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Timelike Factorable Surfaces in Minkowski Space-Time

Sezgin Büyükkütük*

ABSTRACT

In this study, we discuss timelike factorable surfaces in Minkowski 4 – space IE_1^4 . We calculate Gaussian and mean curvatures of these surfaces and classify timelike flat and minimal factorable surfaces in Minkowski space-time.

Keywords: factorable surface, timelike surface, Minkowski 4 – space

1. INTRODUCTION

The Minkowski space defined by Lorentzian inner product is the mathematical structure in which Einstein's special relativity theory is the most appropriately represented. Since the inner product is not always positively defined, curves and surfaces vary in this space. Spacelike vectors, curves and surfaces show similarity to the Euclidean space structure.

In n – dimensional semi-Euclidean space, the Lorentzian inner product with t – index is defined by

$$g(X, Y) = -\sum_{i=1}^t x_i y_i + \sum_{j=t+1}^n x_j y_j \quad (1)$$

where $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ [1]. The semi-Euclidean space defined by this metric is denoted by IE_t^n . Especially, for 4 – dimensional case with index $t = 1$, the semi-Euclidean space IE_1^4 is called Minkowski space-time. In this case, the Lorentzian metric (1) is expressed in the form of

$$g(X, Y) = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4. \quad (2)$$

Any arbitrary vector $X = (x_1, \dots, x_4)$ is called timelike, null or spacelike if the Lorentzian inner product g is negative definite, zero or positive definite, respectively. Then, the length of the vector $X \in IE_1^4$ is calculated by

$$\|X\| = \sqrt{|g(X, X)|} \quad (3)$$

where $X = (x_1, \dots, x_4) \in IE_1^4$.

Let S be a surface in four-dimensional Minkowski space IE_1^4 . Then, the surface is called timelike if the induced metric g on S is a metric with index 1. Minkowski 4 – space can be written by the direct sum of the tangent space and the normal space of S at each point p :

$$IE_1^4 = T_p S \oplus T_p^\perp S \quad (4)$$

Represented by $\tilde{\nabla}$ and ∇ is the Levi-Civita connections on IE_1^4 and S , respectively. Let X_1 and X_2 indicate the tangent vector fields and let ξ indicates the normal vector field on S . Then, Gauss and Weingarten formulas are given by the followings:

$$\begin{aligned} \tilde{\nabla}_{X_1} X_2 &= \nabla_{X_1} X_2 + h(X_1, X_2), \\ \tilde{\nabla}_{X_1} \xi &= -A_\xi X_1 + D_{X_1} \xi, \end{aligned} \quad (5)$$

where h , D and A_ξ are the second fundamental tensor, the normal connection and the shape operator with regard to ξ , respectively [2].

Let S be a timelike surface in IE_1^4 given by the parameterization $F(u, v)$, $(u, v) \in U$ ($U \in IE^2$) and

* Corresponding Author: Department of Mathematics, Kocaeli University, Kocaeli, Turkey, sezginbuyukkutuk@gmail.com

$T_p S = \text{span}\{F_u, F_v\}$ is the tangent space at any point $p = F(u, v)$. In the chart (u, v) the coefficients of the first fundamental form of S are given by

$$E = g(F_u, F_u), \quad F = g(F_u, F_v), \quad G = g(F_v, F_v). \quad (6)$$

We suppose that $g(F_u, F_u) < 0$, $g(F_v, F_v) > 0$. Thus, $E < 0, G > 0$ and for the later use we set $W = \sqrt{F^2 - EG}$. We choose an orthonormal frame field $\{\xi_1, \xi_2\}$ of the normal bundle, i.e., $g(\xi_1, \xi_1) = 1$, $g(\xi_2, \xi_2) = 1$. Therefore, we give the following derivative formulas:

$$\begin{aligned} \tilde{\nabla}_{F_u} F_u &= F_{uu} = -\Gamma_{11}^1 F_u + \Gamma_{11}^2 F_v + c_{11}^1 \xi_1 + c_{11}^2 \xi_2, \\ \tilde{\nabla}_{F_u} F_v &= F_{uv} = -\Gamma_{12}^1 F_u + \Gamma_{12}^2 F_v + c_{12}^1 \xi_1 + c_{12}^2 \xi_2, \\ \tilde{\nabla}_{F_v} F_v &= F_{vv} = -\Gamma_{22}^1 F_u + \Gamma_{22}^2 F_v + c_{22}^1 \xi_1 + c_{22}^2 \xi_2, \end{aligned} \quad (7)$$

