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Certain Subclasses of Bi-Univalent Functions Associated with Generalized Mersenne Polynomials

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Highlights:
- A new subclass of bi-univalent functions related to generalized
- Mersenne polynomials is introduced
- Fekete–Szegö type inequalities for this subclass are obtained

Keywords:
- Bi-univalent functions
- Mersenne polynomials
- Fekete–Szegö inequality
- Coefficient estimates
- Analytic functions

ABSTRACT:

The study of bi-univalent functions, whose defining property requires both the function and its inverse to be univalent in the unit disk, has received considerable attention due to its intrinsic analytical complexity. In this paper, a novel subclass of analytic and bi-univalent functions is formulated in relation to generalized Mersenne polynomials. These polynomials are employed as an effective tool for constructing operators related to this class of functions. The main focus of the paper is to obtain coefficient estimates for the Taylor–Maclaurin expansion, with particular emphasis on the initial coefficients. Using subordination techniques and standard methods from geometric function theory, several Fekete–Szegö type inequalities are derived. In addition, a number of special cases and related results are discussed to establish connections with previously known subclasses. The results presented here extend and unify various earlier coefficient inequalities in the literature and highlight the applicability of Mersenne polynomials in coefficient problems for bi-univalent function classes.

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INTRODUCTION

Let $\mathcal{A}$ be the set of analytic functions defined in the open unit disk $\mathbb{E} = \{m \in \mathbb{C} : |m| < 1\}$ that fulfill the normalization conditions $l(0) = 0$ and $l'(0) = 1$. With these conditions, each function $l \in \mathcal{A}$ can be expressed by a Taylor–Maclaurin series expansion given by

$$l(m) = m + \sum_{n=2}^{\infty} a_n m^n, \quad (m \in \mathbb{E}) \tag{1}$$

Furthermore, we denote by S the class of all functions belonging to $\mathcal{A}$ which are univalent in $\mathbb{E}$. Consider the functions l and g defined and analytic in $\mathbb{E}$. The function l is called subordinate to g, and this relationship is denoted by

$l(m) \prec g(m)$, if there exists an analytic function ω defined in $\mathbb{E}$ such that

$\omega(0) = 0$ and $|\omega(m)| < 1$, $(m \in \mathbb{E})$,

and the following condition holds:

$l(m) = g(\omega(m))$, $(m \in \mathbb{E})$.

Moreover, if the function g is univalent in $\mathbb{E}$, the following equivalence is satisfied:

$l(m) \prec g(m) \Leftrightarrow l(0) = g(0)$ and $l(\mathbb{E}) \subset g(\mathbb{E})$.

For every function $l \in S$, there exists inverse l^{-1} with the property that

$$l^{-1}(l(m)) = m, \quad (m \in \mathbb{E})$$

and

$$l(l^{-1}(\omega)) = \omega, \ (|\omega| < r_0(f)), \ r_0(f) \geq \frac{1}{4}.$$

If the inverse function l^{-1} can be analytically continued to the whole unit disk $\mathbb{E}$, then l is defined as bi-univalent in $\mathbb{E}$.

Suppose that both l and l^{-1} are univalent in $\mathbb{E}$. Under this assumption, define

$$g(\omega) = l^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 + \cdots \tag{2}$$

and introduce Σ to represent the family of bi-univalent mappings defined on $\mathbb{E}$ as described in (1). Notable instances of functions belonging to Σ include

$$l_1(m) = \frac{m}{m-1}, \quad l_2(m) = \frac{1}{2}\log\frac{1+m}{1-m}, \quad l_3(m) = -\log(1-m),$$

without altering the general properties of the class.

Even though the Koebe function is excluded from Σ, other frequently encountered analytic functions within $\mathbb{E}$, such as

$$\frac{2m-m^2}{2} \text{ and } \frac{m}{1-m^2}$$

do not belong to the class Σ. The class of bi-univalent functions was studied by Lewin (1967), who established the estimate $|a_2| < 1.51$. This result was later refined through the conjecture of Brannan and Clunie (1980), asserting that $|a_2| < \sqrt{2}$. In contrast, Netanyahu (1969) determined the sharp bound $\max_{f \in \Sigma}|a_2| = 4/3$. The problem of obtaining precise bounds for the Taylor-Maclaurin coefficients $|a_n|$, for $n \in \mathbb{N}$ with $n \geq 3$, remains unsolved.

