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Research Paper

On Soft Topology

Serdar Enginoğlu[a,*], Naim Çağman[b], Serkan Karataş[c], Tuğçe Aydın[a]

[a]Department of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, 17100, Çanakkale, Turkey
[b]Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpaşa University, 60250, Tokat, Turkey
[c]Department of Mathematics, Faculty of Arts and Sciences, Ordu University, 52100, Ordu, Turkey
[*]serdarenginoglu@gmail.com, naim.cagman@gop.edu.tr, serkankaratas@odu.edu.tr, aydinttugce@gmail.com

Abstract: Soft set theory was introduced by Molodtsov in 1999. Until now many versions of it have been developed and applied to a lot of areas from algebra to decision making problems. One of these areas is *Soft Topology* [Çağman, N., Karataş, S., Enginoglu, S., Soft topology, Computers and Mathematics with Applications, 62, 351-358, 2011]. However, it has some difficulties and mistakes. In this paper, for further study on the soft topology, we have made fit this concept which is important for development of the concept of soft sets by decontaminating from its own inconsistencies. We finally discuss this concept later on works.

Keywords: Soft sets; soft topology; soft open sets; soft single point set; soft limit point; soft Hausdorff space.

1. Introduction

The concept of soft sets was firstly introduced by Molodtsov [23] in 1999 as a general mathematical tool for dealing with some kinds of uncertainty. Then many versions of it have been developed and applied to a lot of areas from algebra to decision making problems such as [1-3,6,8,13-16,18-20,24,26,28,30]. One of these areas is *Soft Topology* [7] propounding by using the soft sets given by Çağman and Enginoğlu [5] and defining on a soft set by using the soft subsets of it. In the same period, Shabir and Naz [31] introduced the concept of soft topology defining on a classical set by using the soft sets over it. Afterwards, a lot of papers have been presented on this concept such as [4,9-12,17,21,22,25,27,29,32,33].

Although the concept of the soft topology is important for development of the soft sets, it has some own difficulties arising from some definitions such as the definition of soft closed set and the theorems related with this definition. This situation necessitates to arrange some parts of it. So we have revised the paper [7] by defining the soft single point set preventing the confusion in the notions of soft limit point, soft interior point, etc. In addition to this case, we should emphasize that the soft topology has become consistent in itself. In other words, some arranges can require when the other types of the soft topology are taken into consideration such as fuzzy parameterized fuzzy soft topology.

2. Preliminary

In this section, we have presented the basic definitions and results of soft set theory which may be found in earlier studies [5,18,23].

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Throughout this work, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subseteq E$.

Definition 1. *A soft set F_A on the universe U is defined by the set of ordered pairs*

$$F_A = \{(x, f_A(x)) : x \in E\}$$

where $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

Here, the value of $f_A(x)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection.

Note that the set of all soft sets with the parameter set E over U will be denoted by $S(U)$.

Example 1. *Suppose that there are six houses in the universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ under consideration and that $E = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of parameters. The x_i ($i \in \{1,2,3,4,5\}$) stand for the parameters "expensive", "beautiful", "wooden", "cheap" and "in green surroundings", respectively.*

Suppose that $A = \{x_1, x_3, x_4\} \subseteq E$ and $f_A(x_1) = \{h_2, h_4\}$, $f_A(x_3) = U$, and $f_A(x_4) = \emptyset$. Then

$$F_A = \{(x_1, \{h_2, h_4\}), (x_2, \emptyset), (x_3, U), (x_4, \emptyset), (x_5, \emptyset)\}$$

or briefly

$$F_A = \{(x_1, \{h_2, h_4\}), (x_3, U)\}$$

Definition 2. *Let $F_A \in S(U)$. If $f_A(x) = \emptyset$ for all $x \in A$, then F_A is called an empty soft set, denoted by F_Φ.*

Definition 3. *Let $F_A \in S(U)$. If $f_A(x) = U$ for all $x \in A$, then F_A is called A-universal soft set, denoted by $F_{\tilde{A}}$. If $A = E$, then the A-universal soft set is called universal soft set denoted by $F_{\tilde{E}}$.*

Definition 4. *Let $F_A, F_B \in S(U)$. Then F_A is a soft subset of F_B, denoted by $F_A \tilde{\subseteq} F_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

Remark 1. *It should be note that $F_A \tilde{\subseteq} F_B$ does not imply the condition "Each element of F_A is also an element of F_B". In other words, the concept of classical subset differs from the concept of soft subset.*

Definition 5. *Let $F_A, F_B \in S(U)$. Then F_A and F_B are soft equal, denoted by $F_A = F_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$.*

Definition 6. *Let $F_A, F_B \in S(U)$. Then soft union $F_A \tilde{\cup} F_B$, soft intersection $F_A \tilde{\cap} F_B$ and soft difference $F_A \tilde{\setminus} F_B$ of F_A and F_B are defined by, respectively,*

$$f_{A\tilde{\cup}B}(x) = f_A(x) \cup f_B(x), \qquad f_{A\tilde{\cap}B}(x) = f_A(x) \cap f_B(x), \qquad f_{A\tilde{\setminus}B}(x) = f_A(x) \setminus f_B(x)$$

and the soft complement $F_A^{\tilde{c}}$ of F_A is defined by

$$f_{A^{\tilde{c}}}(x) = f_A{}^c(x)$$

where $f_A{}^c(x)$ is complement of the set $f_A(x)$, that is, $f_A{}^c(x) = U \setminus f_A(x)$ for all $x \in E$.

It is easy to see that $\left(F_A^{\tilde{c}}\right)^{\tilde{c}} = F_A$ and $F_\Phi{}^{\tilde{c}} = F_{\tilde{E}}$.

Proposition 1. *Let $F_A \in S(U)$. Then*

i. $F_A \tilde{\cup} F_A = F_A$ *and* $F_A \tilde{\cap} F_A = F_A$
ii. $F_A \tilde{\cup} F_\Phi = F_A$ *and* $F_A \tilde{\cap} F_\Phi = F_\Phi$
iii. $F_A \tilde{\cup} F_{\tilde{E}} = F_{\tilde{E}}$ *and* $F_A \tilde{\cap} F_{\tilde{E}} = F_A$
iv. $F_A \tilde{\cup} F_A^{\tilde{c}} = F_{\tilde{E}}$ *and* $F_A \tilde{\cap} F_A^{\tilde{c}} = F_\Phi$

Proposition 2. *Let* $F_A, F_B, F_C \in S(U)$. *Then*

i. $F_A \tilde{\cup} F_B = F_B \tilde{\cup} F_A$ *and* $F_A \tilde{\cap} F_B = F_B \tilde{\cap} F_A$
ii. $(F_A \tilde{\cup} F_B)^{\tilde{c}} = {F_A}^{\tilde{c}} \tilde{\cap} {F_B}^{\tilde{c}}$ *and* $(F_A \tilde{\cap} F_B)^{\tilde{c}} = {F_A}^{\tilde{c}} \tilde{\cup} {F_B}^{\tilde{c}}$
iii. $(F_A \tilde{\cup} F_B) \tilde{\cup} F_C = F_A \tilde{\cup} (F_B \tilde{\cup} F_C)$ *and* $(F_A \tilde{\cap} F_B) \tilde{\cap} F_C = F_A \tilde{\cap} (F_B \tilde{\cap} F_C)$
iv. $F_A \tilde{\cup} (F_B \tilde{\cap} F_C) = (F_A \tilde{\cup} F_B) \tilde{\cap} (F_A \tilde{\cup} F_C)$ *and* $F_A \tilde{\cap} (F_B \tilde{\cup} F_C) = (F_A \tilde{\cap} F_B) \tilde{\cup} (F_A \tilde{\cap} F_C)$

Proposition 3. *Let* $F_A, F_B \in S(U)$. *Then* $F_A \tilde{\setminus} F_B = F_A \tilde{\cap} F_B^{\tilde{c}}$

3. Soft Topology

In this section, we recall some basic notions with updates in soft topology [7].