where Γ_{ij}^k are the Christoffel's symbols and the functions $c_{ij}^k, i, j, k = 1, 2$ are given by

$$\begin{aligned} c_{11}^1 &= g(F_{uu}, \xi_1), & c_{11}^2 &= g(F_{uu}, \xi_2), \\ c_{12}^1 &= g(F_{uv}, \xi_1), & c_{12}^2 &= g(F_{uv}, \xi_2), \\ c_{22}^1 &= g(F_{vv}, \xi_1), & c_{22}^2 &= g(F_{vv}, \xi_2). \end{aligned} \quad (8)$$

The second fundamental form of the timelike surface $S : F(u, v)$ is expressed as follows:

$$\begin{aligned} h(F_u, F_u) &= c_{11}^1 \xi_1 + c_{11}^2 \xi_2, \\ h(F_u, F_v) &= c_{12}^1 \xi_1 + c_{12}^2 \xi_2, \\ h(F_v, F_v) &= c_{22}^1 \xi_1 + c_{22}^2 \xi_2, \end{aligned} \quad (9)$$

where F_u and F_v are tangent vectors.

By the use of Gram-Schmidt orthonormalization method, we can obtain the orthonormal tangent vectors as:

$$\begin{aligned} X_1 &= \frac{F_u}{\sqrt{|E|}}, \\ X_2 &= \frac{\sqrt{|E|}}{W} \left(F_v - \frac{F}{E} F_u \right). \end{aligned} \quad (10)$$

Hence, with the help of the orthonormal tangent vectors, the second fundamental form can be written as

$$\begin{aligned} h(X_1, X_1) &= h_{11}^1 \xi_1 + h_{11}^2 \xi_2, \\ h(X_1, X_2) &= h_{12}^1 \xi_1 + h_{12}^2 \xi_2, \\ h(X_2, X_2) &= h_{22}^1 \xi_1 + h_{22}^2 \xi_2 \end{aligned} \quad (11)$$

where the functions $h_{ij}^k, i, j, k = 1, 2$ are given by

$$\begin{aligned} h_{11}^k &= -\frac{c_{11}^k}{E}, \\ h_{12}^k &= \frac{E c_{12}^k - F c_{11}^k}{EW}, \\ h_{22}^k &= -\frac{E^2 c_{22}^k - 2E F c_{12}^k + F^2 c_{11}^k}{EW^2}. \end{aligned} \quad (12)$$

These coefficients are entries of shape operator matrices. Gaussian curvature of a timelike surface $S : F(u, v)$ by using the second fundamental form coefficients is defined by

$$K = \sum_{k=1}^2 h_{11}^k h_{22}^k - (h_{12}^k)^2, \quad (13)$$

(see, [3]).

The mean curvature vector field can be calculated by

$$\begin{aligned} H &= \frac{1}{2} \text{tr} h. \text{ Therefore, if } S \text{ is a timelike surface, then} \\ \text{the mean curvature vector field is} \\ H &= \frac{1}{2} (-h(X_1, X_1) + h(X_2, X_2)), \end{aligned} \quad (14)$$

where $\{X_1, X_2\}$ is a local orthonormal frame of the tangent bundle such that $g(X_1, X_1) = -1$, $g(X_2, X_2) = 1$ [2].

Any surface is said to be flat (minimal), if its Gaussian curvature (mean curvature vector) vanishes [4].

Factorable surfaces (also known homothetical surfaces) in IE^3 can be parameterized, locally, as $F(u, v) = (u, v, f(u)g(v))$, where f and g are smooth functions [5, 6]. Some authors have considered factorable surfaces in Euclidean space and in semi-Euclidean spaces [6, 7, 8, 9, 10, 11, 12]. In [5], Van de Woestyne proved that the only minimal factorable non-degenerate surfaces in L^3 are planes and helicoids.

In [13, 14], the authors gave the surface parametrization as

$$F(u, v) = (u, v, z(u, v), w(u, v)) \quad (15)$$

and called it Monge patch in IE^4 . Furthermore, in [14], the authors characterized this surface and gave some examples. Also, some surfaces and curves in four dimensional spaces can be found in [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

In the present study, we consider timelike factorable surfaces in Minkowski 4-space. We characterize

such surfaces in terms of their Gaussian curvature and mean curvatures and give the conditions for such surfaces to become flat and minimal.

