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On a Class of Recursive Sequence

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Abstract In this paper, we study the following type of a recursive sequence

$$x(n+1)=\frac{x(n-pk+1)}{1+\prod_{j=1}^{p-1}x(n-jk+1)},\, n=0,1,2,...,$$

where p ≥ 2, k ≥ 1 are fixed integers, with the initial values x(n) > 0 for n = -pk+1,-pk+2,…,0. Our results generalize some results in the literature. We give illustrating examples of which solutions are calculated and plotted by the MatLab programming.

Keywords *Convergence, pk periodic solution, recursive sequence.*

Azalan Dizilerin Bir Sınıfı Üzerine

Özet Bu çalışmada, n = -pk+1,-pk+2,…,0 için x(n) > 0 başlangıç şartı ile p ≥ 2, k ≥ 1 sabit tamsayıları için aşağıdaki azalan dizilerin bir türünü

$$x(n+1)=\frac{x(n-pk+1)}{1+\prod_{j=1}^{p-1}x(n-jk+1)},\, n=0,1,2,...,$$

çalıştık. Bizim sonuçlarımız literatürdeki bazı sonuçların genelleştirilmesidir. Matlab proğramı tarafından örneklerin çözümlerinin hesaplanmasını ve çizimini gösterdik.

Anahtar Kelimeler: *Yakınsaklık, pk peryotlu çözümler, azalan diziler.*

INTRODUCTION

Difference equations are always attracting very much interest, because these equations appear in the mathematical models of some problems in biology, ecology and physics, and numerical solutions of differential equations (see [9]).

Recently there has been a lot of studies on the periodic nature of nonlinear difference equations. We refer readers to [1, 6, 7], for some recent results concerning among other problems and the periodicity of scalar nonlinear difference equations.

Gibbons posed the problem in [2] that whether there is a solution of the following difference equation

$$x(n+1)=\frac{\beta x(n-1)}{\beta+x(n)}\ 1 \text{ for } n=0, \tag{1}$$

where $x(-1),x(0) > 0$ and $\beta > 0$ is a constant, which converges to zero at infinity. [8] includes a particular answer given by Stević to Gibbon's problem by letting $\beta = 1$ in (1) , the following equation

$$x(n+1)=\frac{x(n-1)}{1+x(n)} \text{ for } n=0,1,\ldots, \tag{2}$$

where $x(-1),x(0) > 0$. Also, Stević's result was generalized to the equation having the following form

$$x(n+1)=\frac{x(n-1)}{g(x(n))} \text{ for } n=0,1,\ldots,$$

where $x(-1),x(0) > 0$ in the same paper.

Dağıstan et. al., studied the following problems with positive initial values

$$x(n+1)=\frac{x(n-3)}{1+x(n-1)} \tag{3}$$

$$x(n+1)=\frac{x(n-5)}{1+x(n-2)}$$

$$x(n+1)=\frac{x(n-5)}{1+x(n-1)x(n-3)} \tag{4}$$

for $n=0,1,\ldots,$ in [1,3,5] respectively.

Here, in this paper, we study the following rational difference equation

$$x(n+1)=\frac{x(n-pk+1)}{1+\prod_{j=1}^{p-1}x(n-jk+1)},\ n=0,1,2,\ldots, \tag{5}$$

where $p \geq 2$, $k \geq 1$ are fixed integers, with the initial values $x(n) > 0$ for $n = -pk+1, -pk+2, \ldots, 0$. Letting $p = 2$, $k = 1$ and $p = 2$, $k = 2$ and $p = 3$, $k = 2$ (5) reduces to (2),(3) and (4) respectively. Hence, our results generalize and extend the results in [3,5,8].

MAIN RESULT

Theorem 1. Let p, k be fixed integers and the initial values $x(n) > 0$ for $n = -pk+1, -pk+2, \ldots, 0$. If $\{x(n)\}_{n=1}^{\infty}$ is a solution of the difference equation (5), then the followings are true:

(a) for each $\ell=1,2\ldots,pk$ the subsequence $\{x(pkn+\ell)\}_{n=1}^{\infty}$ is decreasing and there exist pk nonnegative constants L_ℓ for $\ell=1,2,\ldots,pk$, satisfying

$$\lim_{n\to\infty} x(pkn+\ell) = L_\ell \quad \text{for} \quad \ell=1,2,\ldots,pk.$$

(b) The semi-cycle $L_1, L_2, \ldots, L_{pk}, L_1, L_2, \ldots, L_{pk}, \ldots$ is a solution of equation (5) with the period pk.

(c) $\prod_{j=1}^{p} L_{jk+i} = 0$ holds for each $i = 1, \ldots, k$. Here the convenience $L_{pk+\ell} = L_\ell$ $(\ell=1,2,\ldots,pk)$ is used.