In (Lewin, 1967), by means of Loewner's method, the Fekete-Szegö inequality for the coefficients of $l \in S$ is that

$$|a_3 - \mu a_2^2| \leq 1 + 2exp\left(\frac{-2\mu}{1-\mu}\right)\ for\ \ 0 \leq \mu < 1.$$

As $\mu \to 1^{-}$, the elementary inequality $|a^3 - a_2^2| \leq 1$ is obtained. Moreover, the coefficient functional

$$F_\mu(l) = a_3 - \mu a_2^2$$

In geometric function theory, normalized analytic functions l in the open unit disk $\mathbb{E}$ occupy a fundamental position. The problem concerned with maximizing the modulus of the functional $F_\mu(l)$ is known as the Fekete-Szegö problem.

The Fekete–Szegö inequalities, originally introduced by Fekete and Szegö (1933), have constituted an important subject of investigation within several subclasses of univalent functions, as documented in Dziok (2013), Kanas (2012), Malik, Mahmood, Raza, Farman, and Zainab (2018), and Wanas and Cotîrla (2021). In light of these developments, related forms of these inequalities were obtained for bi-univalent functions, and later investigations indicate that this line of research continues to produce noteworthy results (Al-Hawary, Amourah, & Frasin, 2021; Amourah, Frasin, & Abdeljaward, 2021; Zaprawa, 2014; Aktaş & Hamarat, 2023; Aktaş & Karaman, 2023; Orhan, Aktaş, & Arıkan, 2023; Öztürk & Aktaş, 2024).

Ma (1992), a systematic treatment of certain subclasses of starlike and convex functions was formulated through subordination principles. In this framework, the expressions $\frac{ml'(m)}{l(m)}$ and $1 + \frac{ml''(m)}{l'(m)}$ are required to be subordinate to an analytic function ϕ satisfying the normalization conditions $\phi(0) = 1$, $\phi'(0) > 0$. The function ϕ maps the domain $\mathbb{E}$ onto a region that is starlike with respect to the point 1 and symmetric about the real axis. Under these assumptions, the subclasses

$$S_\phi^* = \left\{l \in \mathcal{A}: \frac{ml'(m)}{l(m)} \prec \phi(m), \quad m \in \mathbb{E}\right\}$$

and

$$\mathcal{C}_\phi = \left\{l \in \mathcal{A}: 1 + \frac{ml''(m)}{l'(m)} \prec \phi(m), \quad m \in \mathbb{E}\right\}$$

were defined. For the particular choice $\phi(m) = \frac{1+m}{1-m}$, these classes reduce to well-known fundamental subclasses of starlike and convex functions. In addition, a considerable number of investigations have been devoted to functions belonging to the class Σ (Al-Rawashdeh, 2024; Adegani, Jafari, Bulboacă, et al., 2023; Frasin & Aouf, 2011; Khan, Srivastava, Tahir, et al., 2021; Murugusundaramoorthy, Magesh, Prameela, et al., 2013; Sharma, Sivasubramanian, & Cho, 2023; Srivastava, Eker, & Ali, 2015; Aarthy & Srutha Keerthi, 2025).

Building upon the classical subclasses $S^*(\rho)$ and $\mathcal{K}(\rho)$, which correspond to starlike and convex functions of order ρ with $0 \leq \rho \leq 1$, **Brannan and Taha (1986)** (see also **Taha, 1981**) introduced analogous subclasses within the framework of the bi-univalent function class Σ. These subclasses, namely $S_\Sigma^*(\rho)$ and $\mathcal{K}_\Sigma(\rho)$, describe bi-starlike and bi-convex functions of order ρ respectively. Moreover, they obtained coefficient bounds for the initial Taylor–Maclaurin terms $|a_2|$ and $|a_3|$; however, the results were not shown to be sharp.

MATERIALS AND METHODS

Mersenne Polynomials

Mersenne numbers M_n are sequences of integers of the form $2^n - 1$ for non-negative integers n. Kumari, Tanti, and Prasad (2023) introduced the polynomial version of the Mersenne numbers defined by the recurrence relation

$$M_n(x) = 3M_{n-1}(x) - 2M_{n-2}(x),$$

with the initial conditions

$M_0(x) = 0$ and $M_1(x) = 1$.