2. TIMELIKE FACTORABLE SURFACES IN \mathbb{IE}_1^4

Definition 1: Let $S \subset \mathbb{IE}_1^4$ be a surface in 4-dimensional Minkowski space. If in (15), we take

$$z = f_1(u)g_1(v), \quad w = f_2(u)g_2(v),$$

where f_1, f_2, g_1, g_2 are differentiable functions in \mathbb{IE}_1^4 , then we can define a surface parameterization (Monge patch) which is called factorable surface in Minkowski 4-space.

Therefore, the parameterization of the factorable surface can be written as

$$F(u, v) = (u, v, f_1(u)g_1(v), f_2(u)g_2(v)). \quad (16)$$

Let S be a timelike factorable surface given by the parameterization (16) in \mathbb{IE}_1^4 . Then, the tangent space of the surface is spanned by the vector fields

$$F_u = \frac{\partial F(u, v)}{\partial u} = (1, 0, f_1'(u)g_1(v), f_2'(u)g_2(v)),$$

$$F_v = \frac{\partial F(u, v)}{\partial v} = (0, 1, f_1(u)g_1'(v), f_2(u)g_2'(v)). \quad (17)$$

The first fundamental form coefficients are obtained as

$$E = g(F_u, F_u) = -1 + (f_1'g_1)^2 + (f_2'g_2)^2,$$

$$F = g(F_u, F_v) = f_1'f_1g_1'g_1 + f_2'f_2g_2'g_2, \quad (18)$$

$$G = g(F_v, F_v) = 1 + (f_1g_1')^2 + (f_2g_2')^2.$$

Here, we suppose $E < 0$, namely the surface S is timelike, and so $EG - F^2 < 0$.

The second derivatives of $F(u, v)$ are

$$F_{uu} = (0, 0, f_1''(u)g_1(v), f_2''(u)g_2(v)),$$

$$F_{uv} = (0, 0, f_1'(u)g_1'(v), f_2'(u)g_2'(v)), \quad (19)$$

$$F_{vv} = (0, 0, f_1(u)g_1''(v), f_2(u)g_2''(v)).$$

Normal space of the timelike surface S is spanned by the orthonormal vector fields

$$\xi_1 = \frac{1}{\sqrt{A}}(f_1'g_1, -f_1g_1', 1, 0),$$

$$\xi_2 = \frac{1}{\sqrt{AD}} \begin{pmatrix} Af_2'g_2 - Bf_1'g_1, \\ Bf_1g_1' - Af_2g_2', -B, A \end{pmatrix}, \quad (20)$$

where

$$A = 1 - (f_1'g_1)^2 + (f_1g_1')^2,$$

$$B = -f_1'f_2'g_1g_2 + f_1f_2g_1'g_2',$$

$$C = 1 - (f_2'g_2)^2 + (f_2g_2')^2, \quad (21)$$

$$D = AC - B^2.$$

Since S is timelike surface in \mathbb{IE}_1^4 with respect to chosen orthonormal frame, A and D are positive definite. By the use of the equations (8), (19), and (20), we calculate the second fundamental form coefficients

$$c_{11}^1 = \frac{f_1''g_1}{\sqrt{A}}, \quad c_{22}^1 = \frac{f_1g_1''}{\sqrt{A}},$$

$$c_{12}^1 = \frac{f_1'g_1'}{\sqrt{A}}, \quad c_{12}^2 = \frac{Af_2'g_2' - Bf_1'g_1'}{\sqrt{AD}},$$

$$c_{11}^2 = \frac{Af_2''g_2 - Bf_1''g_1}{\sqrt{AD}}, \quad (22)$$

$$c_{22}^2 = \frac{Af_2g_2'' - Bf_1g_1''}{\sqrt{AD}}.$$

With the help of (12) and (22), the shape operator matrices are

$$\begin{bmatrix} h_{11}^1 & h_{12}^1 \\ h_{12}^1 & h_{22}^1 \end{bmatrix}, \quad \begin{bmatrix} h_{11}^2 & h_{12}^2 \\ h_{12}^2 & h_{22}^2 \end{bmatrix},$$

where the functions h_{ij}^k are given by

$$h_{11}^1 = \frac{-f_1''g_1}{E\sqrt{A}}, \quad h_{12}^1 = \frac{f_1'g_1'E - f_1''g_1F}{EW\sqrt{A}},$$

