_n$ α and β-parameter gamma. Let us accept that the G does not include the α and β parameters, which is the censored random variable. In this case; the following equations are found:

$$L(\alpha, \beta) = \prod_{i \in D} f(x_i)\prod_{i \in C}\left(1 - F(x_i)\right)$$

$$= \prod_{i \in D}\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x_i^{\alpha - 1} e^{-x_i/\beta}\prod_{i \in C}\left(1 - F(x_i)\right)$$

and

$$\ln L(\alpha, \beta) = \sum_{i \in D}\left[(\alpha - 1)\ln x_i - \frac{x_i}{\beta} - \ln\Gamma(\alpha) - \alpha\ln\beta\right] + \prod_{i \in C}\left(1 - F(x_i)\right)$$

$$= (\alpha - 1)\sum_{i \in D}\ln x_i - \frac{1}{\beta}\sum_{i \in D} x_i - S(D)\ln\Gamma(\alpha) - S(D)\alpha\ln\beta + \prod_{i \in C}\left(1 - F(x_i)\right)$$

Here, $S(D)$ shows the number of the elements of the D set. In order to find α and β's maximum likelihood estimators, the partial derivation of this expression according to α and β, the following are contained:

$$\frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = \sum_{i \in D}\ln x_i - \frac{S(D) d\ln\Gamma(\alpha)}{d\alpha} - S(D)\ln\beta + \sum_{i \in C}\frac{\dfrac{d\ln\Gamma(\alpha)}{d\alpha}\displaystyle\int_0^{x_i/\beta}\frac{e^{-t}t^{\alpha-1}}{\Gamma(\alpha)}dt - \frac{1}{\Gamma(\alpha)}\int_0^{x_i/\beta} t^{\alpha-1}e^{-t}\ln t\,dt}{1 - F(\frac{x_i}{\beta}; \alpha, \beta = 1)}$$

and

$$\frac{\partial \ln L(\alpha, \beta)}{\partial \beta} = \frac{1}{\beta^2}\sum_{i \in D}\ln x_i - \frac{S(D)\alpha}{\beta} + \alpha\sum_{i \in C}\frac{f(x_i; \alpha + 1, \beta)}{1 - F(x_i)}$$

In this case, our equation system become

$$\sum_{i \in D}\ln x_i - \frac{S(D) d\ln\Gamma(\alpha)}{d\alpha} - S(D)\ln\beta + \sum_{i \in C}\frac{\dfrac{d\ln\Gamma(\alpha)}{d\alpha}\displaystyle\int_0^{x_i/\beta}\frac{e^{-t}t^{\alpha-1}}{\Gamma(\alpha)}dt - \frac{1}{\Gamma(\alpha)}\int_0^{x_i/\beta} t^{\alpha-1}e^{-t}\ln t\,dt}{1 - F(\frac{x_i}{\beta}; \alpha, \beta = 1)} = 0$$

$$\frac{1}{\beta^2}\sum_{i \in D}\ln x_i - \frac{S(D)\alpha}{\beta} + \alpha\sum_{i \in C}\frac{f(x_i; \alpha + 1, \beta)}{1 - F(x_i)} = 0$$

There is no common analytical solution of these two equations. The maximum likelihood estimations of the α and β may be calculated with the Newton-Raphson Method. The mathematical expressions required in working this method are as follows:

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\,\partial\beta} == \frac{-S(D)}{\beta} - \alpha g(\alpha)\sum_{i\in C}\frac{f(x_i,\alpha+1,\beta)}{1-F(x_i)} + \alpha\sum_{i\in C}\frac{f(x_i,\alpha+1,\beta)}{1-F(x_i)}\ln\left(x_i/\beta\right)$$

$$-\alpha g(\alpha)\sum_{i\in C}\frac{f(x_i,\alpha+1,\beta)F(x_i)}{\left[1-F(x_i)\right]^2} + \alpha\sum_{i\in C}\frac{f(x_i,\alpha+1,\beta)}{\left[1-F(x_i)\right]^2}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta^2} = -\frac{S(D)\alpha}{\beta^2} - \frac{2}{\beta^3}\sum_{i\in D}x_i + \frac{\alpha}{\beta}\sum_{i\in C}\frac{f(x_i;\alpha+1,\beta)\left[\frac{x_i}{\beta}-(\alpha+1)\right]}{\left[1-F(x_i)\right]} - \alpha^2\sum_{i\in C}\frac{f^2(x_i;\alpha+1,\beta)}{\left[1-F(x_i)\right]^2}$$