Kumari, Prasad, and Tanti (2024) generalized the Mersenne polynomial, by allowing arbitrary initial values, which are defined as follows:

Definition 1. (Kumari, Prasad, & Tanti, 2024). The Generalized Mersenne polynomials $\{\mathcal{W}_n(x)\}_{n\geq 0}$ are defined by the recurrence relation,

$$\mathcal{W}_n(x) = 3x\mathcal{W}_{n-1}(x) - 2\mathcal{W}_{n-2}(x), \quad n \geq 2 \tag{3}$$

Let a, b and c be arbitrary real or complex constants, not simultaneously zero. The sequence is initiated by

$$\mathcal{W}_0(x) = a, \tag{4}$$

$$\mathcal{W}_1(x) = bx + c. \tag{5}$$

Lemma 1. (Kumari, Prasad, & Tanti, 2024). An explicit representation of the ordinary generating function of the generalized Mersenne polynomials $\{\mathcal{W}_n(x)\}_{n\geq 0}$ takes the form

$$\mathfrak{F}(m) = \frac{a + m[(b-3a)x + c]}{1 - 3mx + 2m^2}. \tag{6}$$

Investigations into Some New Families of Bi-univalent Functions

This section is devoted to the introduction of two new subclasses of holomorphic and bi-univalent functions within the framework of the class Σ, formulated by means of generalized Mersenne polynomials. Among these, the first subclass is the family of bi-starlike functions, denoted by $S_M^*(x)$, whose definition is given below.

Definition 2. Let $l(m) \in \Sigma$ with representation (1). The function is said to be in the class $S_M^*(x)$ whenever the conditions stated below are fulfilled:

$$\frac{2ml'(m)}{l(m) - l(-m)} \prec \mathcal{W}(x, m) \tag{7}$$

and

$$\frac{2\omega g'(\omega)}{g(\omega) - g(-\omega)} \prec \mathcal{W}(x, \omega) \tag{8}$$

where $m, \omega \in \mathbb{E}$, and g denotes the inverse of l and has the form given in (2).

The second subclass considered in this study is the family of bi-convex functions, denoted by $\mathcal{C}_M(x)$, which is defined as follows.

Definition 3. Let $l(m) \in \Sigma$ be represented in the form (1). The function l is said to belong to the class $\mathcal{C}_M(x)$ if it satisfies the following subordination conditions:

$$\frac{2[ml'(m)]'}{[l(m)-l(-m)]'} \prec \mathcal{W}(x,m) \tag{9}$$

and

$$\frac{2[\omega g'(\omega)]'}{[g(\omega)-g(-\omega)]'} \prec \mathcal{W}(x,\omega) \tag{10}$$

Here $m,\omega \in \mathbb{E}$, and g denotes the inverse of l and has the form given in (2).

In this paper, our primary objective is to establish sharp upper bounds for the Taylor–Maclaurin coefficients of functions in the subclasses $S_M^*(x)$ and $\mathcal{C}_M(x)$. A comprehensive account of the class Σ can be found in the pioneering work of **Srivastava, Mishra, and Gochhayat (2010)**.

RESULTS AND DISCUSSION

Coefficient Inequalities Associated with the Class $S_M^*(x)$

In the present section, initial coefficient bounds are established for functions in the corresponding subclass $S_M^*(x)$.

Theorem 1. Let $l(m) \in S_M^*(x)$ be a function of the form (1). Then the initial Taylor–Maclaurin coefficients satisfy the following inequalities:

$$|a_2| \leq \frac{|bx+c|\sqrt{|bx+c|}}{\sqrt{|2(bx+c)^2-4[3x(bx+c)-2a]|}} \tag{11}$$

and

$$|a_3| \leq \frac{|bx+c|}{2}+\frac{(bx+c)^2}{4} \tag{12}$$

where a, b and c are the initial parameters of the generalized Mersenne polynomials defined in Definition 1, provided that $(bx+c)^2 \neq 2[3x(bx+c)-2a]$.