$$h_{22}^1 = -\frac{f_1g_1''E^2 - 2f_1'g_1'EF + f_1''g_1F^2}{EW^2\sqrt{A}},$$

$$h_{11}^2 = -\frac{Af_2''g_2 - Bf_1''g_1}{E\sqrt{AD}}, \quad (23)$$

$$h_{12}^2 = \frac{\left(f_2' g_2' A - f_1' g_1' B \right) E - \left(f_2'' g_2 A - f_1'' g_1 B \right) F}{EW \sqrt{AD}},$$

$$h_{22}^2 = - \frac{\left(\begin{array}{l} \left(f_2 g_2'' A - f_1 g_1'' B \right) E^2 \\ - 2 \left(f_2' g_2' A - f_1' g_1' B \right) EF \\ + \left(f_2'' g_2 A - f_1'' g_1 B \right) F^2 \end{array} \right)}{EW^2 \sqrt{AD}}.$$

2.1. Timelike flat factorable surfaces

Theorem 1. Let S be a timelike factorable surface with the parametrization (16) in IE_1^4 . Then, its Gaussian curvature is given by

$$K = \frac{\left(\begin{array}{l} \left(f_1'' f_1 g_1'' g_1 - f_1'^2 g_1'^2 \right) C \\ - \left(\begin{array}{l} f_1'' f_2 g_1 g_2'' \\ + f_1 f_2'' g_1'' g_2 - 2 f_1' f_2' g_1' g_2' \end{array} \right) B \\ + \left(f_2'' f_2 g_2'' g_2 - f_2'^2 g_2'^2 \right) A \end{array} \right)}{DW^2}. \quad (24)$$

Proof. By the use of the equations (13) and (23), we obtain the desired result.

Theorem 2. Let S be a timelike factorable surface with the parameterization (16) in IE_1^4 . If S is given by one of the following parameterizations, then it is a flat surface:

- (1) $F(u, v) = (u, v, a_1 g_1(v), a_2 g_2(v)),$
- (2) $F(u, v) = (u, v, b_1 f_1(u), b_2 f_2(u)),$
- (3) $F(u, v) = (u, v, a_1 g_1(v), a_2 f_2(u)),$
- (4) $F(u, v) = (u, v, b_1 f_1(u), b_2 g_2(v)),$
- (5) $F(u, v) = \left(u, v, a_1 b_1, \exp(a_2 u + b_2) \exp(a_3 v + b_3) \right),$
- (6) $F(u, v) = \left(u, v, a_1 b_1, (a_2 u + b_2)^{\frac{1}{1-\lambda}} (a_3 v + b_3)^{\frac{\lambda}{\lambda-1}} \right),$

$$(7) F(u, v) = \left(u, v, \exp(a_1 u + b_1) \exp(a_2 v + b_2), \exp(a_3 u + b_3) \exp(a_3 \frac{a_1}{a_j} v + b_4) \right),$$

$$(8) F(u, v) = (u, v, f_1(u) \cos v, f_1(u) \sin v),$$

the function $f_1(u)$ satisfies

$$u = \pm \int \sqrt{\frac{a_1 f_1^2(u) + 1}{f_1^2(u) + 1}} df_1(u)$$

where $i, j = 1, 2, i \neq j$ and $a_k, b_k, k = 1, \dots, 4$ are real constants.

Proof. Let S be a timelike factorable surface given with the parameterization (16) in IE_1^4 . If $f_1'(u) = 0, f_2'(u) = 0$ or $g_1'(v) = 0, g_2'(v) = 0$ or $f_1'(u) = 0, g_2'(v) = 0$ ($f_2'(u) = 0, g_1'(v) = 0$),

then we obtain the cases (1), (2), (3) and (4).

If $f_1'(u) = 0, g_1'(v) = 0$, then we have

$$f_2'' f_2 g_2'' g_2 - f_2'^2 g_2'^2 = 0. \quad (25)$$

Let $p(u) = \frac{df_2}{du}$ and $q(v) = \frac{dg_2}{dv}$. By the use of (25), we can write

$$f_2(u) p(u) \frac{dp}{df_2} g_2(v) q(v) \frac{dq}{dg_2} - (p(u) q(v))^2 = 0, \quad (26)$$

If $p(u) \neq 0, q(v) \neq 0$, from (26), we get

$$f_2(u) \frac{dp}{df_2} g_2(v) \frac{dq}{dg_2} = p(u) q(v).$$

Then we have differential equation

$$\frac{f_2(u) \frac{dp}{df_2}}{p(u)} = \frac{q(v)}{g_2(v) \frac{dq}{dg_2}} = \lambda, \quad (27)$$

where λ is constant.