and

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha^2} = -S(D)\frac{d}{d\alpha}g(\alpha) + \frac{d}{d\alpha}g(\alpha)\sum_{i\in C}\frac{F(x_i)}{1-F(x_i)} + g(\alpha)\sum_{i\in C}\frac{I_i - g(\alpha)F(x_i)}{\left[1-F(x_i)\right]^2} - \sum_{i\in C}\frac{K_i}{1-F(x_i)} - \sum_{i\in C}\frac{I_i\left(I_i-g(\alpha)\right)}{\left[1-F(x_i)\right]^2}$$

here

$$I_i = \int_0^{x_i/\beta}\frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}\ln t\,dt \qquad i = 1,2,\ldots,n,$$

$$K_i = \int_0^{x_i/\beta}\frac{t^{\alpha-1}e^{-t}(\ln t)^2}{\Gamma(\alpha)}dt \qquad i = 1,2,\ldots,n$$

and

$$g(\alpha) = \frac{d\left(\ln\Gamma(\alpha)\right)}{d\alpha} = \frac{\frac{d\Gamma(\alpha)}{d\alpha}}{\Gamma(\alpha)}$$

defined as above. Let

$$U(\alpha,\beta) = \begin{bmatrix}\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha}\\ \frac{\partial \ln L(\alpha,\beta)}{\partial\beta}\end{bmatrix}$$

and

$$V(\alpha,\beta) = \begin{bmatrix}\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha^2} & \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\partial\beta}\\ \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta\partial\alpha} & \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta^2}\end{bmatrix}$$

The following iteration operation is obtained by using the Newton Raphson method with $\alpha(1)$ and $\beta(1)$ initial values:

$$\begin{bmatrix}\alpha(m+1)\\ \beta(m+1)\end{bmatrix} = \begin{bmatrix}\alpha(m)\\ \beta(m)\end{bmatrix} - V^{-1}\left(\alpha(m),\beta(m)\right)U\left(\alpha(m),\beta(m)\right), \quad m = 1,2,\ldots$$

If $\alpha(m)$ and $\beta(m)$ are close to $\alpha(m+1)$ and $\beta(m+1)$, respectively, at a sufficient level, the iteration is terminated. The maximum likelihood estimation values of α and β, which is $\hat{\alpha} = \alpha(m+1)$ and $\hat{\beta} = \beta(m+1)$, are achieved.

3. Parametric Estimation of GAMMA Renewal Function in Censored Data

Let $\{N(t),\, t \geq 0\}$ be the intervals between the renewals, $\alpha > 0$ and $\beta > 0$ be a gamma renewal process with Gamma Distribution. Generally, the M Gamma Renewal Function Data are required in the application that are related with this process. In case α and β parameters are known, it is known that this function may be obtained either analytically or numerically depending on the status of α form parameter. However, ın case one of the α and β parameters is not known, the gamma renewal function must be estimated from the data.

In case $X_1,\ldots,X_n$ α and β parameter gamma distribution with n-unit Random Right Censored Sampling is known as $\alpha = k \in N$, by taking the $\hat{\beta}$ maximum likelihood estimator instead of β in the expression of M function (1.5), a parametric estimator is obtained for $M(t)$ so long as for each $t \geq 0$, $c = e^{2\pi i/k}$

$$\hat{M}_n(t) = \frac{t}{k\hat{\beta}} + \frac{1}{k}\sum_{r=1}^{k-1}\frac{c^r}{1-c^r}\left(1-e^{-t(1-c^r)/\hat{\beta}}\right) \tag{3.1}$$

In case $\alpha = 1$, $\hat{\beta} = \frac{\sum_{i=1}^{n}X_i}{S(D)}$, and

$$\hat{M}_n(t) = \frac{S(D)}{\sum_{i=1}^{n}X_i}t \tag{3.2}$$

If the known α parameter is not a natural number, a $\hat{M}_n(t)$ parametric estimator cannot be expressed analytically. Generally, if α and β are not known, a $\hat{M}_n(t)$ parametric estimator is achieved for $M(t)$ by using the $\hat{\alpha}$ and $\hat{\beta}$, which are the maximum likelihood estimators of α and β:

For F Gamma Distribution Function, let us write $F = F(\alpha,\beta)$.