Definition 7. [7] *Let* $F_A \in S(U)$. *Soft power set of* F_A *is defined by*

$$\tilde{P}(F_A) = \{F_{A_i} : F_{A_i} \tilde{\subseteq} F_A,\ i \in I \subseteq \mathbb{N}\}$$

and its cardinality is defined by

$$\left|\tilde{P}(F_A)\right| = 2^{\sum_{x \in E}|f_A(x)|}$$

where $|f_A(x)|$ *is cardinality of* $f_A(x)$.

Example 2. [7] *Let* $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$, $A = \{x_1, x_2\} \subseteq E$ *and* $F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2, u_3\})\}$. *Then all soft subsets of* F_A *as follows,*

$$F_{A_1} = \{(x_1, \{u_1\})\} \qquad F_{A_9} = \{(x_1, \{u_1\}), (x_2, \{u_2, u_3\})\}$$
$$F_{A_2} = \{(x_1, \{u_2\})\} \qquad F_{A_{10}} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\}$$
$$F_{A_3} = \{(x_1, \{u_1, u_2\})\} \qquad F_{A_{11}} = \{(x_1, \{u_2\}), (x_2, \{u_3\})\}$$
$$F_{A_4} = \{(x_2, \{u_2\})\} \qquad F_{A_{12}} = \{(x_1, \{u_2\}), (x_2, \{u_2, u_3\})\}$$
$$F_{A_5} = \{(x_2, \{u_3\})\} \qquad F_{A_{13}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}$$
$$F_{A_6} = \{(x_2, \{u_2, u_3\})\} \qquad F_{A_{14}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\})\}$$
$$F_{A_7} = \{(x_1, \{u_1\}), (x_2, \{u_2\})\} \qquad F_{A_{15}} = F_A$$
$$F_{A_8} = \{(x_1, \{u_1\}), (x_2, \{u_3\})\} \qquad F_{A_{16}} = F_\Phi$$

Note that $\left|\tilde{P}(F_A)\right| = 2^4 = 16$.

Definition 8. [7] *Let* $F_A \in S(U)$. *A soft topology on* F_A, *denoted by* $\tilde{\tau}$, *is a collection of soft subsets of* F_A *having following properties:*

i. $F_\Phi, F_A \in \tilde{\tau}$
ii. $\{F_{A_i} \tilde{\subseteq} F_A : i \in I \subseteq \mathbb{N}\} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcup}_{i \in I} F_{A_i} \in \tilde{\tau}$
iii. $\{F_{A_i} \tilde{\subseteq} F_A : 1 \le i \le n,\ n \in \mathbb{N}\} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcap}_{i=1}^{n} F_{A_i} \in \tilde{\tau}$

The pair $(F_A, \tilde{\tau})$ is called a soft topological space.

Example 3. [7] *Let's consider the soft subsets of F_A that are given in Example 2. Then* $\tilde{\tau}_1 = \{F_\Phi, F_A\}$, $\tilde{\tau}_2 = \tilde{P}(F_A)$ *and* $\tilde{\tau}_3 = \{F_\Phi, F_A, F_{A_2}, F_{A_{11}}, F_{A_{13}}\}$ *are soft topologies on* F_A.

Here, $\{F_\Phi, F_A\}$ and $\tilde{P}(F_A)$ are called indiscrete and discrete soft topology on F_A, respectively.

Definition 9. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space. Then every element of $\tilde{\tau}$ is called a soft open set or briefly soft open in $\tilde{\tau}$. Clearly, F_Φ and F_A are soft open sets in $\tilde{\tau}$.*

Definition 10. [7] *Let $(F_A, \tilde{\tau}_1)$ and $(F_A, \tilde{\tau}_2)$ be soft topological spaces. Then*

i. *If $\tilde{\tau}_2 \supseteq \tilde{\tau}_1$, it is called that $\tilde{\tau}_2$ is soft finer than $\tilde{\tau}_1$*
ii. *If $\tilde{\tau}_2 \supset \tilde{\tau}_1$, it is called that $\tilde{\tau}_2$ is soft strictly finer than $\tilde{\tau}_1$*
iii. *If either $\tilde{\tau}_2 \supseteq \tilde{\tau}_1$ or $\tilde{\tau}_2 \subseteq \tilde{\tau}_1$, it is called $\tilde{\tau}_1$ is comparable with $\tilde{\tau}_2$*

Example 4. [7] *Let's consider the soft topologies on F_A that are given in Example 3. Then $\tilde{\tau}_2$ is soft finer than $\tilde{\tau}_1$ and $\tilde{\tau}_3$, and $\tilde{\tau}_3$ is soft finer than $\tilde{\tau}_1$. So $\tilde{\tau}_1$, $\tilde{\tau}_2$ and $\tilde{\tau}_3$ are comparable soft topologies.*

Definition 11. **[7]** *Let $(F_A, \tilde{\tau})$ be a soft topological space and $\tilde{\mathfrak{B}} \subseteq \tilde{\tau}$. If every element of $\tilde{\tau}$ can be written as the soft union of element of $\tilde{\mathfrak{B}}$, then $\tilde{\mathfrak{B}}$ is called a soft basis for $\tilde{\tau}$. Each element of $\tilde{\mathfrak{B}}$ is called soft basis element.*

Example 5. [7] *Let's consider the Example 2 and Example 3. Then $\tilde{\mathfrak{B}} = \{F_\Phi, F_{A_1}, F_{A_2}, F_{A_4}, F_{A_5}\}$ is a soft basis for $\tilde{\tau}_2$.*

Theorem 1. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space and $\tilde{\mathfrak{B}}$ be a soft basis for $\tilde{\tau}$. Then $\tilde{\tau}$ equals the collection of all soft unions of elements of $\tilde{\mathfrak{B}}$.*

Proof. It is clearly seen from Definition 11.

Definition 12. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then the collection*

$$\tilde{\tau}_{F_B} = \{F_{A_i} \tilde{\cap} F_B \, : \, F_{A_i} \in \tilde{\tau}, \; i \in I \subseteq \mathbb{N}\}$$

is called a soft subspace topology on F_B.

Hence $(F_B, \tilde{\tau}_{F_B})$ is called a soft topological subspace of $(F_A, \tilde{\tau})$.

Theorem 2. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then a soft subspace topology on F_B is a soft topology.*

Proof. [7] Indeed, it contains F_Φ and F_B because $F_\Phi \tilde{\cap} F_B = F_\Phi$ and $F_A \tilde{\cap} F_B = F_B$, where $F_\Phi, F_A \in \tilde{\tau}$. Since $\tilde{\tau} = \{F_{A_i} \, : \, F_{A_i} \tilde{\subseteq} F_A, \; i \in I\}$, it is closed under finite soft intersections and arbitrary soft unions;

$$\tilde{\bigcap_{i=1}^{n}} (F_{A_i} \tilde{\cap} F_B) = \left(\tilde{\bigcap_{i=1}^{n}} F_{A_i} \right) \tilde{\cap} F_B$$

$$\tilde{\bigcup_{i \in I}} (F_{A_i} \tilde{\cap} F_B) = \left(\tilde{\bigcup_{i \in I}} F_{A_i} \right) \tilde{\cap} F_B$$

Example 6. [7] *Let's consider the soft topology $\tilde{\tau}_3$ on F_A given in Example 3. If $F_B = F_{A_9}$, then $\tilde{\tau}_{F_B} = \{F_\Phi, F_{A_5}, F_{A_7}, F_{A_9}\}$ and so $(F_B, \tilde{\tau}_{F_B})$ is a soft topological subspace of $(F_A, \tilde{\tau}_3)$.*

Theorem 3. [7] *Let $(F_A, \tilde{\tau})$ and $(F_A, \tilde{\tau}')$ be soft topological spaces, and $\widetilde{\mathfrak{B}}$ and $\widetilde{\mathfrak{B}}'$ be soft bases for $\tilde{\tau}$ and $\tilde{\tau}'$, respectively. If $\widetilde{\mathfrak{B}}' \subseteq \widetilde{\mathfrak{B}}$, then $\tilde{\tau}$ is soft finer than $\tilde{\tau}'$.*

Proof. [7] Let $\widetilde{\mathfrak{B}}' \subseteq \widetilde{\mathfrak{B}}$. Then for each $F_B \in \tilde{\tau}'$ and $F_C \in \widetilde{\mathfrak{B}}'$,

$$F_B = \widetilde{\bigcup_{F_C \in \widetilde{\mathfrak{B}}'}} F_C = \widetilde{\bigcup_{F_C \in \widetilde{\mathfrak{B}}}} F_C$$

Therefore $F_B \in \tilde{\tau}$, hence $\tilde{\tau}' \subseteq \tilde{\tau}$.

