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Solution of Linear Volterra – Stieltjes Integral Equation of the Second Kind Using Generalized Midpoint Rule

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Abstract: *In this study, the numerical solution of linear Volterra – Stieltjes equations of the second kind by using the generalized midpoint rule is established and investigated. The conditions on estimation of the error are determined and proved. One example is solved employing the proposed method.*

Keywords: *Volterra – Stieltjes integral equation, linear integral equation of the second kind, generalized midpoint rule, error estimation.*

İkinci Tip Lineer Volterra-Stieltjes İntegral Denkleminin Genelleşmiş Orta Nokta Kuralı ile Cözümü

Özet: *Bu çalışmada İkinci türden lineer Volterra –Stieljes integral denklemi için orta nokta kuralı kullanılarak, sayısal çözüm kurulmuş ve incelenmiştir. Ayrıca hata tahmini ile ilgili koşullar belirlenmiş ve ispat edilmiştir. Önerilen yöntemle bir örnek çözülmüştür.*

Anahtar Kelimeler: *lineer Volterra –Stieljes integral denklemi, İkinci türden lineer integral denklemi, geneleştirilmiş orta nokta kuralı, hata tahmini*

INTRODUCTİON

Various issues concerning Volterra and Volterra – Stieltjes integral equations were studied in [1-4,6,7,9]. Some practical and theoretical investigations were made in [1-3]. In [7], the regularization operator for the solution of the Volterra integral equation of the first kind was constructed. The definition of derivative of a function by means of a strictly increasing function was given in [8]. In [9], Volterra – Stieltjes integral equation of the second kind and the first kind by means of a strictly increasing function was studied. In [5], the generalized midpoint rule was proposed to evaluate the Stieltjes integral approximation by employing the definition of derivative of a function by means of a strictly increasing function. In [6], the numerical solution of linear Fredholm – Stieltjes integral equations of the second kind by using midpoint rule were investigated. In this study, we investigate the numerical solution of linear Volterra – Stieltjes integral equations of the second kind by using the generalized midpoint rule.

Volterra – Stieltjes Integral Equations

Consider the linear integral equation of the second kind

$$v(x) = \int_a^x K(x,s)v(s)du(s) + f(x),\ \ x \in [a,b] \tag{1}$$

Where $K(x,s)$ is given continuous function on $G = \{(t,s): a \le s \le t \le b\}$, $f(x)$ is given continuous function on $[a,b]$, $u(x)$ is given bounded variation on $[a,b]$ where it can be expressed as a difference of two strictly increasing continuous functions; that is, $u(x) = \varphi(x) - \psi(\mathrm{x})$ for $[a,b]$. $v(x)$ is a function to be determined. We need the following theorem which is given in [5].

Theorem 1: Let $f(x)$ be the continuous function in $[a,b]$, $u(x)$ be the function of bounded variation in $[a,b]$. Then

$$|I - A_n| \le \omega_f(h)[\varphi(b) - \varphi(a) + \psi(b) - \psi(a)] \tag{2}$$

Where $\omega_f(h)$ is the modulus of continuity of function $f(x)$,

$I = \int_a^b f(x)du(x)$, $u(x) = \varphi(x) - \psi(x)$ for $x \in [a,b]$,

$\varphi(x)$ and $\psi(x)$ are the known increasing continuous functions on $[a,b]$.

$A_n = \sum_{i=1}^{n} f(x_{2i-1})\left[u(x_{2i}) - u(x_{2i-2})\right]$,

$h = \dfrac{b-a}{2n}$, $n \in \mathbb{N}$, $x_i = a + ih$, $i = 0,1,...,2n$.

Corollary:

Let $f(x)\in H^{\alpha}[a,b]$, $0<\alpha\le 1$; $u(x)$ is the function of bounded variation in $[a,b]$, where $H^{\alpha}[a,b]$ is the Holder space; that is, $\forall x_1,x_2\in[a,b]$ the estimate $\left|f(x_1)-f(x_2)\right|\le c|x_1-x_2|^{\alpha}$ holds, c is a positive constant dependent on $f(x)$, but independent on x_1,x_2. Then, the estimate

$$|I-A_n|\le ch^{\alpha}\left[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)\right] \tag{3}$$

holds.