Proof. Let the function $l(m) \in S_M^*(x)$ and $g = l^{-1}$ given by (2). By Definition 2, the subordinations (7) and (8) imply that

$$\frac{2ml'(m)}{l(m)-l(-m)} = \mathcal{W}\big(x,\tau(m)\big) \tag{13}$$

and

$$\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)} = \mathcal{W}(x,\varphi(\omega)) \tag{14}$$

Here $\tau(m) = k_1 m + k_2 m^2 + \cdots$ and $\varphi(\omega) = \varphi_1\omega + \varphi_2\omega^2 + \cdots$ are Schwarz functions satisfying $\tau(0) = \varphi(0) = 0$, $|\tau(m)| < 1$ and $|\varphi(\omega)| < 1$ for all $m, \omega \in \mathbb{E}$. Moreover, it is well known that these conditions $|\tau(m)| < 1$ and $|\varphi(\omega)| < 1$ yield the following results.

$$|\tau_j| \leq 1 \tag{15}$$

and

$$|\varphi_j| \leq 1 \tag{16}$$

for all $j \in \mathbb{N}$. Basic computations yield that

$$\begin{aligned}\mathcal{W}(x,\tau(m)) = {} & \mathcal{W}_0(x) + [\mathcal{W}_1(x)k_1]m + [\mathcal{W}_1(x)k_2 + \mathcal{W}_2(x)k_1^2]m^2 \\ & + [\mathcal{W}_1(x)k_3 + 2\mathcal{W}_2(x)k_1k_2 + \mathcal{W}_3(x)k_1^3]m^3 + \cdots\end{aligned} \tag{17}$$

and

$$\begin{aligned}\mathcal{W}(x,\varphi(\omega)) = {} & \mathcal{W}_0(x) + [\mathcal{W}_1(x)\varphi_1]\omega + [\mathcal{W}_1(x)\varphi_2 + \mathcal{W}_2(x)\varphi_1^2]\omega^2 \\ & + [\mathcal{W}_1(x)\varphi_3 + 2\mathcal{W}_2(x)\varphi_1\varphi_2 + \mathcal{W}_3(x)\varphi_1^3]\omega^3 + \cdots\end{aligned} \tag{18}$$

By performing direct computations, the following series representations are obtained:

$$\frac{2ml'(m)}{l(m) - l(-m)} = 1 + 2a_2m + 2a_3m^2 + 4(a_4 - a_2a_3)m^3 + \cdots \tag{19}$$

and

$$\frac{2\omega g'(\omega)}{g(\omega) - g(-\omega)} = 1 - 2a_2\omega + (4a_2^2 - 2a_3)\omega^2 + (4a_4 - 2a_2a_3)\omega^3 \ldots \tag{20}$$

A term-by-term comparison of the coefficients in (19) and (20) with those given in (17) and (18), respectively, yields the following relations,

$$2a_2 = \mathcal{W}_1(x)k_1, \tag{21}$$

$$-2a_2 = \mathcal{W}_1(x)\varphi_1, \tag{22}$$

$$2a_3 = \mathcal{W}_1(x)k_2 + \mathcal{W}_2(x)k_1^2 \tag{23}$$

$$4a_2^2 - 2a_3 = \mathcal{W}_1(x)\varphi_2 + \mathcal{W}_2(x)\varphi_1^2 \tag{24}$$

Now, from equations (21) and (22) we get

$$k_1 = -\varphi_1, \tag{25}$$

and

$$\frac{8a_2^2}{[\mathcal{W}_1(x)]^2} = k_1^2 + \varphi_1^2 \tag{26}$$

Furthermore, by adding equations (23) and (24), one obtains

$$4a_2^2 = \mathcal{W}_1(x)(k_2 + \varphi_2) + \mathcal{W}_2(x)(k_1^2 + \varphi_1^2), \tag{27}$$

By substituting equation (26) in equation (27) we get

$$a_2^2 = \frac{[\mathcal{W}_1(x)]^3(k_2 + \varphi_2)}{4(\mathcal{W}_1(x))^2 - 8\mathcal{W}_2(x)} \tag{28}$$

By applying the recurrence relation stated in Definition 1 (see equation (3)), the following expressions are obtained:

$$\mathcal{W}_1(x) = bx + c, \tag{29}$$

$$\mathcal{W}_2(x) = 3x(bx + c) - 2a. \tag{30}$$

If we replace equations (29) and (30) in (28) we get

$$a_2^2 = \frac{(bx+c)^3(k_2+\varphi_2)}{4(bx+c)^2 - 8(3x(bx+c)-2a)}. \tag{31}$$

By means of the triangle inequality and inequalities (15) and (16), the following estimate is derived.

$$|a_2|^2 \leq \frac{|bx+c|^3}{|2(bx+c)^2 - 4(3x(bx+c)-2a)|}. \tag{32}$$

Upon taking the square root of both sides of the above inequality, relation (11) follows.