Theorem 4. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space. If $\widetilde{\mathfrak{B}}$ is a soft basis for $\tilde{\tau}$, then collection $\widetilde{\mathfrak{B}}_{F_B} = \{F_{A_i} \tilde{\cap} F_B : F_{A_i} \in \widetilde{\mathfrak{B}},\ i \in I \subseteq \mathbb{N}\}$ is a soft basis for $\tilde{\tau}_{F_B}$.*

Proof. [7] Given each $F_{A_i} \in \tilde{\tau}_{F_B}$. From definition of soft subspace topology; $F_C = F_D \tilde{\cap} F_B$, where $F_D \in \tilde{\tau}$. Because of $F_D \in \tilde{\tau}$, $F_D = \tilde{\bigcup}_{F_{A_i} \in \widetilde{\mathfrak{B}}} F_{A_i}$. Therefore,

$$F_C = \left(\widetilde{\bigcup_{F_{A_i} \in \widetilde{\mathfrak{B}}}} F_{A_i} \right) \tilde{\cap} F_B = \widetilde{\bigcup_{F_{A_i} \in \widetilde{\mathfrak{B}}}} \left(F_{A_i} \tilde{\cap} F_B \right)$$

Hence $\widetilde{\mathfrak{B}}_{F_B}$ is a soft basis for $\tilde{\tau}_{F_B}$.

Remark 2. *It is seen that the condition $F_C \in \tilde{\tau}_{F_B} \Rightarrow F_C \in \tilde{\tau}$ given in* [7], *Theorem 5, does not hold. Let's update of it as follows.*

Theorem 5. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of it. If F_C is soft open in $\tilde{\tau}_{F_B}$, then there exists at least one element F_D of $\tilde{\tau}$ such that $F_C \tilde{\subseteq} F_D$.*

Proof. It is clearly seen from Definition 12.

Remark 3. *It is seen that the proposition "The universal soft set $F_{\tilde{E}}$ and $F_A{}^{\tilde{c}}$ are soft closed sets", Theorem 6 (i.), given in* [7] *does not hold according to the Definition 13 in the same paper. On the other hand, the soft complement according to F_A of a soft set in the soft topological space $(F_A, \tilde{\tau})$ is more useful and meaningful than the soft complement according to the universal soft set $F_{\tilde{E}}$. Let's update of it as follows.*

Definition 13. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then F_B is called a soft closed set or briefly soft closed according to $\tilde{\tau}$, if the soft set $F_A \tilde{\setminus} F_B$ is soft open set in $\tilde{\tau}$.*

Theorem 6. *Let $(F_A, \tilde{\tau})$ be a soft topological space. Then the following conditions hold:*

i. *The empty soft set F_Φ and F_A are soft closed sets.*
ii. *Arbitrary soft intersections of the soft closed sets are soft closed.*
iii. *Finite soft unions of the soft closed sets are soft closed.*

Proof.

i. By the definition of soft closed set, $F_A \tilde{\setminus} F_A = F_\Phi$ and $F_A \tilde{\setminus} F_\Phi = F_A$ are soft open. Then F_A and F_Φ are soft closed.

ii. If $\{F_{A_i} : F_A \tilde{\setminus} F_{A_i} \in \tilde{\tau},\ i \in I \subseteq \mathbb{N}\}$ is a given collection of soft closed sets, then

$$F_A \tilde{\setminus} \left(\tilde{\bigcap_{i\in I}} F_{A_i} \right) = \tilde{\bigcup_{i\in I}} (F_A \tilde{\setminus} F_{A_i})$$

is soft open. Therefore, $\tilde{\bigcap}_{i\in I} F_{A_i}$ is soft closed.

iii. Similarly, if F_{A_i} is a soft closed for $i = 1,2,\dots,n$, then

$$F_A \tilde{\setminus} \left(\tilde{\bigcup_{i=1}^{n}} F_{A_i} \right) = \tilde{\bigcap_{i=1}^{n}} (F_A \tilde{\setminus} F_{A_i})$$

is soft open. Hence $\tilde{\bigcup}_{i=1}^{n} F_{A_i}$ is soft closed.

Remark 4. *It is seen that the Theorems 12-17 given in* [7] *have some incompatibilities to the other some definitions in the same paper. To overcome these difficulties, let's give a definition of a single point soft set and update the theorems mentioned above.*

Definition 14. *Let $F_A \in S(U)$ and $F_B \tilde{\subseteq} F_A$. If $f_B(x)$ is a single point set for only one $x \in B$ and $f_B(y) = \emptyset$ for $y \in E \setminus \{x\}$, then F_B is called soft single point set or soft element of F_A and is denoted by $F_B \tilde{\in} F_A$ or $(x, f_B(x)) \tilde{\in} F_A$ or briefly $\alpha \tilde{\in} F_A$.*

Example 7. *Let's consider the soft subsets of F_A that are given in Example 2. Then $F_{A_1}, F_{A_2}, F_{A_4}$ and F_{A_5} are soft single point sets of F_A, briefly, can be shown $\alpha_1, \alpha_2, \alpha_4$ and α_5, respectively.*

Theorem 7. *Let $F_A, F_B \in S(U)$ and $F_B \tilde{\subseteq} F_A$. Then*

$$\alpha \tilde{\in} F_B \Leftrightarrow \alpha \tilde{\notin} (F_A \tilde{\setminus} F_B)$$

Proof. Let $F_B \tilde{\subseteq} F_A$ and $\alpha = (x, f_C(x))$.

$$\begin{aligned} \alpha \tilde{\in} F_B &\Leftrightarrow \alpha \tilde{\in} F_A \wedge \alpha \tilde{\in} F_B \\ &\Leftrightarrow \forall x \in E,\ f_C(x) \subseteq f_A(x) \wedge \forall x \in E,\ f_C(x) \subseteq f_B(x) \\ &\Leftrightarrow \forall x \in E,\ f_C(x) \subseteq f_A(x) \wedge \exists x \in E,\ f_C(x) \nsubseteq f_A(x) \setminus f_B(x) \\ &\Leftrightarrow \alpha \tilde{\notin} F_A \tilde{\setminus} F_B \end{aligned}$$

Theorem 8. *Let $F_A, F_B \in S(U)$. Then*

$$(F_A = F_B) \Leftrightarrow (\alpha \tilde{\in} F_A \Leftrightarrow \alpha \tilde{\in} F_B)$$

Proof. The proof is trivial.

