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On Generalized (Ψ,φ)-Almost Weakly Contractive Maps in Generalized Fuzzy Metric Spaces

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Abstract – In this paper, we come out with the approach of generalized (Ψ,φ)-almost weakly contractive maps in the context of generalized fuzzy metric spaces. We prove theorem to show the existence of a fixed point and also provide an example in support to our result.

Keywords – *(Ψ,φ)-almost weakly contractive map, Fuzzy metric space, Generalized fuzzy metric spaces.*

1 Introduction

In Mathematics, the concept of fuzzy set was introduced by Zadeh [15]. It is a new way to represent vagueness in our daily life. In 1975 Kramosil and Michalek [3] introduced the concept of fuzzy metric spaces which opened a new way for further development of analysis in such spaces. George and Veeramani [2] modified the concept of fuzzy metric space. After that several fixed point theorems have been proved in fuzzy metric spaces. In 2008, Dutta and Choudary [8] introduced (Ψ,φ) – weakly contractive maps and showed the existence of fixed points in complete metric spaces. In 2009, Doric [7] unfolded it to a pair of maps by broadening the result that was proposed by Zhang and Song [14] Harjani and Sadarangani [9], Presented some fixed point results in a complete metric space bestowed with a partial order for weakly C-contractive mappings. Saha [12] established a weakened version of contraction mappings principle in fuzzy metric space with a partial ordering. In the present work, we insinuate the concept of (Ψ,φ)-almost weakly contractive maps in the panorama of fuzzy metric spaces and observe few results.

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2 Preliminaries

Definition 2.1. A 3 – tuple (X, $\mathcal{M}$, *) is called generalized fuzzy metric space if X is an arbitrary non – empty set, * is a continuous t – norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions; for each x, y, z, a $\in$ X and t, s > 0

(GFM – 1) $\mathcal{M}(x, y, z, t) > 0$,
(GFM – 2) $\mathcal{M}(x, y, z, t) = 1$, if x = y= z,
(GFM – 3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,
(GFM – 4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
(GFM – 5) $\mathcal{M}(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,
(GFM – 6) $\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1$.

Definition 2.2. If $\{x_n\}$ is a sequence in a generalized fuzzy metric spaces such that $\mathcal{M}(x_n, x, x, t) \to 1$ whenever $n \to \infty$, then $\{x_n\}$ is said to converges to $x \in X$.

(i) A sequence $\{x_n\}$ in X is said to be a converge to a point x in X if and only if for each $\varepsilon > 0$, t > 0 there exists $n_0 \in N$ such that $\mathcal{M}(x_n, x_m, x_m, t) > 1-\varepsilon$ for all $n \geq n_0$.
(ii) A generalized fuzzy metric space (X , $\mathcal{M}$,*) is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.3. Let (X , $\mathcal{M}$, *) be a complete generalized fuzzy metric space. Let C be a subset of X. Let T: C $\to$ C be a self mapping which satisfies the following inequality:

$\Psi(\mathcal{M}(Tx, Ty, Tz, t) \leq \Psi(\mathcal{M}(x, y, z, t)) - \varphi(\mathcal{M}(x, y, z, t)$ where x, y, z $\in$ X, t > 0, Ψ and $\varphi : (0, 1] \to [0, \infty)$ are two functions such that,

(i) Ψ is continuous and monotone decreasing with $\Psi(t) = 0 \Leftrightarrow t = 1$
(ii) φ is continuous with $\varphi(s) = 0 \Leftrightarrow s = 1$

Then T is said to be a weak contraction on C.

Definition 2.4. Let (X , $\mathcal{M}$,*) be a generalized fuzzy metric space. Let there exists $\Psi, \varphi : (0,1] \to [0, \infty)$ such that

(i) Ψ is continuous and monotonically decreasing,
(ii) $\Psi(t) = 0 \Leftrightarrow t= 1$
(iii) φ is continuous with $\varphi(s) = 0 \Leftrightarrow s = 1$

Then T: X $\to$ X be a self map satisfying the inequality:

$\Psi(\mathcal{M}(Tx, Ty, Tz, t) \leq \Psi(\mathcal{M}(x, y, z, t)) - \varphi(\mathcal{M}(x, y, z, t) + L\{1-m(x,y, z)\}$ for all x,y, z $\in$ X, t > 0, $L \geq 0$, where m(x, y, z)= max $\{\mathcal{M}(x, Tx, z, t), \mathcal{M}(x, Ty, Tz, t), \mathcal{M}(y, Ty, Tz, t), \mathcal{M}(Tx, y, z, t)\}$. Then T is said to be a (Ψ, φ) - almost weakly contractive map on X.

3 Main Result

Theorem 3.1. Let $(X, \mathcal{M}, *)$ be a complete generalized fuzzy metric space. Let $T: X \to X$ be a (Ψ, φ)- almost weakly contractive map. Then, T has a fixed point in X which is unique.