Moreover, by subtracting (24) from (23) and employing (25), one arrives at

$$a_3 = \frac{\mathcal{W}_1(x)(k_2-\varphi_2)}{4} + a_2^2. \tag{33}$$

By substituting relation (26) into equation (33) and carrying out routine computations, we obtain

$$a_3 = \frac{\mathcal{W}_1(x)(k_2-\varphi_2)}{4} + \frac{[\mathcal{W}_1(x)]^2(k_1^2+\varphi_1^2)}{8} \tag{34}$$

Furthermore, applying equation (29) together with the triangle inequality and inequalities (15) and (16) to relation (34) leads to the desired estimate stated in (12).

Fekete-Szegö Type Results for the Class $S_M^*(x)$

By applying the result of Zaprawa (2014), the Fekete–Szegö inequality for $S_M^*(x)$ is established.

Theorem 2. Suppose that the function $l(m) \in S_M^*(x)$ and has the form (1), Then, provided that $(bx+c)^2 \neq 4[3x(bx+c)-2a]$.

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|bx+c|}{2}, & if \quad |1-\mu| \leq \frac{(bx+c)^2 - 2|3x(bx+c)-2a|}{(bx+c)^2}, \\ \frac{|bx+c|^3|1-\mu|}{2|(bx+c)^2 - 4(3x(bx+c)-2a)|}, & if \quad |1-\mu| \geq \frac{(bx+c)^2 - 2|3x(bx+c)-2a|}{(bx+c)^2}, \end{cases} \tag{35}$$

Proof. Suppose that $l(m) \in S_M^*(x)$ is given by (1), from (28) and (33), we have

$$\begin{aligned} a_3 - \mu {a_2}^2 &= \frac{\mathcal{W}_1(x)(k_2-\varphi_2)}{4} + a_2^2 - \mu a_2^2 \\ &= \frac{\mathcal{W}_1(x)(k_2-\varphi_2)}{4} + (1-\ \mu)a_2^2 \\ &= \frac{\mathcal{W}_1(x)(k_2-\varphi_2)}{4} + \frac{(1-\ \mu)[\mathcal{W}_1(x)]^3(k_2+\varphi_2)}{4\left(\mathcal{W}_1(x)\right)^2 - 8\mathcal{W}_2(x)} \\ &= \mathcal{W}_1(x)\left\{\left(h_1(\mu)+\frac{1}{4}\right)k_2 + \left(h_1(\mu)-\frac{1}{4}\right)\varphi_2\right\}, \end{aligned} \tag{36}$$

where $h_1(\mu) = \frac{(1-\ \mu)[\mathcal{W}_1(x)]^2}{4\left(\mathcal{W}_1(x)\right)^2 - 8\mathcal{W}_2(x)}$. Finally, by taking the modulus and applying the triangle inequality along with relations (15), (16), (29), and (30) to equation (36), the proof is thus completed.

Coefficient Estimates for the Class $\mathcal{C}_M(x)$

This section is devoted to deriving estimates for the first coefficients of functions that are members of the subclass $\mathcal{C}_M(x)$.

Theorem 3. Suppose that the function $l(m) \in \mathcal{C}_M(x)$ and has the form (1), Then

$$|a_2| \leq \frac{|bx+c|\sqrt{|bx+c|}}{\sqrt{|6(bx+c)^2 - 16[3x(bx+c) - 2a]|}} \tag{37}$$

and

$$|a_3| \leq \frac{|bx+c|}{6} + \frac{(bx+c)^2}{16} \tag{38}$$

where a, b and c are the initial parameters of the generalized Mersenne polynomials defined in Definition 1, provided that $3(bx+c)^2 \neq 8[3x(bx+c) - 2a]$.