Theorem 9. *Let $F_A, F_B, F_C, F_D \in S(U)$. Then*

i. $(F_A \tilde{\subseteq} F_B \wedge F_C \tilde{\subseteq} F_D) \Rightarrow (F_A \tilde{\cap} F_C \tilde{\subseteq} F_B \tilde{\cap} F_D)$

ii. $(F_A \tilde{\subseteq} F_B \wedge F_C \tilde{\subseteq} F_D) \Rightarrow (F_A \tilde{\cup} F_C \tilde{\subseteq} F_B \tilde{\cup} F_D)$

Proof.

i. Let $F_A \tilde{\subseteq} F_B \wedge F_C \tilde{\subseteq} F_D$.

$$\alpha \tilde{\in} F_A \tilde{\cap} F_C \Rightarrow \alpha \tilde{\in} F_A \wedge \alpha \tilde{\in} F_C$$
$$\Rightarrow \alpha \tilde{\in} F_A \tilde{\subseteq} F_B \wedge \alpha \tilde{\in} F_C \tilde{\subseteq} F_D$$
$$\Rightarrow \alpha \tilde{\in} F_B \tilde{\cap} F_D$$

Hence $F_A \tilde{\cap} F_C \tilde{\subseteq} F_B \tilde{\cap} F_D$.

ii. Let $F_A \tilde{\subseteq} F_B \wedge F_C \tilde{\subseteq} F_D$.

$$\alpha \tilde{\in} F_A \tilde{\cup} F_C \Rightarrow \alpha \tilde{\in} F_A \vee \alpha \tilde{\in} F_C$$
$$\Rightarrow \alpha \tilde{\in} F_A \tilde{\subseteq} F_B \vee \alpha \tilde{\in} F_C \tilde{\subseteq} F_D$$
$$\Rightarrow \alpha \tilde{\in} F_B \tilde{\cup} F_D$$

Hence $F_A \tilde{\cup} F_C \tilde{\subseteq} F_B \tilde{\cup} F_D$.

Definition 15. *Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \tilde{\subseteq} F_A$ and $\alpha \tilde{\in} F_B$. If there exists $\exists F_C \in \tilde{\tau}$ such that $\alpha \tilde{\in} F_C \tilde{\subseteq} F_B$, then α is called a soft interior point of F_B, and the soft union of all soft interior points of F_B, denoted by F_B°, is called soft interior of F_B.*

Note that the soft interior of F_B is also defined as the soft union of all soft open subsets of F_B. In other words, F_B° is the biggest soft open set that contained by F_B.

Example 8. [7] *Let's consider the soft topology $\tilde{\tau}_3$ given in Example 3. If $F_B = F_{A_{12}} = \{(x_1, \{u_2\}), (x_2, \{u_2, u_3\})\}$, then $F_B^{\circ} = F_{\Phi} \tilde{\cup} F_{A_2} \tilde{\cup} F_{A_{11}} = F_{A_{11}}$*

Theorem 10. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. F_B is soft open if and only if $F_B = F_B^{\circ}$.*

Proof. The proof is trivial.

Theorem 11. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then*

i. $\left(F_B^{\circ}\right)^{\circ} = F_B^{\circ}$

ii. $F_B \tilde{\subseteq} F_C \Rightarrow F_B^{\circ} \tilde{\subseteq} F_C^{\circ}$

iii. $F_B^{\circ} \tilde{\cap} F_C^{\circ} = (F_B \tilde{\cap} F_C)^{\circ}$

iv. $F_B^{\circ} \tilde{\cup} F_C^{\circ} \tilde{\subseteq} (F_B \tilde{\cup} F_C)^{\circ}$

Proof.

i. Let $F_B^{\circ} = F_D$. Then $F_D \in \tilde{\tau}$ if and only if $F_D = F_D^{\circ}$. Therefore, $\left(F_B^{\circ}\right)^{\circ} = F_B^{\circ}$.

ii. Let $F_B \tilde{\subseteq} F_C$. From the definition of soft interior; $F_B^{\circ} \tilde{\subseteq} F_B$ and $F_C^{\circ} \tilde{\subseteq} F_C$. F_C° is the biggest soft open set that contained by F_C. Hence $F_B \tilde{\subseteq} F_C \Rightarrow F_B^{\circ} \tilde{\subseteq} F_C^{\circ}$.

iii. By the definition of soft interior; $F_B^{\circ} \tilde{\subseteq} F_B$ and $F_C^{\circ} \tilde{\subseteq} F_C$. Then $F_B^{\circ} \tilde{\cap} F_C^{\circ} \tilde{\subseteq} F_B \tilde{\cap} F_C$. $(F_B \tilde{\cap} F_C)^{\circ}$ is the biggest soft open set that contained by $F_B \tilde{\cap} F_C$, therefore $F_B^{\circ} \tilde{\cap} F_C^{\circ} \tilde{\subseteq} (F_B \tilde{\cap} F_C)^{\circ}$. Conversely, $F_B \tilde{\cap} F_C \tilde{\subseteq} F_B$ and $F_B \tilde{\cap} F_C \tilde{\subseteq} F_C$. Then $(F_B \tilde{\cap} F_C)^{\circ} \tilde{\subseteq} F_B^{\circ}$ and $(F_B \tilde{\cap} F_C)^{\circ} \tilde{\subseteq} F_C^{\circ}$. Therefore, $(F_B \tilde{\cap} F_C)^{\circ} \tilde{\subseteq} F_B^{\circ} \tilde{\cap} F_C^{\circ}$. Hence $F_B^{\circ} \tilde{\cap} F_C^{\circ} = (F_B \tilde{\cap} F_C)^{\circ}$.

iv. By the definition of soft interior; $F_B^{\circ} \tilde{\subseteq} F_B$ and $F_C^{\circ} \tilde{\subseteq} F_C$. Then $F_B^{\circ} \tilde{\cup} F_C^{\circ} \tilde{\subseteq} F_B \tilde{\cup} F_C$. $(F_B \tilde{\cup} F_C)^{\circ}$ is the biggest soft open set that contained by $F_B \tilde{\cup} F_C$. Hence $F_B^{\circ} \tilde{\cup} F_C^{\circ} \tilde{\subseteq} (F_B \tilde{\cup} F_C)^{\circ}$.

Definition 16. [7] *Let $(F_A,\tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then soft closure of F_B, denoted $\overline{F}_B$, is defined as the soft intersection of all soft closed supersets of F_B.*

Note that $\overline{F}_B$ is the smallest soft closed set that containing F_B.

Example 9. *Let's consider the soft topology $\tilde{\tau}_3$ that is given in Example 3. If $F_B = F_{A_9} = \{(x_1,\{u_1\}),(x_2,\{u_2,u_3\})\}$, then $F_{A_9} = \{(x_1,\{u_1\}),(x_2,\{u_2,u_3\})\}$ and $F_A = \{(x_1,\{u_1,u_2\}),(x_2,\{u_2,u_3\})\}$ are soft closed supersets of F_B. Hence $\overline{F}_B = F_{A_9} \tilde{\cap} F_A = F_{A_9}$.*

Theorem 12. *Let $(F_A,\tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. F_B is a soft closed set if and only if $F_B = \overline{F}_B$.*

Proof. The proof is trivial.

Theorem 13. *Let $(F_A,\tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then $F_B^{\circ} \tilde{\subseteq} F_B \tilde{\subseteq} \overline{F}_B$.*

Proof. Indeed, $F_B^{\circ} = \tilde{\bigcup}\{F_{B_i} :\ F_{B_i} \in \tilde{\tau}, F_{B_i} \tilde{\subseteq} F_B, i \in I \subseteq \mathbb{N}\}$. Then $f_{B_i}(x) \subseteq f_B(x)$ and $\bigcup_{i\in I} f_{B_i}(x) \subseteq f_B(x)$ for all $x \in E$. So $F_B^{\circ} \tilde{\subseteq} F_B$.

$\overline{F}_B = \tilde{\bigcap}\{F_{A_i} :\ F_A \tilde{\setminus} F_{A_i} \in \tilde{\tau}, F_B \tilde{\subseteq} F_{A_i}, i \in J \subseteq \mathbb{N}\}$. Then $f_B(x) \subseteq f_{A_i}(x)$ and $f_B(x) \subseteq \bigcap_{i\in J} f_{A_i}(x)$ for all $x \in E$. So $F_B \tilde{\subseteq} \overline{F}_B$.