Theorem 2: Let $K(x,s)\in(G)$, $\forall(x_1,s),(x_2,s),(x,s_1),(x,s_2)\in G$ the estimates

$$\left|K(x_1,s)-K(x_2,s)\right|\le c_1|x_1-x_2|^{\alpha},\ \left|K(x,s_1)-K(x,s_2)\right|\le c_2|s_1-s_2|^{\alpha},$$

for $0<\alpha\le 1$, holds, c_1 and c_2 are the positive constants, $f(x)\in H^{\alpha}[a,b]$; that is, $\forall x_1,x_2\in[a,b]$ the estimate

$$\left|f(x_1)-f(x_2)\right|\le c|x_1-x_2|^{\alpha},\ v(x)\in N[a,b]$$

be the solution of the integral equation (1). Then $f(x)\in H^{\alpha}[a,b]$, $K(x,s)v(s)\in H^{\alpha}[a,x]$; that is, $\forall x_1,x_2\in[a,b]$ and $\forall(x,s_1),(x,s_2)\in G$ the estimates

$$\left|v(x_1)-v(x_2)\right|\le c_3|x_1-x_2|^{\alpha},\ \left|K(x,s_1)v(s_1)-K(x,s_2)v(s_2)\right|\le c_4|s_1-s_2|^{\alpha}$$

hold, where $c_3=c_0+\left[c_1(b-a)+(b-a)^{1-\alpha}M\right]\|v(x)\|_c$, $c_4=c_2\|v(t)\|_c+c_3M$, and $M=\sup\limits_{(x,s)\in G}|K(x,s)|$.

Proof: $\forall x_1,x_2\in[a,b]$. From (1) we have

$$\left|v(x_1)-v(x_2)\right|\le\int_a^{x_1}\left|K(x_1,s)-K(x_2,s)\right|\left|v(s)\right|ds+\left|\int_{x_1}^{x_2}K(x_2,s)v(s)ds\right|+\left|f(x_1)-f(x_2)\right|\le c_3\,.$$

For $\forall(x,s_1),(x,s_2)\in G$ we obtain

$$\left|K(x,s_1)v(s_1)-K(x,s_2)v(s_2)\right|=\left|\left[K(x,s_1)-K(x,s_2)\right]v(s_1)+K(x,s_2)\left[v(s_1)-v(s_2)\right]\right|\le$$

$$\leq c_2 |s_1 - s_2|^{\alpha} \|v(t)\|_c + Mc_3 |s_1 - s_2|^{\alpha} \leq c_4 |s_1 - s_2|^{\alpha}$$

Theorem 2 is proved.

Thus, from now on, we will assume that all conditions of Theorem 2 holds.

Numerical Solution

In order to approximate solution of Equation (1), we employ the generalized midpoint rule given in [5] to the int Let $n \in N$, $h = \dfrac{b-a}{2n}$, $x_k = a + kh$, where $k = 0,1,2,...,2n$. Let us substitute $x = x_{2k}$ and $x = x_{2j+1}$ in the integral equation (1) and examine the following system of equations

$$\begin{cases} v(x_0) = f(x_0),\ x_0 = a \\ v(x_{2k}) = \int\limits_a^{x_{2k}} K(x_{2k}, s) v(s) du(s) + f(x_{2k}),\ k = 1,2,...,n \\ v(x_1) = \int\limits_a^{x_1} K(x_1, s) v(s) du(s) + f(x_1), \\ v(x_{2j+1}) = \int\limits_a^{x_{2j+1}} K(x_{2j+1}, s) v(s) du(s) + f(x_{2j+1}),\ j = 1,2,...,n-1 \end{cases} \tag{4}$$

integral equation (1). To evaluate the integral in equation (4), we employ the generalized midpoint rule given in [5]; that is, Theorem 1 with the nodes $x_0, x_1, x_2, ..., x_{2k}$. So we get