Proof. Let the function $l(m) \in \mathcal{C}_M(x)$ and $g = l^{-1}$ given by (2). By Definition 3, the subordinations (9) and (10) imply that

$$\frac{2[ml'(m)]'}{[l(m) - l(-m)]'} = \mathcal{W}\big(x, p(m)\big) \tag{39}$$

and

$$\frac{2[\omega g'(\omega)]'}{[g(\omega) - g(-\omega)]'} = \mathcal{W}(x, d(\omega)). \tag{40}$$

Let $p(m) = p_1 m + p_2 m^2 + \cdots$ and $d(\omega) = d_1\omega + d_2\omega^2 + \cdots$ be Schwarz functions satisfying $p(0) = d(0) = 0$, $|p(m)| < 1$ and $|d(\omega)| < 1$ for all $m, \omega \in \mathbb{E}$. It is a well-established fact that the conditions $|p(m)| < 1$ and $|d(\omega)| < 1$ imply

$$|p_j| \leq 1 \tag{41}$$

and

$$|d_j| \leq 1 \tag{42}$$

for all $j \in \mathbb{N}$. By performing straightforward computations, one obtains

$$\begin{aligned}\mathcal{W}\big(x, p(m)\big) = {} & \mathcal{W}_0(x) + [\mathcal{W}_1(x)p_1]m + [\mathcal{W}_1(x)p_2 + \mathcal{W}_2(x)p_1^2]m^2 \\ & + [\mathcal{W}_1(x)p_3 + 2\mathcal{W}_2(x)p_1p_2 + \mathcal{W}_3(x)p_1^3]m^3 + \cdots\end{aligned} \tag{43}$$

and

$$\begin{aligned}\mathcal{W}\big(x, d(\omega)\big) = {} & \mathcal{W}_0(x) + [\mathcal{W}_1(x)d_1]\omega + [\mathcal{W}_1(x)d_2 + \mathcal{W}_2(x)d_1^2]\omega^2 \\ & + [\mathcal{W}_1(x)d_3 + 2\mathcal{W}_2(x)d_1d_2 + \mathcal{W}_3(x)d_1^3]\omega^3 + \cdots\end{aligned} \tag{44}$$

On the other hand, by performing straightforward computations, one can express

$$\frac{2[ml'(m)]'}{[l(m) - l(-m)]'} = 1 + 4a_2 m + 6a_3 m^2 + 4(4a_4 - 3a_2a_3)m^3 + \cdots \tag{45}$$

and

$$\frac{2[\omega g'(\omega)]'}{[g(\omega)-g(-\omega)]'}=1-4a_2\omega+(12a_2^2-6a_3)\omega^2+\cdots \tag{46}$$

A term-by-term comparison of the coefficients in (45) and (46) with those in equations (43) and (44), respectively, yields

$$4a_2=\mathcal{W}_1(x)p_1, \tag{47}$$

$$-4a_2=\mathcal{W}_1(x)d_1, \tag{48}$$

$$6a_3=\mathcal{W}_1(x)p_2+\mathcal{W}_2(x)p_1^2 \tag{49}$$

$$12a_2^2-6a_3=\mathcal{W}_1(x)d_2+\mathcal{W}_2(x)d_1^2 \tag{50}$$

Now, from equations (47) and (48) we get

$$p_1=-d_1, \tag{51}$$

and

$$\frac{32a_2^2}{[\mathcal{W}_1(x)]^2}=p_1^2+d_1^2 \tag{52}$$

Also, by summing equations (49) and (50), we readily obtain

$$12a_2^2=\mathcal{W}_1(x)(p_2+d_2)+\mathcal{W}_2(x)(p_1^2+d_1^2), \tag{53}$$

By substituting equation (52) in equation (53) we get

$$a_2^2=\frac{[\mathcal{W}_1(x)]^3(p_2+d_2)}{12(\mathcal{W}_1(x))^2-32\mathcal{W}_2(x)} \tag{54}$$

Furthermore, by applying the recurrence relation outlined in Definition 1 (see equation (3)), the following expressions are obtained:

$$\mathcal{W}_1(x)=bx+c, \tag{55}$$

$$\mathcal{W}_2(x)=3x(bx+c)-2a. \tag{56}$$

If we replace equations (55) and (56) in (54) we get

$$a_2^2=\frac{(bx+c)^3(p_2+d_2)}{12(bx+c)^2-32(3x(bx+c)-2a)}. \tag{57}$$

Now, by applying the triangle inequality to the inequalities (41) and (42), we obtain

$$|a_2|^2\le\frac{|bx+c|^3}{|6(bx+c)^2-16[3x(bx+c)-2a]|}. \tag{58}$$

Taking the square root both sides of the last inequality, we have (37).