Proof: Let $\{x_n\}$ be a sequence in X such that $Tx_n = x_{n+1}$. If $x_n = x_{n+1}$, then the theorem is obvious.If $x_n \neq x_{n+1}$, consider

$$\Psi(\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)) = \Psi(\mathcal{M}(Tx_{n-1}, Tx_n, Tx_n, t)) \leq \Psi\begin{pmatrix}\mathcal{M}(x_{n-1}, x_n, x_n, t)) - \varphi\,(\mathcal{M}\,(x_{n-1}, x_n, x_n, t))\ + \\ L\{1 - m(x_{n-1}, x_n, x_n)\end{pmatrix} \quad (3.1.1)$$

$$\begin{aligned} m(x_{n-1}, x_n, x_n) &= \max\begin{Bmatrix}\mathcal{M}(x_{n-1}, Tx_{n-1}, x_n, t), \mathcal{M}(Tx_{n-1}, Tx_n, Tx_n, t), \\ \mathcal{M}(x_n, Tx_n, Tx_n, t), \mathcal{M}(Tx_{n-1}, x_n, x_n, t),\end{Bmatrix} \\ &= \max\begin{Bmatrix}\mathcal{M}(x_{n-1}, x_n, x_n, t), \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t), \\ \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t), \mathcal{M}(x_{n+1}, x_n, x_n, t)\}\end{Bmatrix} \\ &= \max\{\mathcal{M}(x_{n-1}, x_n, x_n, t), 1, \mathcal{M}(x_{n-1}, x_{n+1}, x_{n+1}, t), \mathcal{M}(x_n, x_{n+1}, x_n, t)\} \\ &= 1 \end{aligned}$$

$$m(x_{n-1}, x_n, x_n) = 1 \quad (3.1.2)$$

from (3.1.1) and (3.1.2) , we get that

$$\Psi(\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)) \leq \Psi(\mathcal{M}(x_{n-1}, x_n, x_n, t)) - \varphi\,(\mathcal{M}(x_{n-1}, x_n, x_n, t)) \quad (3.1.3)$$

$$\Psi(\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)) < \Psi(\mathcal{M}(x_{n-1}, x_n, x_n, t)) \quad (3.1.4)$$

We know Ψ is monotonically decreasing $\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) > \mathcal{M}(x_{n-1}, x_n, x_n, t)$ (3.1.5)

$\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)\}$ is an increasing sequence of non-negative real numbers.

Let $\lim_{n\to\infty}\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) = r$ then taking limit as $n \to \infty$ in (3.1.3)

$\Longrightarrow \psi(r) \leq \psi(r) - \varphi(r)$
$\Longrightarrow \varphi(r) \leq 0 \Longrightarrow \varphi(r) = 0$.
$\Leftrightarrow r = 1$ (from definition (2.4)

Therefore $\lim_{n\to\infty}\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) = 1$. (3.1.6)

To prove that $\{x_n\}$is a Cauchy sequence.

Let $\{x_n\}$ is not Cauchy, then, for any given $\varepsilon > 0$, we can find subsequences $\{x_{n_k}\}$,$\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k$ such that

$$\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) \leq 1 - \varepsilon \quad (3.1.7)$$

then, we have

$\mathcal{M}(x_{n_{k-1}}, x_{m_k}, x_{m_k}, t) > 1 - \varepsilon, \quad \mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) > 1 - \varepsilon.$ (3.1.8)

Consider

$1 - \varepsilon \geq \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)$
$1 - \varepsilon \geq \lim_{k\to\infty} \sup \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)$ (3.1.9)

$$\begin{aligned}\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) &\geq \mathcal{M}\left(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}, \frac{t}{2}\right) * \mathcal{M}\left(x_{n_{k-1}}, x_{m_k}, x_{m_k}, \frac{t}{2}\right)\\ &> \mathcal{M}\left(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}, \frac{t}{2}\right) * 1-\varepsilon \text{ (from (3.1.8))}\\ &> 1 * 1-\varepsilon \text{ as } k \to \infty \quad \text{(from (3.1.6))}\\ &\Rightarrow \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) > 1-\varepsilon \end{aligned} \quad (3.1.10)$$

Therefore

$\lim_{k\to\infty} \inf \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) > 1-\varepsilon,$ (3.1.11)

from (3.1.9) and (3..11) we see that

$1-\varepsilon < \lim_{k\to\infty} \inf \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) \leq \lim_{k\to\infty} \sup \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) < 1-\varepsilon$

$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)$ exists and is equal to 1- ε
$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) = 1-\varepsilon.$ (3.1.12)

Consider

$$\begin{aligned}\Psi(\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t)) &= \Psi(\mathcal{M}(Tx_{n_{k-1}}, Tx_{m_{k-1}}, Tx_{m_{k-1}}, t))\\ &\leq \Psi((\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) -\\ &\quad \varphi(\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) +\\ &\quad L\{1-m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t)\})\end{aligned} \quad (3.1.13)$$

from definition (2.4) , (3.1.8), and since we know that Ψ is a decreasing function, we have

$\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) > 1-\varepsilon \Rightarrow \Psi(\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) \leq \Psi(1-\varepsilon).$ (3.1.14)