$$\begin{cases} \int\limits_a^{x_{2k}} K(x_{2k}, s) v(s) du(s) = \sum\limits_{m=1}^{k} K(x_{2k}, x_{2m-1}) v(x_{2m-1}) \left[u(x_{2m}) - u(x_{2m-2})\right] + \sum\limits_{i=1}^{k} R_i^{(2n)}(v),\ k = 1,2,...,n \\ \int\limits_a^{x_1} K(x_1, s) v(s) du(s) = K(x_1, x_0) v(x_0) \left[u(x_1) - u(x_0)\right] + R_0^{(0)}(v), \\ \int\limits_a^{x_{2j+1}} K(x_{2j+1}, s) v(s) du(s) = \int\limits_a^{x_1} K(x_{2j+1}, s) v(s) du(s) + \int\limits_{x_1}^{x_{2j+1}} K(x_{2j+1}, s) v(s) du(s) = \\ \qquad = K(x_{2j+1}, x_0) v(x_0) \left[u(x_1) - u(x_0)\right] + R_j^{(0)}(v) + \\ \qquad + \sum\limits_{m=1}^{j} K(x_{2j+1}, x_{2m}) v(x_{2m}) \left[u(x_{2m+1}) - u(x_{2m-1})\right] + \sum\limits_{m=1}^{j} R_m^{(2n-1)}(v),\ j = 1,2,...,n-1 \end{cases} \tag{5}$$

where

$$R_0^{(0)}(v)=\int_a^{x_1}\left[K(x_1,s)-K(x_1,x_0)\right]v(s)\,du(s)+\int_a^{x_1}K(x_1,x_0)\left[v(s)-v(x_0)\right]du(s) \tag{6}$$

$$R_j^{(0)}(v)=\int_a^{x_1}\left[K(x_{2j+1},s)-K(x_{2j+1},x_0)\right]v(s)\,du(s)+\int_a^{x_1}K(x_{2j+1},x_0)\left[v(s)-v(x_0)\right]du(s),\ \ j=1,2,...,n-1 \tag{7}$$

On the strength of Theorem 2 and Corollary of Theorem 1, for $R_k^{(2n)}(v)$ and $R_j^{(2n-1)}(v)$ from (5), we obtain the following estimates:

$$\begin{cases}\left|R_k^{(2n)}(v)\right|\le c_4h^{\alpha}\left[\varphi(x_{2k})-\varphi(x_{2k-2})+\psi(x_{2k})-\psi(x_{2k-2})\right]\\ \left|R_j^{(2n-1)}(v)\right|\le c_4h^{\alpha}\left[\varphi(x_{2j+1})-\varphi(x_{2j-1})+\psi(x_{2j+1})-\psi(x_{2j-1})\right]\end{cases} \tag{8}$$

where $k=1,2,...,n,\ j=1,2,...,n-1$. Then taking into account (6),(7), and (8), we have

$$\begin{cases}\left|\sum_{i=1}^{k}R_i^{(2n)}(v)\right|\le c_4h^{\alpha}\left[\varphi(x_{2k})-\varphi(a)+\psi(x_{2k})-\psi(a)\right],\ k=1,2,...,n,\\ \left|R_0^{(0)}(v)\right|\le c_4h^{\alpha}\left[\varphi(x_1)-\varphi(a)+\psi(x_1)-\psi(a)\right],\\ \left|R_j^{(0)}(v)\right|\le c_4h^{\alpha}\left[\varphi(x_1)-\varphi(a)+\psi(x_1)-\psi(a)\right],\\ \left|\sum_{m=1}^{j}R_m^{(2n-1)}(v)\right|\le c_4h^{\alpha}\left[\varphi(x_{2j+1})-\varphi(x_1)+\psi(x_{2j+1})-\psi(x_1)\right],\ j=1,2,...,n-1.\end{cases} \tag{9}$$

On the strength (5) from (4) we obtain

$$\begin{cases}v(x_0)=f(x_0),\ x_0=a\\ v(x_{2k})=\sum_{m=1}^{k}K(x_{2k},x_{2m-1})v(x_{2m-1})\left[u(x_{2m})-u(x_{2m-2})\right]+f(x_{2k})+\sum_{i=1}^{k}R_i^{(2n)}(v)+f(x_{2k}),\ k=1,2,...,n,\\ v(x_1)=K(x_1,x_0)v(x_0)\left[u(x_1)-u(x_0)\right]+f(x_1)+R_0^{(0)}(v),\\ v(x_{2j+1})=K(x_{2j+1},x_0)v(x_0)\left[u(x_1)-u(x_0)\right]+\sum_{m=1}^{j}K(x_{2j+1},x_{2m})v(x_{2m})\left[u(x_{2m+1})-u(x_{2m-1})\right]+\\ \qquad +f(x_{2j+1})+R_j^{(0)}(v)+\sum_{m=1}^{j}R_m^{(2n-1)}(v),\ j=1,2,...,n-1\end{cases} \tag{10}$$