Furthermore, by subtracting equation (50) from equation (49) and taking equation (51) into account, we find

$$a_3=\frac{\mathcal{W}_1(x)(p_2-d_2)}{12}+{a_2}^2. \tag{59}$$

Considering the equation (52) in (59) and performing straightforward computations gives

$$a_3 = \frac{\mathcal{W}_1(x)(p_2 - d_2)}{12} + \frac{[\mathcal{W}_1(x)]^2(p_1^2 + d_1^2)}{32} \tag{60}$$

Finally, by employing equation (55) together with the triangle inequality applied to (41) and (42) in (60), we deduce the desired inequality (38).

On a Fekete–Szegö Type Inequality Associated with the Class $\mathcal{C}_M(x)$

A Fekete–Szegö type estimate for functions in the class $\mathcal{C}_M(x)$ is presented in the next result.

Theorem 4. Suppose that the function $l(m) \in \mathcal{C}_M(x)$ and has the form (1), Then, provided that $3(bx+c)^2 \neq 8[3x(bx+c) - 2a]$.

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|bx+c|}{6}, & if \quad |1-\mu| \leq \frac{3(bx+c)^2 - 8|3x(bx+c)-2a|}{3(bx+c)^2}, \\ \frac{|bx+c|^3|1-\mu|}{4|3(bx+c)^2 - 8(3x(bx+c)-2a)|}, & if \quad |1-\mu| \geq \frac{3(bx+c)^2 - 8|3x(bx+c)-2a|}{3(bx+c)^2}, \end{cases} \tag{61}$$

Proof. Suppose that $l(m) \in \mathcal{C}_M(x)$ is given by (1), from (54) and (59), we have

$$\begin{aligned} a_3 - \mu {a_2}^2 &= \frac{\mathcal{W}_1(x)(p_2 - d_2)}{12} + a_2^2 - \mu a_2^2 \\ &= \frac{\mathcal{W}_1(x)(p_2 - d_2)}{12} + (1-\ \mu)a_2^2 \\ &= \frac{\mathcal{W}_1(x)(p_2 - d_2))}{12} + \frac{(1-\ \mu)[\mathcal{W}_1(x)]^3(p_2 + d_2)}{12\big(\mathcal{W}_1(x)\big)^2 - 32\mathcal{W}_2(x)} \\ &= \mathcal{W}_1(x)\left\{\left(h_2(\mu) + \frac{1}{12}\right)k_2 + \left(h_2(\mu) - \frac{1}{12}\right)\varphi_2\right\}, \end{aligned} \tag{62}$$

where $h_2(\mu) = \frac{(1-\ \mu)[\mathcal{W}_1(x)]^2}{12(\mathcal{W}_1(x))^2 - 32\mathcal{W}_2(x)}$. By taking the modulus and invoking the triangle inequality together with (41), (42), (55) and (56) in (62), the proof is completed.

CONCLUSION

In the present study, we have introduced and investigated two novel subclasses of the bi-univalent function class Σ associated with generalized Mersenne polynomials, namely the bi-starlike function class $S_M^*(x)$ and the bi-convex function class $\mathcal{C}_M(x)$. Using subordination techniques and standard methods from geometric function theory, we derived explicit estimates for the first Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ associated with functions in these subclasses. Furthermore, the Fekete–Szegö inequalities have been derived for both classes, providing a general framework that encompasses previously known results. Several special cases and corollaries were also examined to demonstrate the connections with earlier studied subclasses defined via alternative polynomial frameworks. These findings highlight the effectiveness of generalized Mersenne polynomials in addressing coefficient problems in the theory of bi-univalent functions and extend the analytical tools available for further investigations in this area.

Conflict of Interest

There is no conflict of interest.

Author's Contributions

The author contributed to all stages of the study.

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