(1) If $\lambda = 1$, from (27) we have

$$\begin{aligned} f_2(u) &= \exp(a_2 u + b_2), \\ g_2(v) &= \exp(a_3 v + b_3), \end{aligned} \quad (28)$$

which gives the case (5).

(2) If $\lambda \neq 1$, from (27) we have

$$f_2(u) = (a_2 u + b_2)^{\frac{1}{1-\lambda}}, \quad (29)$$

$$g_2(v) = (a_3 v + b_3)^{\frac{\lambda}{\lambda-1}},$$

which gives the case (6).

Further, we assume $f_i'' f_i g_i'' - f_i' g_i'^2 = 0$ holds for $i = 1$ and $i = 2$. Then we get

$$f_1(u) = \exp(a_1 u + b_1), \quad g_1(v) = \exp(a_2 v + b_2), \quad (30)$$

$$f_2(u) = \exp(a_3 u + b_3), \quad g_2(v) = \exp(a_4 v + b_4),$$

Substituting these functions into $B = 0$ and $f_1'' f_2 g_1 g_2'' + f_1 f_2'' g_1'' g_2 - 2f_1' f_2' g_1' g_2' = 0$, we

have $a_4 = \frac{a_3 a_i}{a_j}$, $i, j = 1, 2$ ($i \neq j$) which vanish

Gaussian curvature of the surface. Thus, we obtain the case (7).

Also, if $f_1(u) = f_2(u)$ and $g_1(v) = \cos v$, $g_2(v) = \sin v$, then we get

$$-f_1''(u)f_1(u)(f_1'(u)^2 + 1) + (f_1'(u))^2((f_1'(u))^2 - 1) = 0$$

which gives the case (8).

Example 1. The surface with the parameterization

$$F(u, v) = \left(\begin{array}{l} u, v, \exp(2u + 3) \exp(3v + 4), \\ \exp(u + 1) \exp\left(\frac{2}{3}v + 2\right) \end{array} \right) \quad (31)$$

is timelike flat factorable surface in IE_1^4 and one can plot its projection to 3-dimensional space with the help of the maple command

$$\text{plot3d}([u + v, z, w], u = a..b, v = c..d). \quad (32)$$

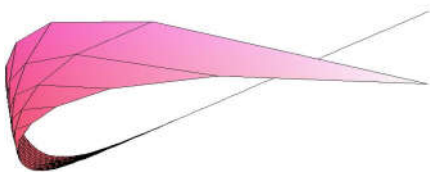


Figure 1 Projection of timelike flat factorable surface given by (31)

2.2. Timelike minimal factorable surfaces

Theorem 2. Let S be a timelike factorable surface with the parameterization (16) in IE_1^4 . Then its mean curvature vector is given by

$$\bar{H} = -\frac{f_1'' g_1 G + f_1 g_1'' E - 2f_1' g_1' F}{2\sqrt{AW^2}} \xi_1 - \frac{\left(\begin{array}{l} A(f_2'' g_2 G + f_2 g_2'' E - 2f_2' g_2' F) \\ -B(f_1'' g_1 G + f_1 g_1'' E - 2f_1' g_1' F) \end{array} \right)}{2\sqrt{ADW^2}} \xi_2. \quad (33)$$

Proof. By the use of the equations (11), (14) and (23), we obtain the desired result.

Theorem 3. Let S be a timelike factorable surface with the parameterization (16) in IE_1^4 . Then S is a minimal surface if and only if

$$f_i'' g_i G + f_i g_i'' E - 2f_i' g_i' F = 0, \quad i=1,2 \quad (34)$$

Proof. Let S be a factorable surface in IE_1^4 . Since we can write the mean curvature vector as $H = -H_1 \xi_1 - H_2 \xi_2$, for a minimal surface $H_1 = 0, H_2 = 0$. With reference to the previous theorem, we get (34). The converse statement is trivial.