Omitting the terms $\sum_{i=1}^{k} R_i^{(2n)}(v), R_0^{(0)}(v), R_j^{(0)}(v)$, and $\sum_{m=1}^{j} R_m^{(2n-1)}(v)$ appearing in each equation of the system (10) and writing the sought solution $v(x)$ at the nodes x_k, we get the linear algebraic system of equations in terms v_k:

$$\begin{cases} v_0 = f(x_0),\ x_0 = a \\ v_1 = K(x_1, x_0)\left[u(x_1) - u(x_0)\right]v_0 + f(x_1), \\ v_2 = K(x_2, x_1)\left[u(x_2) - u(x_0)\right]v_1 + f(x_2), \\ v_3 = K(x_3, x_0)\left[u(x_1) - u(x_0)\right]v_0 + K(x_3, x_2)\left[u(x_3) - u(x_1)\right]v_2 + f(x_3), \\ v_4 = \sum_{m=1}^{2} K(x_4, x_{2m-1})\left[u(x_{2m}) - u(x_{2m-2})\right]v_{2m-1} + f(x_4), \\ v_5 = K(x_5, x_0)\left[u(x_1) - u(x_0)\right]v_0 + \sum_{m=1}^{2} K(x_5, x_{2m})\left[u(x_{2m+1}) - u(x_{2m-1})\right]v_{2m} + f(x_5), \\ v_{2k-1} = K(x_{2k-1}, x_0)\left[u(x_1) - u(x_0)\right]v_0 + \sum_{m=2}^{k} K(x_{2k-1}, x_{2m-2})\left[u(x_{2m-1}) - u(x_{2m-3})\right]v_{2m-2} + f(x_{2k-1}), \\ v_{2k} = \sum_{m=1}^{k} K(x_{2k}, x_{2m-1})\left[u(x_{2m}) - u(x_{2m-2})\right]v_{2m-1} + f(x_{2k}),\ k = 2, 3, \ldots, n \end{cases} \tag{11}$$

where $v_0 = v(x_0), v_i \approx v(x_i),\ i = 1, 2, \ldots, 2n$.

Example 1.

Let us take the integral equation (1) for $a = 0$ and It is easily seen that $v(x) = \sqrt{x},\ x \in [0, 2]$ is the unique solution of the integral equation (1) and the conditions of the Theorem 2 for $\alpha = \frac{1}{2}$. Using the proposed method of this study, we get the following results: Here $n = 20$. In Table 1, we give the values of the approximate solution obtained by the proposed method of this study and the error in absolute values at the given nodes.

$b = 2$ with $u(x) = \sqrt{x} - \sqrt[3]{x},\ K(x, s) = \frac{1}{20}\sqrt{xs},\ f(x) = \sqrt{x} - \frac{1}{60}x^2 + \frac{1}{80}x^{\frac{11}{6}}$.