Hence $F_B^{\circ} \tilde{\subseteq} F_B \tilde{\subseteq} \overline{F}_B$.

Theorem 14. *Let $(F_A,\tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then*

i. $\left(\overline{\overline{F}_B}\right) = \overline{F}_B$
ii. $\left(F_A \tilde{\setminus} \overline{F}_B\right) = \left(F_A \tilde{\setminus} F_B\right)^{\circ}$
iii. $F_B \tilde{\subseteq} F_C \Rightarrow \overline{F}_B \tilde{\subseteq} \overline{F}_C$
iv. $\overline{(F_B \tilde{\cap} F_C)} \tilde{\subseteq} \overline{F}_B \tilde{\cap} \overline{F}_C$
v. $\overline{F}_B \tilde{\cup} \overline{F}_C = \overline{(F_B \tilde{\cup} F_C)}$

Proof.

i. Let $\overline{F}_B = F_D$. Then F_D is a soft closed set. Therefore, F_D and $\overline{F}_D$ are equal. Hence $\left(\overline{\overline{F}_B}\right) = \overline{F}_B$.
ii. If we consider the definitions of the soft closure and soft interior, we obtain

$$\left(F_A \tilde{\setminus} \overline{F}_B\right) = F_A \tilde{\setminus} \left(\tilde{\bigcap_{\substack{F_{A_i} \tilde{\supseteq} F_B \\ F_A \tilde{\setminus} F_{A_i} \in \tilde{\tau}}}} F_{A_i} \right) = \tilde{\bigcup_{\substack{F_A \tilde{\setminus} F_B \tilde{\supseteq} F_A \tilde{\setminus} F_{A_i} \\ F_A \tilde{\setminus} F_{A_i} \in \tilde{\tau}}}} \left(F_A \tilde{\setminus} F_{A_i}\right) = \left(F_A \tilde{\setminus} F_B\right)^{\circ}$$

iii. Let $F_B \tilde{\subseteq} F_C$. By the definition of soft closure; $F_B \tilde{\subseteq} \overline{F}_B$ and $F_C \tilde{\subseteq} \overline{F}_C$. In other words, $F_B \tilde{\subseteq} \overline{F}_B$ and $F_B \tilde{\subseteq} \overline{F}_C$. Since $\overline{F}_B$ is the smallest soft closed set that containing F_B, the inclusion $F_B \tilde{\subseteq} \overline{F}_B \tilde{\subseteq} \overline{F}_C$ is hold. Hence $\overline{F}_B \tilde{\subseteq} \overline{F}_C$.

iv. $\overline{F}_B$ and $\overline{F}_C$ are soft closed sets. So $\overline{F}_B \tilde{\cap} \overline{F}_C$ is a soft closed set. Since $F_B \tilde{\cap} F_C \tilde{\subseteq} \overline{F}_B \tilde{\cap} \overline{F}_C$ and $\overline{(F_B \tilde{\cap} F_C)}$ is the smallest soft closed set that containing $F_B \tilde{\cap} F_C$, then $\overline{(F_B \tilde{\cap} F_C)} \tilde{\subseteq} \overline{F}_B \tilde{\cap} \overline{F}_C$.

v. By the definition of soft closure; $F_B \tilde{\subseteq} \overline{F}_B$ and $F_C \tilde{\subseteq} \overline{F}_C$. Then $F_B \tilde{\cup} F_C \tilde{\subseteq} \overline{F}_B \tilde{\cup} \overline{F}_C$. Since $\overline{(F_B \tilde{\cup} F_C)}$ is the smallest soft closed set that containing $F_B \tilde{\cup} F_C$, then $\overline{(F_B \tilde{\cup} F_C)} \tilde{\subseteq} \overline{F}_B \tilde{\cup} \overline{F}_C$. Conversely, $F_C \tilde{\subseteq} \overline{F}_C \tilde{\subseteq} \overline{(F_B \tilde{\cup} F_C)}$ and $F_B \tilde{\subseteq} \overline{F}_B \tilde{\subseteq} \overline{(F_B \tilde{\cup} F_C)}$. Therefore, $\overline{F}_B \tilde{\cup} \overline{F}_C \tilde{\subseteq} \overline{(F_B \tilde{\cup} F_C)}$. Hence $\overline{F}_B \tilde{\cup} \overline{F}_C = \overline{(F_B \tilde{\cup} F_C)}$.

Theorem 15. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then*

i. $\alpha \tilde{\in} \overline{F}_B$ if and only if, for all $F_C \in \tilde{\tau}, F_B \tilde{\cap} F_C \neq F_\Phi$ such that $\alpha \tilde{\in} F_C$.

ii. Let $\tilde{\mathfrak{B}}$ be a soft basis for $\tilde{\tau}$. Then $\alpha \tilde{\in} \overline{F}_B$ if and only if, for all $F_D \in \tilde{\mathfrak{B}}, F_B \tilde{\cap} F_D \neq F_\Phi$ such that $\alpha \tilde{\in} F_D$.

Proof.

i. As logically, hypothesis is equivalent to; $\alpha \tilde{\notin} \overline{F}_B$ if and only if there exists a soft open set F_C such that $\alpha \tilde{\in} F_C$ and $F_B \tilde{\cap} F_C = F_\Phi$.
(⇒:) If $\alpha \tilde{\notin} \overline{F}_B$, then $\alpha \tilde{\in} \left(F_A \tilde{\setminus} \overline{F}_B\right)$. Because of $\alpha \tilde{\in} \left(F_A \tilde{\setminus} \overline{F}_B\right) = \left(F_A \tilde{\setminus} F_B\right)^\circ$ from Theorem 14 (*ii.*), we obtain $\alpha \tilde{\in} \left(F_A \tilde{\setminus} F_B\right)^\circ$. By the definition of soft interior; there exists $\exists F_C \in \tilde{\tau}$ such that $\alpha \tilde{\in} F_C \tilde{\subseteq} \left(F_A \tilde{\setminus} F_B\right)$. Hence there exists $\exists F_C \in \tilde{\tau}$ such that $\alpha \tilde{\in} F_C$ and $F_B \tilde{\cap} F_C = F_\Phi$.
(⇐:) If there exists a soft open set F_C such that $\alpha \tilde{\in} F_C$ and $F_B \tilde{\cap} F_C = F_\Phi$, then $F_A \tilde{\setminus} F_C$ is a soft closed set such that $F_B \tilde{\subseteq} \left(F_A \tilde{\setminus} F_C\right)$. By the definition of the soft closure, $\overline{F}_B \tilde{\subseteq} \left(F_A \tilde{\setminus} F_C\right)$. Therefore, $\alpha \tilde{\notin} \overline{F}_B$.

ii. Let $\alpha \tilde{\in} \overline{F}_B$ and $F_D \in \tilde{\mathfrak{B}}$ such that $\alpha \tilde{\in} F_D$. By the definition of soft basis and Theorem 15 (*i.*), for all $F_D \in \tilde{\mathfrak{B}}$, $F_B \tilde{\cap} F_D \neq F_\Phi$ such that $\alpha \tilde{\in} F_D$. Conversely, if for all $F_D \in \tilde{\mathfrak{B}}$, $F_B \tilde{\cap} F_D \neq F_\Phi$ such that $\alpha \tilde{\in} \overline{F}_B$, so does for all $F_C \in \tilde{\tau}$, $F_B \tilde{\cap} F_C \neq F_\Phi$ such that $\alpha \tilde{\in} F_C$. Hence $\alpha \tilde{\in} \overline{F}_B$.