Theorem 4. Let S be a timelike factorable surface with the parametrization (16) in IE_1^4 . If S is given by one of the following parameterizations, then it is minimal:

$$(1) F(u, v) = (u, v, (a_1 u + a_2) b_1, (a_3 u + a_4) b_2),$$

$$(2) F(u, v) = (u, v, a_1 (b_1 v + b_2), a_2 (b_3 v + b_4)),$$

$$(3) F(u, v) = (u, v, (a_1 u + a_2) b_1, a_3 (b_3 v + b_4)),$$

$$(4) F(u, v) = \left(\begin{array}{l} u, v, a_1 b_1, \\ (u + a_2) \frac{-1 - \exp(b_2 v + b_3)}{-1 + \exp(b_2 v + b_3)} \end{array} \right),$$

$$(5) F(u, v) = (u, v, a_1 b_1, \tan(a_2 u + a_3)(v + b_2)),$$

$$(6) F(u, v) = \left(\begin{array}{l} u, v, \\ \frac{-1 - a_1^2 + \exp(\pm 2a_1(a_1 u + a_2))}{2a_1 \exp(\pm 2a_1(a_1 u + a_2))} \cos v, \\ \frac{-1 - a_1^2 + \exp(\pm 2a_1(a_1 u + a_2))}{2a_1 \exp(\pm 2a_1(a_1 u + a_2))} \sin v \end{array} \right),$$

$$(7) F(u, v) = \left(\begin{array}{l} u, v, (u + a_1) \frac{-1 - \exp(b_1 v + b_2)}{-1 + \exp(b_1 v + b_2)}, \\ (u + a_1) \frac{-1 - \exp(b_1 v + b_2)}{-1 + \exp(b_1 v + b_2)} \end{array} \right),$$

$$(8) F(u, v) = \left(\begin{array}{l} u, v, \tan(a_1 u + a_2)(v + b_1), \\ \tan(a_1 u + a_2)(v + b_1) \end{array} \right),$$

$$(9) F(u, v) = (u, v, a_1 b_1, f_2(u)g_2(v)),$$

$$(10) F(u, v) = (u, v, f_1(u)g_1(v), f_1(u)g_1(v)),$$

the functions $f_i(u), g_i(v), i = 1, 2$ satisfy the equations

$$u = \int \frac{df_i(u)}{\sqrt{2m \ln f_i(u) + a_1}}, v = \int \frac{dg_i(v)}{\sqrt{a_2 g_i^4(v) - \frac{n}{2}}},$$

$$u = \int \frac{df_i(u)}{\sqrt{a_1 f_i^4(u) - \frac{m}{2}}}, v = \int \frac{dg_i(v)}{\sqrt{2n \ln g_i(v) + a_2}},$$

$$u = \int \frac{df_i(u)}{\sqrt{a_1 f_i^{2(1+c)}(u) - a_2}}, v = \int \frac{dg_i(v)}{\sqrt{a_3 g_i^{2(1-c)}(v) - a_4}},$$

where $c, m, n, a_k, b_k, k = 1, \dots, 4$ are real constants and $c \neq 1$.

Proof. Let S be a timelike factorable surface with the parameterization (16) in IE_1^4 . By the use of (34) with (18), we get,

$$\begin{aligned} & f_i'' g_i \left(1 + f_1'^2 g_1'^2 + f_2'^2 g_2'^2 \right) \\ & + f_i g_i'' \left(-1 + f_1'^2 g_1'^2 + f_2'^2 g_2'^2 \right) \\ & - 2f_i' g_i' \left(f_1' f_1 g_1' g_1 + f_2' f_2 g_2' g_2 \right) = 0, \quad i = 1, 2. \end{aligned} \quad (35)$$

If $g_1'(v) = 0, g_2'(v) = 0$ or $f_1'(u) = 0, f_2'(u) = 0$, we obtain the cases (1) and (2), respectively.

If $f_i'(u) = 0, g_j'(v) = 0, i, j = 1, 2, i \neq j$, then

$$f_1'' g_1 \left(1 + f_1'^2 g_1'^2 + f_2'^2 g_2'^2 \right) = 0, \quad (36)$$

$$f_2 g_2'' \left(-1 + f_1'^2 g_1'^2 + f_2'^2 g_2'^2 \right) = 0. \quad (37)$$

Since $E < 0$ and $G > 0$, then we get $f_1''(u) = 0, g_1'(v) = 0$ and $g_2''(v) = 0, f_2'(u) = 0$ which congruent the case (3).