(d) If there exists $n_0 \in N$ such that $x(n-(p-1)k) \geq x(n)$ for all $n \geq n_0$, then $\lim_{n\to\infty} x(n) = 0$ holds.

(e) For each $\ell=1,2\ldots,pk$, the following formulation holds:

$$x(pkn+\ell) = x(-pk+\ell)\left[1 + \frac{1}{w(0,\ell)} - \sum_{i=0}^{n}\prod_{j=1}^{pi}\frac{1}{w(jk,\ell)}\right],$$

where

$$w(n,\ell) = 1 + \prod_{j=1}^{p-1} x(n-jk+\ell),$$

for all $n \geq 1$.

Proof.

We start the proof by **(a)**.

Since $\prod_{j=1}^{p-1} x(n-jk+1) > 0$ for all $n \in N$ (5) indicates that $x(n+1) < x(n-pk+1)$ holds for all $n \in N$. Hence for each $\ell=1,2,\ldots,pk$ the subsequences $\{x(pkn+\ell)\}_{n=1}^{\infty}$ are decreasing and bounded below by 0. Therefore there exist pk nonnegative constants L_ℓ for $\ell=1,2,\ldots,pk$, satisfying

$$\lim_{n\to\infty} x(pkn+\ell) = L_\ell \quad \text{for} \quad \ell=1,2,\ldots,pk.$$

The proof of **(a)** is completed.

(b) Proof directly follows from the discussion in **(a)**.

Replacing *n* with $pkn+i-1 (i=1,2,...,k)$ in (5), we obtain

$$x(pkn+i)=\frac{x(pkn-pk+i)}{1+\prod_{j=1}^{p-1}x(kpn-jk+i)}$$

$$=\frac{x(pk(n-1)+i)}{1+\prod_{j=1}^{p-1}x(k(pn-j)+i)} \tag{6}$$

for $n\in N$. Limiting on both sides of (6) as $n\to\infty$ and considering the convenience

$L_\ell=L_{pk+\ell}=\lim_{n\to\infty}x(pk(n+1)+\ell)=\lim_{n\to\infty}x(pkn+\ell)=\lim_{n\to\infty}x(pk(n-1)+\ell)$ for $\ell=1,2,...,pk$

we get

$$L_{pk+i}=\frac{L_{pk+i}}{1+\prod_{j=1}^{p-1}L_{jk+i}}. \tag{7}$$

Then from (7), we see that

$$L_{pk+i}=0 \text{ or } \prod_{j=1}^{p}L_{jk+i}=0 \text{ for } i=1,2,...,k.$$

Hence the proof of (c) is completed.

(d) Let $n_0\in N$ satisfy $x(n-(p-1)k)\geq x(n)$ for all $n\geq n_0$. It suffices to prove that $L_\ell=0$ for all $\ell=1,2,...,pk$. Now we prove that $L_{jk+i}=0$ for $j=1,2,...,p$ and $i=1,2,...,k$. We have

$x(pkn+i) \geq x(pkn+(p-1)k+i)$
$\geq x(pkn+2(p-1)k+i)$
.
.
.
$\geq x(pkn+(p-1)(p-1)k+i)$
$\geq x(pkn+p(p-1)k+i)$

for all $n\geq n_0$ and $i=1,2,...,k$. Limiting on both sides as $n\to\infty$, we see that

$$L_i\geq L_{(p-1)k+i}\geq L_{2(p-1)k+i}\geq ...\geq L_{(p-1)(p-1)k+i}\geq L_{p(p-1)k+i}=L_i,$$

which implies that

$$L_i=L_{jk+i} \text{ for all } j=1,2,...,p \text{ and } i=1,2,...,k \tag{8}$$

Using (c), we get

$$L_i^p = \prod_{j=1}^{p} L_{jk+i} = 0,$$

which implies $L_i = 0$ for all $i=1,2,\ldots,k$. And hence $L_\ell = 0$ for all $\ell=1,2,\ldots,pk$ by (8), which is equivalent to

$\lim_{n\to\infty} x(n) = 0$.

(e) From (5), we obtain

$$x(n+1) - x(n-pk+1) = \left[\frac{1}{w(n,1)} - 1\right] x(n-pk+1)$$

$$= -\frac{\prod_{j=1}^{p} x(n-jk+1)}{w(n,1)}$$

$$= -\frac{x(n-k+1)}{w(n,1)} \prod_{j=1}^{p-1} x(n-(j+1)k+1) \tag{9}$$

for all $n = 0,1,\ldots$. Using (5) and (9), we get

$$x(n+1) - x(n-pk+1) = \frac{x(n-k+1)}{w(n,1)} \left[1 - \frac{x(n-(p+1)k+1)}{x(n-k+1)}\right]$$