Definition 17. *Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \tilde{\subseteq} F_A$ and $\alpha \tilde{\in} F_A$. If there is a soft open set F_C such that $\alpha \tilde{\in} F_C \tilde{\subseteq} F_B$, then F_B is called soft neighborhood of α. Set of all soft neighborhoods of α, denoted by $\tilde{\mathcal{N}}(\alpha)$, is called family of soft neighborhoods of α, that is*

$$\tilde{\mathcal{N}}(\alpha) = \{F_B : F_C \in \tilde{\tau} \text{ and } \alpha \tilde{\in} F_C \tilde{\subseteq} F_B\}$$

In particular,

$$\tilde{\mathcal{V}}(\alpha) = \{F_C \in \tilde{\tau} : \alpha \tilde{\in} F_C\}$$

is called family of soft open neighborhood of α.

Example 10. *Let's consider the $(F_A, \tilde{\tau}_3)$ soft topological space in Example 3 and $\alpha_5 = (x_2, \{u_3\}) \tilde{\in} F_A$. Then $\tilde{\mathcal{N}}(\alpha_5) = \{F_A, F_{A_{11}}, F_{A_{12}}, F_{A_{14}}\}$ and $\tilde{\mathcal{V}}(\alpha_5) = \{F_A, F_{A_{11}}\}$.*

Definition 18. *Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B, F_C \tilde{\subseteq} F_A$ and $\alpha \tilde{\in} F_A$. Then α is called a soft limit point of F_B, if $F_C \tilde{\cap} \left(F_B \tilde{\setminus} \alpha\right) \neq F_\Phi$ for all $F_C \in \tilde{\mathcal{V}}(\alpha)$. Here, the soft union of all soft limit points of F_B is denoted by F_B'.*

Example 11. *Let's consider $(F_A, \tilde{\tau}_3)$ in Example 3. Then*

$$F_{A_{13}}' = \tilde{\bigcup}\{F_{A_1}, F_{A_4}, F_{A_5}\} = F_{A_9}$$

since

$F_{A_{13}} \tilde{\cap} \left(F_{A_{13}} \tilde{\setminus} F_{A_1}\right) = F_{A_{13}} \tilde{\cap} F_{A_{10}} = F_{A_{10}} \neq F_\Phi$ $F_A \tilde{\cap} \left(F_{A_{13}} \tilde{\setminus} F_{A_1}\right) = F_A \tilde{\cap} F_{A_{10}} = F_{A_{10}} \neq F_\Phi$
$F_{A_2} \tilde{\cap} \left(F_{A_{13}} \tilde{\setminus} F_{A_2}\right) = F_{A_2} \tilde{\cap} F_{A_7} = F_\Phi$
$F_{A_{13}} \tilde{\cap} \left(F_{A_{13}} \tilde{\setminus} F_{A_4}\right) = F_{A_{13}} \tilde{\cap} F_{A_3} = F_{A_3} \neq F_\Phi$ $F_A \tilde{\cap} \left(F_{A_{13}} \tilde{\setminus} F_{A_4}\right) = F_A \tilde{\cap} F_{A_3} = F_{A_3} \neq F_\Phi$
$F_{A_{11}} \tilde{\cap} \left(F_{A_{13}} \tilde{\setminus} F_{A_5}\right) = F_{A_{11}} \tilde{\cap} F_{A_{13}} = F_{A_2} \neq F_\Phi$ $F_A \tilde{\cap} \left(F_{A_{13}} \tilde{\setminus} F_{A_5}\right) = F_A \tilde{\cap} F_{A_{13}} = F_{A_{13}} \neq F_\Phi$

Theorem 16. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then $F_B \tilde{\cup} F_B' = \overline{F}_B$*

Proof. If $\alpha \tilde{\in} F_B \tilde{\cup} F_B'$, then $\alpha \tilde{\in} F_B$ or $\alpha \tilde{\in} F_B'$. In this case, if $\alpha \tilde{\in} F_B$, then $\alpha \tilde{\in} \overline{F}_B$. If $\alpha \tilde{\in} F_B'$, then $F_C \tilde{\cap} \left(F_B \tilde{\setminus} \alpha\right) \neq F_\Phi$ for all $F_C \in \tilde{\mathcal{V}}(\alpha)$ and so $F_C \tilde{\cap} F_B \neq F_\Phi$ for all $F_C \in \tilde{\mathcal{V}}(\alpha)$, hence $\alpha \tilde{\in} \overline{F}_B$ from Theorem 15. Conversely, if $\alpha \tilde{\in} \overline{F}_B$, then $\alpha \tilde{\in} F_B$ or $\alpha \tilde{\notin} F_B$. In this case, if $\alpha \tilde{\in} F_B$, it is trivial that $\alpha \tilde{\in} F_B \tilde{\cup} F_B'$. If $\alpha \tilde{\notin} F_B$, then $F_C \tilde{\cap} F_B = F_C \tilde{\cap} \left(F_B \tilde{\setminus} \alpha\right) \neq F_\Phi$ for all $F_C \in \tilde{\mathcal{V}}(\alpha)$ from Theorem 15 and Definition 18. Therefore, $\alpha \tilde{\in} F_B'$ so that $\alpha \tilde{\in} F_B \tilde{\cup} F_B'$. Hence $F_B \tilde{\cup} F_B' = \overline{F}_B$.

Theorem 17. [7] *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then F_B is soft closed if and only if $F_B' \tilde{\subseteq} F_B$.*

Proof. F_B is a soft closed $\Leftrightarrow F_B = \overline{F}_B \Leftrightarrow F_B = F_B \tilde{\cup} F_B' \Leftrightarrow F_B' \tilde{\subseteq} F_B$.

Theorem 18. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then*

i. $F_B' \tilde{\subseteq} \overline{F}_B$
ii. $F_B \tilde{\subseteq} F_C \Rightarrow F_B' \tilde{\subseteq} F_C'$
iii. $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_B' \tilde{\cap} F_C'$
iv. $(F_B \tilde{\cup} F_C)' = F_B' \tilde{\cup} F_C'$

Proof.

i. From the definitions of soft closure the proof is trivial.
ii. Let $\alpha \tilde{\in} F_B'$. Then $F_D \tilde{\cap} \left(F_B \tilde{\setminus} \alpha\right) \neq F_\Phi$, for all $F_D \in \tilde{\mathcal{V}}(\alpha)$. Since $F_B \tilde{\subseteq} F_C$, $F_D \tilde{\cap} \left(F_C \tilde{\setminus} \alpha\right) \neq F_\Phi$, for all $F_D \in \tilde{\mathcal{V}}(\alpha)$. In other words, $\alpha \tilde{\in} F_C'$. Hence $F_B' \tilde{\subseteq} F_C'$.

iii. $F_B \tilde{\cap} F_C \tilde{\subseteq} F_B$ and $F_B \tilde{\cap} F_C \tilde{\subseteq} F_C$. Then $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_B'$ and $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_C'$. Therefore, $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_B' \tilde{\cap} F_C'$.

iv. $F_B \tilde{\subseteq} F_B \tilde{\cup} F_C$ and $F_C \tilde{\subseteq} F_B \tilde{\cup} F_C$. Then $F_B' \tilde{\subseteq} (F_B \tilde{\cup} F_C)'$ and $F_B' \tilde{\subseteq} (F_B \tilde{\cup} F_C)'$. Therefore, $F_B' \tilde{\cup} F_C' \tilde{\subseteq} (F_B \tilde{\cup} F_C)'$.
Conversely, for all $F_D \in \tilde{\mathcal{V}}(\alpha)$,

$$\begin{aligned}
\alpha \tilde{\in} (F_B \tilde{\cup} F_C)' &\Leftrightarrow F_D \tilde{\cap} \left[(F_B \tilde{\cup} F_C) \tilde{\setminus} \alpha\right] \neq F_\Phi \\
&\Leftrightarrow F_D \tilde{\cap} \left[\left(F_B \tilde{\setminus} \alpha\right) \tilde{\cup} \left(F_C \tilde{\setminus} \alpha\right)\right] \neq F_\Phi \\
&\Leftrightarrow \left[F_D \tilde{\cap} \left(F_B \tilde{\setminus} \alpha\right)\right] \tilde{\cup} \left[F_D \tilde{\cap} \left(F_C \tilde{\setminus} \alpha\right)\right] \neq F_\Phi \\
&\Leftrightarrow \left[F_D \tilde{\cap} \left(F_B \tilde{\setminus} \alpha\right)\right] \neq F_\Phi \text{ or } \left[F_D \tilde{\cap} \left(F_C \tilde{\setminus} \alpha\right)\right] \neq F_\Phi \\
&\Leftrightarrow \alpha \tilde{\in} F_B' \text{ or } \alpha \tilde{\in} F_C' \\
&\Leftrightarrow \alpha \tilde{\in} F_B' \tilde{\cup} F_C'
\end{aligned}$$

Hence $(F_B \tilde{\cup} F_C)' \tilde{\subseteq} F_B' \tilde{\cup} F_C'$. Thus $(F_B \tilde{\cup} F_C)' = F_B' \tilde{\cup} F_C'$.