If $f_1'(u) = 0, g_1'(v) = 0$, from the equality (35) for $i=2$, we get

$$\begin{aligned} & \frac{f_2''(u)}{f_2(u)} - \frac{g_2''(v)}{g_2(v)} \\ & + \left(f_2''(u) f_2(u) - f_2'^2(u) \right) g_2'^2(v) \\ & + \left(g_2''(v) g_2(v) - g_2'^2(v) \right) f_2'^2(u) = 0 \end{aligned} \quad (38)$$

If $f_2''(u) = 0$ or $g_2''(v) = 0$ in (38), we obtain the cases (4) and (5).

If $f_2''(u) g_2''(v) \neq 0$ in (38), differentiating (38) with respect to u and v , we have

$$\frac{\left(f_2''(u) f_2(u) \right)' - \left(-f_2'^2(u) \right)'}{\left(f_2'^2(u) \right)'} = - \frac{\left(g_2''(v) g_2(v) \right)' - \left(-g_2'^2(v) \right)'}{\left(g_2'^2(v) \right)'} = c \quad (39)$$

Thus, we can write

$$\begin{aligned} & f_2''(u) f_2(u) - (1+c) f_2'^2(u) = m, \\ & g_2''(v) g_2(v) - (1-c) g_2'^2(v) = n. \end{aligned} \quad (40)$$

If $c = 1, c = -1$, and $c \neq \pm 1$, then from the solution of (39), we obtain the case (9).

If $f_1(u) = f_2(u)$ and $g_1(v) = \cos v, g_2(v) = \sin v$, then we get

$$f_i''(u)(1 + f_i'^2(u)) - f_i(u)\left(-1 + f_i'^2(u)\right) = 0.$$

By the solution of this differential equation we obtain the case (6).

If $f_1(u) = f_2(u)$, $g_1(v) = g_2(v)$ in (35), then for $i = 1$ or $i = 2$, we find

$$\begin{aligned} & \frac{f_i''(u)}{f_i(u)} - \frac{g_i''(v)}{g_i(v)} \\ & + \left(f_i''(u)f_i(u) - f_i'^2(u) \right) 2g_i'^2(v) \\ & + \left(g_i''(v)g_i(v) - g_i'^2(v) \right) 2f_i'^2(u) = 0. \end{aligned} \quad (41)$$

If $f_i''(u) = 0$ or $g_i''(v) = 0$ in (40), we obtain the cases (7) and (8). Also, if $f_i''(u)g_i''(v) \neq 0$, we obtain the case (10), which completes the proof.