$$= \frac{1}{w(n,1)} \left[x(n-k+1) - x(n-(p+1)k+1)\right] \tag{10}$$

for all $n = 0,1,\ldots$. Replacing n with $kn + \ell - 1(\ell + = 1,2,\ldots,pk)$ in(10), we get

$$y(n,\ell) = \frac{y(n-1,\ell)}{w(kn,\ell)} \text{ for } n \geq 1, \tag{11}$$

where $y(n,\ell) = x(kn+\ell) - x(k(n-p)+\ell)$. Iterating (11), we deduce the following formula

$$y(n,\ell) = y(0,\ell) \prod_{j=1}^{n} \frac{1}{w(jk,\ell)} \text{ for } n \geq 1. \tag{12}$$

Replacing n with pn in (12), we get

$$y(pn,\ell) = y(0,\ell) \prod_{j=1}^{pn} \frac{1}{w(jk,\ell)} \text{ for } n \geq 1, \tag{13}$$

and summing up (13) from 0 to n, we deduce

$$\sum_{i=0}^{n} y(pi,\ell) = y(0,\ell) \sum_{i=0}^{n} \prod_{j=1}^{pi} \frac{1}{w(jk,\ell)} \text{ for } n \geq 1,$$

which is equivalent to

$$x(pkn+\ell)-x(-pk+\ell)=\left[x(\ell)-x(-pk+\ell)\right]\sum_{i=0}^{n}\prod_{j=1}^{pi}\frac{1}{w(jk,\ell)} \text{ for } n\geq 1. \tag{14}$$

Using (5) and (14), we get

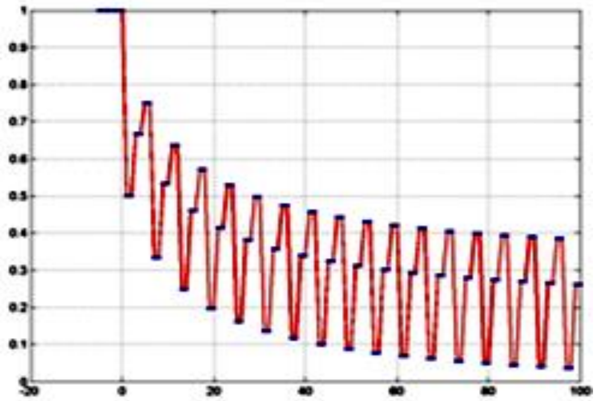
$$x(pkn+\ell)=x(-pk+\ell)+\left[x(\ell)-x(-pk+\ell)\right]\sum_{i=0}^{n}\prod_{j=1}^{pi}\frac{1}{w(jk,\ell)}$$

$$=x(-pk+\ell)+\left[\frac{x(-pk+\ell)}{w(0,\ell)}-x(-pk+\ell)\right]\sum_{i=0}^{n}\prod_{j=1}^{pi}\frac{1}{w(jk,\ell)}$$

$$=x(-pk+\ell)\left[1+\frac{1}{w(0,\ell)}-\sum_{i=0}^{n}\prod_{j=1}^{pi}\frac{1}{w(jk,\ell)}\right],$$

which is the desired equality.
Thus the proof is completed.

Example 1. Consider equation (4), which is a special case of (5) *p=3, k=2*. The following graphic belongs to the solution with the initial values *x(n)=1* for *n = -5,-4,…,0* and of *100* iterates:

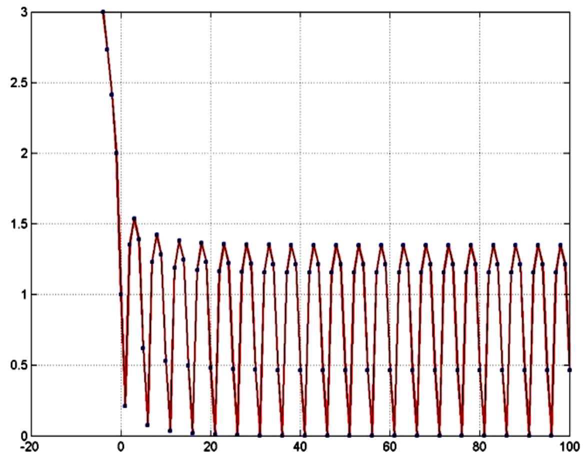


It is not hard to see that Theorem 1(a) holds.

Example 2. Consider the following equation

$$x(n+1)=\frac{x(n-4)}{1+x(n)x(n-1)x(n-2)x(n-3)},\quad n=0,1,2,\ldots,$$

which is a special case of (5) with *p=5, k=1*. The following graphic belongs to the solution with the initial values $x(n)=\sqrt{-n}+1$ for *n = -4,-3,…,0* and of *100* iterates:



It is not hard to see that Theorem 1(c) holds.

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