$\hat{F}_n = F(\hat{\alpha},\hat{\beta})$ is the estimator of F in the result.

In $\hat{F}_n^{k*} = \underbrace{\hat{F}_n * \ldots * \hat{F}_n}_{k\ kez}$,

$M(t)$'s (1.1) Convulution series expression, by taking the $\hat{F}_n^{k*}(t)$ instead of $F^{k*}(t)$, which is its estimator,

$$\hat{M}_n(t) = \sum_{k=1}^{\infty}\hat{F}_n^{k*}(t), \qquad t \geq 0 \tag{3.3}$$

with the above definition, for $\hat{M}_n(t)$ each constant is a parametric estimator of $t \geq 0$, $M(t)$.

If there is an analytical expression exists for M gamma renewal function, the estimator is replaced with the parameter, it is obvious that the given estimator is equal to the above-given estimator as in (3.1).

Let us consider the $\hat{M}_n(t) = \frac{S(D)}{\sum_{i=1}^{n}X_i}t$ estimator, which is given in (3.2). Since $\hat{\beta} = \frac{\sum_{i=1}^{n}X_i}{S(D)}$ is consistent for the maximum likelihood estimator β, $\hat{M}_n(t)$ parametric estimator for $M(t) = \frac{t}{\theta}$.

In general state, if $\hat{\alpha}$ and $\hat{\beta}$ are strongly consistent for α and β parameters, respectively, for each constant $t \geq 0$, and for $\hat{M}_n(t)$ $M(t)$ it is the strong and consistent estimator, and at the same time asymptotically unbiased (Danış 2008).

There is no analytical expression for $\hat{M}_n(t)$ estimator except for some special conditions. In this case, in order to calculate $\hat{M}_n(t)$ the (3.3) convolution series expression is generally not beneficial. For $M(t)$ renewal function, from (1.3) integral equation $\hat{M}_n(t)$, the following can be written:

$$\hat{M}_n(t) = \hat{F}_n(t) + \int_0^t \hat{M}_n(t-x) d\hat{F}_n(x) \quad (3.4)$$

Calculating this integral equation is easier than the convolution series expression of the estimator, and can be calculated with the RS Method of Xie.

4. Simulation

In order to observe how the estimator $\hat{M}_n(t)$ given in (3.3) is calculated, the Random Right Censored observations are used with n=50 units coming from the gamma distribution.

Table 1. Observations that are not censored and observations that are censored

Observations that are not Censored	Observations that are Censored
0.0200 0.0333 0.0371 0.0385 0.0412 0.0453 0.0598 0.0627 0.0684 0.0731	0.0117 0.0569 0.0897 0.1978 0.2094 0.2449 0.3292
0.0773 0.0776 0.0789 0.0900 0.1223 0.1224 0.1252 0.1320 0.1404 0.1454	
0.1509 0.1528 0.1570 0.1592 0.1603 0.1624 0.1626 0.1638 0.1749 0.1859	
0.1901 0.1995 0.2096 0.2107 0.2170 0.2195 0.2209 0.2348 0.2664 0.3456	
0.4160 0.4546 0.4916	

The above observations are not real data, and are produced from the $\alpha = 2$ and $\beta = 1$ gamma distribution via simulation by taking the plain distribution in the (0, 8) censored random variable distribution range.

With the Newton Raphson method, the closeness value is taken as 10^{-5} in stopping the iterations, the maximum likelihood estimation values of α and β parameters found as $\hat{\alpha} = 2.2363$ and $\hat{\beta} = 0.7952$, respectively. Then, for some given t values, the values of the $\hat{M}_n(t)$ estimator is obtained as Table 2 the (3.4) integral equation with the RS method by solving the 0.005 equilibrium step length in a numerical method.

Table 2. Some given *t* values, the values of the *M*$_n$*(t)* estimator

T	0.1	0.4	0.8	1	1.5	2	3	5	10
$\hat{M}_n(t)$	0.0035	0.0615	0.2159	0.3113	0.5737	0.8498	14.106	25.353	53.470

5. Conclusion

In this study, the renewal theory is dealt with. In Random Right Sampling situation, the Gamma Renewal Function is obtained with the RS Method of Xie, and then the parametric estimator has been found with the maximum likelihood method, and the applications has been made on an example. The strong consistency and asymptotical unbiasedness of this estimator has been examined.

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