REFERENCES

- [1] B.Y. Chen and J. Van der Veken, "Marginally trapped surfaces in Lorentzian space forms with positive relative nullity," *Class. Quantum Grav.*, vol. 24, pp. 551–563, 2007.
- [2] G. Ganchev, "Timelike surfaces with zero mean curvature in Minkowski 4 – space," *Israel J. of Math.*, vol. 196, pp. 413–433, 2013.
- [3] B. Bektaş, U. Dursun, "Timelike rotational surface of elliptic, hyperbolic and parabolic types in Minkowski space IE_1^4 with pointwise 1 – type Gauss map," *Filomat*, vol. 29, no. 3, pp. 381–392, 2015.
- [4] B. Y. Chen, "Geometry of Submanifolds," *Marcel Dekker*, New York, 1973.
- [5] I. Van de Woestyne, "A new characterization of helicoids," *Geometry and Topology of Submanifolds V.*, World Sci. Publ. River Edge, NJ, 1993.
- [6] I. Van de Woestyne, "Minimal homothetical hypersurfaces of a semi-Euclidean space," *Results Math*, vol. 27, no. 3, pp. 333–342, 1995.
- [7] R. Lopez, M. Moruz, "Translation and homothetical surfaces in Euclidean spaces with constant curvature," *J. Korean Math. Soc.*, vol. 52, no. 3, pp. 523–535, 2015.
- [8] H. Meng, H. Liu, "Factorable surfaces in 3 – Minkowski space," *Bull. Korean Math. Soc.*, vol. 46, no. 1, pp. 155–169, 2009.
- [9] Y. Yu, H. Liu, "The factorable minimal surfaces," *Proceedings of The Eleventh International Workshop on Diff. Geom.*, vol. 11, pp. 33–39, 2007.
- [10] S. Büyükkütük, "A characterization of factorable surfaces," *PhD Thesis*, Kocaeli University, Kocaeli, Turkey, 2018.
- [11] S. Büyükkütük, G. Öztürk, "A characterization of factorable surfaces in Euclidean 4-space IE^4 ," *Koc. J. Sci. Eng.*, vol. 1, no. 1, pp. 15–20, 2018.
- [12] S. Büyükkütük, G. Öztürk, "Spacelike factorable surfaces in four-dimensional Minkowski space," *Bulletin of Mathematical Analysis and Applications*, vol. 9, no. 4, pp. 12–20, 2017.
- [13] Y. A. Aminov, "Surfaces in IE^4 with a Gaussian curvature coinciding with a Gaussian torsion up to sign," *Mathematical Notes*, vol. 56, pp. 5–6, 1994.
- [14] B. Bulca, K. Arslan, "Surfaces given with the Monge patch in IE^4 ," *Journal of Mathematical Physics, Analysis, Geometry*, vol. 9, no. 4, pp. 435–447, 2013.
- [15] K. Arslan, B. K. Bayram, B. Bulca, G. Öztürk, "Generalized rotational surfaces in IE^4 ," *Results in Mathematics*, vol. 61, no. 3-4, pp. 315–327, 2012.
- [16] K. Arslan, B. K. Bayram, B. Bulca, Y.H. Kim, C. Murathan, G. Öztürk, "Rotational embeddings in IE^4 with pointwise 1-type Gauss map," *Turk J. Math.*, vol. 35, pp. 493–499, 2011.
- [17] K. Arslan, B. Bulca, B. K. Bayram, Y. H. Kim, C. Murathan, G. Öztürk, "Tensor product surfaces with pointwise 1-type Gauss map," *Bull. Korean Math. Soc.*, vol. 48, no. 3, pp. 601–609, 2011.
- [18] K. Arslan, B. K. Bayram, B. Bulca, Y.H. Kim, C. Murathan, G. Öztürk, "Vranceanu surface in IE^4 with pointwise 1-type Gauss map," *Indian J. Pure Appl. Math.*, vol. 42, no. 1, pp. 41–51, 2011.
- [19] E. İyigün, K. Arslan, G. Öztürk, "A characterization of Chen surfaces in IE^4 ," *Bull. Malays. Math. Soc.*, vol. 31, no. 2, pp. 209–215, 2008.
- [20] İ. Kişi, G. Öztürk, "A new approach to canal surface with parallel transport frame," *International Journal of Geometric Methods in Modern Physics.*, vol. 14, no. 2, 2017.
- [21] G. Öztürk, İ. Kişi, S. Büyükkütük "Constant ratio quaternionic curves in Euclidean spaces," *Advances in Applied Clifford Algebras*, vol. 27, no. 2, pp. 1659–1673, 2017.

- [22] İ. Kişi, S. Büyükkütük, Deepmala, G. Öztürk, "AW(k)-type curves according to parallel transport frame in Euclidean space IE^4 ," *Facta Universitatis, Series: Mathematics and Informatics.*, vol. 31, no. 4, pp. 885-905, 2016.
- [23] K. Arslan, B. K. Bayram, B. Bulca, G. Öztürk, "On translation surfaces in 4-dimensional Euclidean space," *Acta et Commentationes Universitatis Tartuensis de Mathematica*, vol. 20, no. 2, pp. 123-133, 2016.
- [24] B. Bulca, K. Arslan, B. Bayram, G. Öztürk, "Spherical product surface in IE^4 ," *Analele Stiintifice ale Universtatii Ovidius Constanta.*, vol. 20, no. 1, pp. 41-54, 2012.
- [25] B. Bayram, B. Bulca, K. Arslan, G. Öztürk, "Superconformal ruled surfaces in IE^4 ," *Mathematical Communications.*, vol. 14, pp. 235-244, 2009.
- [26] B. Bulca, K. Arslan, B. Bayram, G. Öztürk, "Canal surfaces in 4-dimensional Euclidean space," *An International Journal of Optimization and Control Theories & Applications.*, vol. 7, no. 1, pp. 83-89, 2017.
- [27] G. Öztürk, S. Gürpınar, K. Arslan, "A new characterization of curves in Euclidean 4-space IE^4 ," *Buletinul Academici de Ştiinte a Republicii Moldova Matematica*, vol. 83, no. 1, pp. 39-50, 2017.
- [28] G. Öztürk, B. Bulca, B. K. Bayram, K. Arslan, "Meridean surfaces of Weingarten type in 4-dimensional Euclidean space IE^4 ," *Konuralp Journal of Mathematicss.*, vol. 4, no. 1, pp. 239-245, 2016.