Table 1. Table of values

The nodes x_k	Real values at x_k, $v(x_k)$	Approximate values at x_k, v_k	The error of x_k, $\lvert v(x_k)-v_k\rvert$
0	0	0	0.0000000000
0.05	0.223606798	0.223616617	0.0000098192
0.1	0.316227766	0.316127619	0.0001001468
0.15	0.387298335	0.387310659	0.0000123247
0.2	0.447213596	0.447070046	0.0001435494
0.25	0.5	0.500014889	0.0000148890
0.3	0.547722558	0.547546179	0.0001763785
0.35	0.591607978	0.591625212	0.0000172333
0.4	0.632455532	0.63225168	0.0002038524
0.45	0.670820393	0.670839756	0.0000193626
0.5	0.707106781	0.706878836	0.0002279457
0.55	0.741619849	0.741641152	0.0000213032
0.6	0.774596669	0.774347013	0.0002496565
0.65	0.806225775	0.806248853	0.0000230785
0.7	0.836660027	0.836390455	0.0002695720
0.75	0.866025404	0.86605011	0.0000247066
0.8	0.894427191	0.89413912	0.0002880714
0.85	0.921954446	0.921980647	0.0000262013
0.9	0.948683298	0.948377881	0.0003054171
0.95	0.974679435	0.974707008	0.0000275738
1	1	0.999678201	0.0003217990
1.05	1.024695077	1.024723909	0.0000288320
1.1	1.048808848	1.048471488	0.0003373600
1.15	1.072380529	1.072410513	0.0000299840
1.2	1.095445115	1.095092904	0.0003522110
1.25	1.118033989	1.118065024	0.0000310350
1.3	1.140175425	1.139808986	0.0003664390
1.35	1.161895004	1.161926994	0.0000319900
1.4	1.183215957	1.182835842	0.0003801150
1.45	1.204159458	1.204192308	0.0000328500
1.5	1.224744871	1.224351572	0.0003932990
1.55	1.24498996	1.245023582	0.0000336220
1.6	1.264911064	1.264505025	0.0004060390
1.65	1.284523258	1.284557566	0.0000343080
1.7	1.303840481	1.303422106	0.0004183750
1.75	1.322875656	1.322910565	0.0000349090
1.8	1.341640786	1.341210442	0.0004303440
1.85	1.360147051	1.360182478	0.0000354270
1.9	1.378404875	1.3779629	0.0004419750
1.95	1.396424004	1.396459869	0.0000358650
2	1.414213562	1.413760268	0.0004532940

Estimation of the Error

In this section, we investigate the problem of convergence of the approximate solution v_k to the solution $v(x)$ of the integral equation (1) at the nodes x_k as $n \to \infty$.

Theorem 3: Let $u(x)$ be the function of bounded variation in $[a,b]$, $K(x,s) \in C(G)$ $\forall (x_1, s),(x_2, s),(x, s_1),(x, s_2) \in G$ the estimates $\left|K(x_1,s)-K(x_2,s)\right| \le c_1 \left|x_1 - x_2\right|$,

$\left|K(x,s_1)-K(x,s_2)\right|\le c_2\left|s_1-s_2\right|$ hold for positive constants c_1, c_2, and $\forall x_1, x_2\in[a,b]$ the estimate $\left|f(x_1)-f(x_2)\right|\le c\left|x_1-x_2\right|$ holds. Then, the inequality

$$\left|v(x_k)-v_k\right|\le c_6 h,\ k=0,1,2,...,2n, \tag{12}$$

hold where $c_6=c_4\left[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)\right]e^{c_5(b-a)}$, $c_5=M\left[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)\right]$.

Proof: Let the error be denoted by $z_k=v(x_k)-v_k$, $k=0,1,...,2n$. Taking the system of equations (10) and (11) into account, we have the following system of equations

$$\begin{cases} z_0=0 \\ z_1=K(x_1,x_0)\left[u(x_1)-u(x_0)\right]z_0+R_0^{(0)}(v), \\ z_2=K(x_2,x_1)\left[u(x_2)-u(x_0)\right]z_1+R_1^{(2n)}(v), \\ z_3=K(x_3,x_0)\left[u(x_1)-u(x_0)\right]z_0+K(x_3,x_2)\left[u(x_3)-u(x_1)\right]z_2+R_1^{(0)}(v)+R_1^{(2n-1)}(v), \\ z_4=\sum_{m=1}^{2}K(x_4,x_{2m-1})\left[u(x_{2m})-u(x_{2m-2})\right]z_{2m-1}+\sum_{m=1}^{2}R_m^{(2n)}(v), \\ \dots \\ z_{2k-1}=K(x_{2k-1},x_0)\left[u(x_1)-u(x_0)\right]z_0+\sum_{m=2}^{k}K(x_{2k-1},x_{2m-2})\left[u(x_{2m-1})-u(x_{2m-3})\right]z_{2m-2}+ \\ \qquad +R_{k-1}^{(0)}(v)+\sum_{m=1}^{k-1}R_m^{(2n-1)}(v), \\ z_{2k}=\sum_{m=1}^{k}K(x_{2k},x_{2m-1})\left[u(x_{2m})-u(x_{2m-2})\right]z_{2m-1}+\sum_{m=1}^{k}R_m^{(2n)}(v),\ k=2,3,...,n \end{cases} \tag{13}$$

Using the estimates (9) from (13), we get the following inequality for

$$z_k:\begin{cases} \left|z_1\right|\le R(h), \\ \left|z_k\right|\le R(h)+c_5 h\sum_{j=1}^{k-1}\left|z_j\right|,\ k=2,3,...,2n, \end{cases} \tag{14}$$

where $R(h)=c_4 h\left[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)\right]$.