Definition 19. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. For all $F_C \in \tilde{\mathcal{V}}(\alpha)$, if $F_C \tilde{\cap} F_B \neq F_\Phi$ and $F_C \tilde{\cap} \left(F_A \tilde{\setminus} F_B\right) \neq F_\Phi$, then α is called a soft boundary point of F_B, and the soft union of all soft boundary points of F_B, denoted by F_B^b, is called soft boundary of F_B.*

Note that the soft boundary of F_B can also be defined as

$$F_B^b = \overline{F}_B \tilde{\cap} \overline{\left(F_A \tilde{\setminus} F_B\right)}.$$

Example 12. *Let's consider the Example 9. For F_B, $\overline{F}_B = F_{A_9}$ and $\overline{\left(F_A \tilde{\setminus} F_B\right)} = \overline{F}_{A_2} = F_A$. Then $F_B^b = \overline{F}_B \tilde{\cap} \overline{\left(F_A \tilde{\setminus} F_B\right)} = F_{A_9} \tilde{\cap} F_A = F_{A_9}$.*

Theorem 19. *Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then*

i. $F_B^b \tilde{\subseteq} \overline{F}_B$

ii. $F_B^b = \left(F_A \tilde{\setminus} F_B\right)^b$

iii. $F_B^b = \overline{F}_B \tilde{\setminus} F_B^\circ$

Proof.

i. From the definitions of soft boundary the proof is trivial.

ii. Given $\alpha \tilde{\in} F_A^b \Leftrightarrow F_C \tilde{\cap} F_B \neq F_\Phi$ and $F_C \tilde{\cap} \left(F_A \tilde{\setminus} F_B\right) \neq F_\Phi$ for all $F_C \in \tilde{\mathcal{V}}(\alpha) \Leftrightarrow$ $F_C \tilde{\cap} \left[F_A \tilde{\setminus} \left(F_A \tilde{\setminus} F_B\right)\right] \neq F_\Phi$ and $F_C \tilde{\cap} \left(F_A \tilde{\setminus} F_B\right) \neq F_\Phi$ for all $F_C \in \tilde{\mathcal{V}}(\alpha)$. Therefore, $\alpha \tilde{\in} \left(F_A \tilde{\setminus} F_B\right)^b$. Hence $F_B^b = \left(F_A \tilde{\setminus} F_B\right)^b$.

iii. Consider the definitions of the soft closure and soft interior;

$$\overline{F}_B \tilde{\setminus} F_B^\circ = \overline{F}_B \tilde{\cap} \left(F_A \tilde{\setminus} F_B^\circ\right) = \overline{F}_B \tilde{\cap} \left(F_A \tilde{\setminus} \left(\tilde{\bigcup_{\substack{F_{B_i} \tilde{\subseteq} F_B \\ F_{B_i} \in \tilde{\tau}}}} F_{B_i} \right) \right)$$

$$= \overline{F}_B \tilde{\cap} \left(\tilde{\bigcap}_{\substack{F_A \tilde{\setminus} F_B \tilde{\subseteq} F_A \tilde{\setminus} F_{B_i} \\ F_{B_i} \tilde{\in} \tilde{\tau}}} \left(F_A \tilde{\setminus} F_{B_i}\right) \right) = \overline{F}_B \tilde{\cap} \overline{\left(F_A \tilde{\setminus} F_B\right)}$$

$$= F_B^b$$

Example 13. *Let's consider* $(F_A, \tilde{\tau}_3)$ *soft topological space in Example 3. Then*

$$F_{A_{12}}^{\circ} = \tilde{\bigcup}\{F_{A_2}, F_{A_5}\} = F_{A_{11}}$$

since

$$\alpha_2 = (x_1, \{u_2\}), \alpha_2 \tilde{\in} F_{A_{11}} \tilde{\subseteq} F_{A_{12}}$$
$$\alpha_4 = (x_2, \{u_2\}), \alpha_4 \tilde{\in} F_{A_{13}} \tilde{\nsubseteq} F_{A_{12}} \text{ and } \alpha_4 \tilde{\in} F_A \tilde{\nsubseteq} F_{A_{12}}$$
$$\alpha_5 = (x_2, \{u_3\}), \alpha_5 \tilde{\in} F_{A_{11}} \tilde{\subseteq} F_{A_{12}}$$

and

$$\overline{F}_{A_{12}} = F_A$$

since F_A *is the smallest soft closed set which is a superset of* $F_{A_{12}}$

and

$$F'_{A_{12}} = \tilde{\bigcup}\{F_{A_1}, F_{A_4}, F_{A_5}\} = F_{A_9}$$

since

$F_{A_{13}} \tilde{\cap} \left(F_{A_{12}} \tilde{\setminus} F_{A_1}\right) = F_{A_{13}} \tilde{\cap} F_{A_{12}} = F_{A_{10}} \neq F_\Phi$ $F_A \tilde{\cap} \left(F_{A_{12}} \tilde{\setminus} F_{A_1}\right) = F_A \tilde{\cap} F_{A_{12}} = F_{A_{12}} \neq F_\Phi$
$F_{A_2} \tilde{\cap} \left(F_{A_{12}} \tilde{\setminus} F_{A_2}\right) = F_{A_2} \tilde{\cap} F_{A_6} = F_\Phi$
$F_{A_{13}} \tilde{\cap} \left(F_{A_{12}} \tilde{\setminus} F_{A_4}\right) = F_{A_{13}} \tilde{\cap} F_{A_{11}} = F_{A_2} \neq F_\Phi$ $F_A \tilde{\cap} \left(F_{A_{12}} \tilde{\setminus} F_{A_4}\right) = F_A \tilde{\cap} F_{A_{11}} = F_{A_{11}} \neq F_\Phi$
$F_{A_{11}} \tilde{\cap} \left(F_{A_{12}} \tilde{\setminus} F_{A_5}\right) = F_{A_{11}} \tilde{\cap} F_{A_{10}} = F_{A_2} \neq F_\Phi$ $F_A \tilde{\cap} \left(F_{A_{12}} \tilde{\setminus} F_{A_5}\right) = F_A \tilde{\cap} F_{A_{10}} = F_{A_{10}} \neq F_\Phi$

and

$$F_{A_{12}}^b = \tilde{\bigcup}\{F_{A_1}, F_{A_4}\} = F_{A_7}$$

since

$F_{A_{13}}, F_A \in \tilde{\mathcal{V}}(\alpha_1)$, $F_{A_{13}} \tilde{\cap} F_{A_{12}} = F_{A_{10}} \neq F_\Phi$ and $F_{A_{13}} \tilde{\cap} \left(F_A \tilde{\setminus} F_{A_{12}}\right) = F_{A_{13}} \tilde{\cap} F_{A_1} = F_{A_1} \neq F_\Phi$