Since φ is continuous, we have

$\mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) > 1-\varepsilon \Rightarrow \varphi\, \mathcal{M}(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) \geq \varphi(1-\varepsilon)$ (3.1.15)

also,

$$m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t) = max\begin{Bmatrix}\mathcal{M}(x_{n_{k-1}}, Tx_{n_{k-1}}, x_{m_{k-1}}, t),\\ \mathcal{M}(x_{n_{k-1}}, Tx_{m_{k-1}}, Tx_{m_{k-1}}, t),\\ \mathcal{M}(x_{m_{k-1}}, Tx_{m_{k-1}}, Tx_{m_{k-1}}, t),\\ \mathcal{M}(Tx_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}, t)\end{Bmatrix} \quad (3.1.16)$$

$$= \max \begin{Bmatrix} \mathcal{M}(x_{n_{k-1}}, x_{n_k}, x_{m_{k-1}}, t), \\ \mathcal{M}(x_{n_{k-1}}, x_{m_k}, x_{m_k}, t), \\ \mathcal{M}(x_{m_{k-1}}, x_{m_k}, x_{m_k}, t), \\ \mathcal{M}(x_{n_k}, x_{m_{k-1}}, x_{m_k}, t) \end{Bmatrix}$$

Therefore

$$m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}) \to 1 \text{ as } k \to \infty. \quad (3.1.17)$$

Using (3.1.12), (3.1.14), (3.1.15), and (3.1.17) , equation (3.1.13) becomes

$$\Psi(\mathcal{M}(x_{n_k}, x_{m_k}, x_{m_k}, t) \le \Psi(1-\varepsilon) - \varphi(1-\varepsilon) + L\{1 - m(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}})\}.$$

Since, X is complete, we can find a, $z \in X$ such that the sequence $\{x_n\}$ is convergent to z as $n \to \infty$. To prove z is a fixed point of T in X.

$$\begin{aligned} \Psi(\mathcal{M}(x_n, Tz, Tz, t) &= \Psi(\mathcal{M}(Tx_{n-1}, Tz, Tz, t) \\ &\le \Psi(\mathcal{M}(x_{n-1}, z, z, t)) - \varphi(\mathcal{M}(x_{n-1}, z, z, t)) + \\ &\quad L\{1 - m\{(x_{n-1}, z, z)\} \end{aligned} \quad (3.1.18)$$

Where, $m(x_{n-1}, z, z) = \max \begin{Bmatrix} \mathcal{M}(x_{n-1}\ z, z, t), \mathcal{M}(Tx_{n-1}, x_{n-1}, x_{n-1}, t), \\ \mathcal{M}(Tx_{n-1}, z, z, t), \mathcal{M}(Tz, x_{n-1}, x_{n-1}, t), \mathcal{M}(Tz, z, z, t) \end{Bmatrix}$

as $n \to \infty$, (3.1.18) becomes

$$\Psi(\mathcal{M}(z, Tz, Tz, t) \le \Psi(\mathcal{M}(z, z, z, t) - \varphi(\mathcal{M}(z, z., z, t)) + L\{1-1\} = \Psi(1) - \varphi(1) = 0.$$

Therefore, $\Psi(\mathcal{M}(z, Tz, Tz, t) = 0 \Rightarrow \mathcal{M}(z, Tz, Tz, t) = 1$

Thus, $Tz = z \Rightarrow$ z is a fixed point of T in X.

To prove z is unique. If possible, let z, w be two fixed point of T in X, then

$$\begin{aligned} \Psi(\mathcal{M}(z, w, w, t) \le \Psi(\mathcal{M}(Tz, Tw, Tw, t)) &\le \Psi((\mathcal{M}(z, w, w, t) - \varphi(\mathcal{M}(z, w., w, t)) \\ &\quad + L\{1 - m(z, w, w)\}) \\ &= \Psi(\mathcal{M}(z, w, w, t) - \varphi(\mathcal{M}(z, w, w, t)) + L\{0\}. \ \text{(since } m(z, w, w) = 1) \end{aligned}$$

Therefore $\mathcal{M}(z, w, w, t) = 1$ which implies z = w, That is fixed point is unique.

Example 3.2. Let X= [0, 1] and * be the continuous t-norm defined by

$a * b = ab$. $\mathcal{M}(x, y, z, t) = \begin{Bmatrix} 1, & if \ \ either\ x = 0 \ \ or \ \ y = 0\ or\ z = 0 \\ \frac{\min\{x,y,z\}}{\max\{x,y,z\}} & if \ \ x \ne 0 \ \ , y \ne 0\ and\ z \ne 0 \end{Bmatrix}$

Then, clearly $(X, \mathcal{M}, *)$ is a complete generalized fuzzy metric space.

Let T: X→ X be defined by $Tx = \begin{cases} 0 & if\ x = \frac{1}{2} \\ 1 & if\ x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \end{cases}$.

Let ψ and φ on (0, 1] be defined by ψ(s) = 1- s^2 and φ(s) = 1-s. Here, T satisfies the inequality (3.1.8) with any L ≥ 0. Therefore T is a (Ψ, φ) - almost weakly contractive map on X. Thus, T satisfies all the hypothesis of Theorem 3.1 and so, have a unique fixed point in X i.e., at x = 1.

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