Let the term ε_k for $k=2,3,4,...,2n$ be determined by

$$\varepsilon_k=R(h)+c_5 h\sum_{j=1}^{k-1}\left|\varepsilon_j\right|, \tag{15}$$

and $\varepsilon_1 = R(h)$ as an initial condition. It is easily seen that $|z_k| \le \varepsilon_k$ for $k = 1, 2, 3, 4, ..., 2n$. This can be verified by mathematical induction as follows: for $k = 1$, it is obvious. Let $|z_j| \le \varepsilon_j$ for $j = 1, 2, ..., k-1$. Then by using the inequality (14), we get

$$|z_k| \le R(h) + c_5 h \sum_{j=1}^{k-1} \varepsilon_j = \varepsilon_k \ .$$

Let us show that

$$\varepsilon_j = R(h)[1 + c_5 h]^{j-1}, \ j = 1, 2, 3, ..., 2n \tag{16}$$

is the solution of the system of equations (15). Taking into account (16), we get

$$\begin{aligned} R(h) + c_5 h \sum_{j=1}^{k-1} \varepsilon_j &= R(h)\left\{1 + c_5 h \sum_{j=1}^{k-1} (1 + c_5 h)^{j-1}\right\} \\ &= R(h)\left\{1 + \left[(1 + c_5 h)^{k-1} - 1\right]\right\} \\ &= \varepsilon_k, \ k \ge 2 \end{aligned} \tag{17}$$

Here in (17) we use the inequality $(1+\gamma)^{k-1} - 1 = \gamma \sum_{j=1}^{k-1} (1+\gamma)^{j-1}, \ k \ge 2$ where $\gamma = c_5 h$.

Consequently, we get the following estimates for the error z_k for all values $k =, 1, 2, 3, ..., 2n$.

$$|z_k| \le R(h)(1 + c_5 h)^{k-1}.$$

Using the fact that $(1+t)^{1/t}$ is increasing and approaches to the number e as $t \to 0^+$, we get the following chain of inequalities

$(1+c_5 h)^{k-1} \le (1+c_5 h)^{\frac{b-a}{h}} = \left[(1+c_5 h)^{\frac{1}{c_5 h}}\right]^{c_5(b-a)} \le e^{c_5(b-a)}$ for $k \le \dfrac{b-a}{h}$. Hence, we get the estimates (12). Theorem 3 is proved.

Theorem 4: Let $u(x)$ be the function of bounded variation in $[a,b]$, $K(x,s) \in N(G)$, $\forall (x_1, s), (x_2, s), (x, s_1), (x, s_2) \in G$ the estimates

$$|K(x_1, s) - K(x_2, s)| \le c_1 |x_1 - x_2|^{\alpha}, \ |K(x, s_1) - K(x, s_2)| \le c_2 |s_1 - s_2|^{\alpha}$$

hold for positive constants $\alpha \in (0,1), c_1, c_2$, and $\forall x_1, x_2 \in [a,b]$ the estimate

$$|f(x_1) - f(x_2)| \le c |x_1 - x_2|^{\alpha}$$

holds and $\beta = M\left[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)\right]<1$. Then, the inequalities

$$\left|v\left(x_k\right)-v_k\right| \leq \frac{c_4}{1-\beta}\left[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)\right]h^{\alpha},\ k=0,1,2,...,2n, \tag{18}$$

hold.

Proof: Using the estimates (9) from (12), we get the following inequalities for z_k:

$$\left|z_k\right| \leq c_4\left[\varphi(b)-\varphi(\mathrm{a})+\psi(\mathrm{b})-\psi(a)\right]h^{\alpha}+\sup_j\left|z_j\right|M\left[\varphi(b)-\varphi(\mathrm{a})+\psi(\mathrm{b})-\psi(a)\right] \tag{19}$$

for $k=0,1,2,...,2n$. From (19), we obtain the estimates (18). Theorem 4 is proved.

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