$F_A \tilde{\cap} F_{A_{12}} = F_{A_{12}} \neq F_\Phi$ and $F_A \tilde{\cap} (F_A \tilde{\setminus} F_{A_{12}}) = F_A \tilde{\cap} F_{A_1} = F_{A_1} \neq F_\Phi$
$F_{A_2}, F_{A_{11}}, F_{A_{13}}, F_A \in \tilde{\mathcal{V}}(\alpha_2)$, $F_{A_{11}} \tilde{\cap} F_{A_{12}} = F_{A_{11}} \neq F_\Phi$ and $F_{A_{11}} \tilde{\cap} (F_A \tilde{\setminus} F_{A_{12}}) = F_{A_{11}} \tilde{\cap} F_{A_1} = F_\Phi$
$F_{A_{13}}, F_A \in \tilde{\mathcal{V}}(\alpha_4)$, $F_{A_{13}} \tilde{\cap} F_{A_{12}} = F_{A_{10}} \neq F_\Phi$ and $F_{A_{13}} \tilde{\cap} (F_A \tilde{\setminus} F_{A_{12}}) = F_{A_{13}} \tilde{\cap} F_{A_1} = F_{A_1} \neq F_\Phi$ $F_A \tilde{\cap} F_{A_{12}} = F_{A_{12}} \neq F_\Phi$ and $F_A \tilde{\cap} (F_A \tilde{\setminus} F_{A_{12}}) = F_A \tilde{\cap} F_{A_1} = F_{A_1} \neq F_\Phi$
$F_{A_{11}}, F_A \in \tilde{\mathcal{V}}(\alpha_5)$, $F_{A_{11}} \tilde{\cap} F_{A_{12}} = F_{A_{11}} \neq F_\Phi$ and $F_{A_{11}} \tilde{\cap} (F_A \tilde{\setminus} F_{A_{12}}) = F_{A_{11}} \tilde{\cap} F_{A_1} = F_\Phi$

Definition 20. *Let $(F_A, \tilde{\tau})$ be a soft topological space. If $\forall \alpha_1, \alpha_2 \tilde{\in} F_A$ $(\alpha_1 \neq \alpha_2)$, $\exists F_{B_1} \in \tilde{\mathcal{V}}(\alpha_1)$ and $\exists F_{B_2} \in \tilde{\mathcal{V}}(\alpha_2)$ such that $F_{B_1} \tilde{\cap} F_{B_2} = F_\Phi$, then $(F_A, \tilde{\tau})$ is called a soft Hausdorff space.*

Example 14. *Let's consider $(F_A, \tilde{\tau}_2)$ in Example 3.*

If $\alpha_1 = (x_1, \{u_1\})$ and $\alpha_2 = (x_1, \{u_2\})$, then there exists $F_{A_1} \in \tilde{\mathcal{V}}(\alpha_1)$ and $F_{A_2} \in \tilde{\mathcal{V}}(\alpha_2)$ such that $F_{A_1} \tilde{\cap} F_{A_2} = F_\Phi$.

If $\alpha_1 = (x_1, \{u_1\})$ and $\alpha_4 = (x_2, \{u_2\})$, then there exists $F_{A_1} \in \tilde{\mathcal{V}}(\alpha_1)$ and $F_{A_4} \in \tilde{\mathcal{V}}(\alpha_4)$ such that $F_{A_1} \tilde{\cap} F_{A_4} = F_\Phi$.

If $\alpha_1 = (x_1, \{u_1\})$ and $\alpha_5 = (x_2, \{u_3\})$, then there exists $F_{A_1} \in \tilde{\mathcal{V}}(\alpha_1)$ and $F_{A_5} \in \tilde{\mathcal{V}}(\alpha_5)$ such that $F_{A_1} \tilde{\cap} F_{A_5} = F_\Phi$.

If $\alpha_2 = (x_1, \{u_2\})$ and $\alpha_4 = (x_2, \{u_2\})$, then there exists $F_{A_2} \in \tilde{\mathcal{V}}(\alpha_2)$ and $F_{A_4} \in \tilde{\mathcal{V}}(\alpha_4)$ such that $F_{A_2} \tilde{\cap} F_{A_4} = F_\Phi$.

If $\alpha_2 = (x_1, \{u_2\})$ and $\alpha_5 = (x_2, \{u_3\})$, then there exists $F_{A_2} \in \tilde{\mathcal{V}}(\alpha_2)$ and $F_{A_5} \in \tilde{\mathcal{V}}(\alpha_5)$ such that $F_{A_2} \tilde{\cap} F_{A_5} = F_\Phi$.

If $\alpha_4 = (x_2, \{u_2\})$ and $\alpha_5 = (x_2, \{u_3\})$, then there exists $F_{A_4} \in \tilde{\mathcal{V}}(\alpha_4)$ and $F_{A_5} \in \tilde{\mathcal{V}}(\alpha_5)$ such that $F_{A_4} \tilde{\cap} F_{A_5} = F_\Phi$.

Hence $(F_A, \tilde{\tau}_2)$ is a soft Hausdorff space.

Example 15. *Let's consider two soft single point sets $\alpha_1 = (x_1, \{u_1\})$ and $\alpha_2 = (x_1, \{u_2\})$ of F_A in the soft topological space $(F_A, \tilde{\tau}_3)$. Since there does not exist $F_B \in \tilde{\mathcal{V}}(\alpha_1)$ and $F_C \in \tilde{\mathcal{V}}(\alpha_2)$ such that $F_B \tilde{\cap} F_C = F_\Phi$, $(F_A, \tilde{\tau}_3)$ is not a soft Hausdorff space.*

Theorem 20. *Every soft single point set in a soft Hausdorff space is soft closed.*

Proof. Let $(F_A, \tilde{\tau})$ be a soft Hausdorff space. Let α_1 and α_2 be two soft single points of F_A different from each other, then there exist $\exists F_{B_1} \in \tilde{\mathcal{V}}(\alpha_1)$ and $\exists F_{B_2} \in \tilde{\mathcal{V}}(\alpha_2)$ such that $F_{B_1} \tilde{\cap} F_{B_2} = F_\Phi$. Since

$F_{B_1} \tilde{\cap} \alpha_2 = F_\Phi$, we obtain $\alpha_1 \tilde{\notin} \overline{\alpha}_2$ from the Theorem 15. Therefore, for all $\alpha_1 \neq \alpha_2$, $\alpha_1 \tilde{\notin} \overline{\alpha}_2$. That is, $\overline{\alpha}_2 = \alpha_2$. So α_2 is soft closed from the Theorem 12.

4. Conclusion

We firstly thanks Sanjay ROY for his kindness and for warning us about Theorem 6 (*i.*) in [7] by sending an e-mail in 2011. Thus we realized some conceptual confusions such as the complement of a soft open set and the soft limit point set. Then we have revised the paper. So this concept has been become consistent and fit for further study on its. On the other hand, the remark given by Roy and Samanta [29] for the definition of soft topology given in [7] is not valid. Because

$$\{F_{A_i} \tilde{\subseteq} F_A : i \in I \subseteq \mathbb{N}\} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcup}_{i \in I} F_{A_i} \in \tilde{\tau}$$

means that for all subsets $\{F_{A_i} \tilde{\subseteq} F_A : i \in I \subseteq \mathbb{N}\}$ of $\tilde{\tau}$, $\tilde{\bigcup}_{i \in I} F_{A_i} \in \tilde{\tau}$ is true. That is, $\tilde{\tau}$ is closed under arbitrary soft union.

We finally pose a question "Which model of the soft topology is meaningful more than the other?" Which one is more useful than the type 1 soft topology defined on a soft set by using the soft subsets of it or the type 2 topology defined on a classical set by using the soft sets over its? This fair question is important for the development of the concept of soft sets, and people who want to study on this concept should not ignore this detail.

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