Solutions of the Rational Difference Equations

\[ x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}} \]

Dağıstan Şimşek*1,2, Burak Oğul1

1 Department of Applied Mathematics and Informatics, Kyrgyz – Turkish Manas University, Bishkek, Kyrgyzstan
2 Faculty of Engineering, Selcuk University, Konya, Turkey

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Abstract: In this paper the solutions of the following difference equation is examined,

\[ x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad n=0,1,2,... \] (1)

where the initial conditions are positive real numbers.

Keywords: Difference equation, period 2k+2 solution

Rasyonel Fark Denkleminin Çözümleri

Özet: Aşağıdaki Rasyonel fark denkleminin çözümlerini incelendi.

\[ x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad n=0,1,2,... \] (1)

Burada başlangıç şartları reel sayılardır.

Anahtar Kelimeler: Fark denklemleri, 2k+2 periyotlu çözümler

* Corresponding Author: Şimşek D., email: dagistan.simsek@manas.edu.kg , dsimsek@selcuk.edu.tr
1. INTRODUCTION

Recently there has been a lot of interest in studying the periodic nature of non-linear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, [1-24].

Cinar, studied the following problems with positive initial values

\[
x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}
\]

\[
x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}
\]

\[
x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}
\]

for \( n = 0, 1, 2, \ldots \) in [2,3,4], respectively.

In [18] Stevic assumed that \( \beta = 1 \) and solved the following problem

\[
x_{n+1} = \frac{x_{n-1}}{1 + x_n}
\]

for \( n = 0, 1, 2, \ldots \)

Where \( x_{-1}, x_0 \in (0, \infty) \). Also, this results was generalized to the equation of the following form:

\[
x_{n+1} = \frac{x_{n-1}}{g(x_n)}
\]

for \( n = 0, 1, 2, \ldots \)

Where \( x_{-1}, x_0 \in (0, \infty) \).

Simsek et. al., studied the following problems with positive initial values

\[
x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}
\]

\[
x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}
\]

for \( n = 0, 1, 2, \ldots \) in [19,20] respectively.

In this paper we investigated the following nonlinear difference equation

\[
x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad n = 0, 1, 2, \ldots \tag{1}
\]

where \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty) \).
2. **Main Result**

Let $\bar{x}$ be the unique positive equilibrium of Eq. (1), then clearly

$$\bar{x} = \frac{x}{1+x} \Rightarrow x + x^2 = x \Rightarrow x^2 = 0 \Rightarrow x = 0$$

We can obtain $\bar{x} = 0$.

**Theorem 1.** Consider the difference equation (1). Then the following statements are true.

a) The sequences $(x_{(2k+2)n-(2k+1)})$, $(x_{(2k+2)n-(2k)})$, ..., $(x_{(2k+2)n})$ are decreasing and there exist $a_1, a_2, ..., a_{2k+2} \geq 0$ such that

$$\lim_{n \to \infty} x_{(2k+2)n-(2k+1)} = a_1, \quad \lim_{n \to \infty} x_{(2k+2)n-(2k)} = a_2, ..., \lim_{n \to \infty} x_{(2k+2)n-(k)-1} = a_{k-1},$$

$$\lim_{n \to \infty} x_{(2k+2)n-(k)} = a_k, \quad \lim_{n \to \infty} x_{(2k+2)n-(k)+1} = a_{k+1}, ..., \lim_{n \to \infty} x_{(2k+2)n} = a_{2k+2}.$$

b) $(a_1, a_2, ..., a_{2k+2}, a_1, a_2, ..., a_{2k+2}, ...)$ is a solution of equation (1) of period $2k+2$.

c) $\lim_{n \to \infty} x_{(2k+2)n-(2k+1)} \cdot \lim_{n \to \infty} x_{(2k+2)n-(2k)} = 0$, ..., $\lim_{n \to \infty} x_{(2k+2)n-(k)-1} \lim_{n \to \infty} x_{(2k+2)n} = 0$

or

$$a_1a_k = 0, ..., a_{k-1}a_{2k+2} = 0.$$

d) If there exist $n_0 \in N$ such that $x_{n-k} \geq x_{n+1}$ for all $n \geq n_0$, then $\lim_{n \to \infty} x_n = 0$.

e) The following formulas hold:

$$x_{(2k+2)n+1} = x_{(2k+1)} \left(1 - \frac{x_{-k}}{1 + x_{-k}} \sum_{j=0}^{2} \prod_{i=1}^{n} \frac{1}{1 + x_{(k+1)i-k-j}} \right)$$

$$\cdot$$

$$\cdot$$

$$x_{(2k+2)n+k+1} = x_{-(k+1)} \left(1 - \frac{x_0}{1 + x_0} \sum_{j=0}^{2} \prod_{i=1}^{n} \frac{1}{1 + x_{(k+1)i}} \right)$$
\[
X_{(2k+2)n+k+2} = x_{-k} \left( 1 - \frac{x_{-(2k+1)}}{1 + x_{-k}} \right) \sum_{j=0}^{n} \frac{2^{j+1}}{1 + x_{(k+1)j-k}} \left( \frac{1}{1 + x_{(2k+2)n+k+1}} \right)
\]

f) If \( x_{(2k+2)n+1} \to a_1 \neq 0 \) then \( x_{(2k+2)n+k+2} \to 0 \) as \( n \to \infty \),... If \( x_{(2k+2)n+k+1} \to a_{k+1} \neq 0 \) then \( x_{(2k+2)n+2k+2} \to 0 \) as \( n \to \infty \)

**Proof.** a) Firstly, we consider the equation (1). From this equation we obtain
\[
x_{n+1} (1 + x_{n+k}) = x_{-(2k+1)}.
\]

If \( x_{n-k} \in (0, +\infty) \), then \((1 + x_{n-k}) \in (1, +\infty)\). Since \( x_{n+1} < x_{n-(2k+1)} \), \( n \in N \), we obtain that
\[
\lim_{n \to \infty} x_{(2k+2)n-(2k+1)} = a_1, \quad \lim_{n \to \infty} x_{(2k+2)n-(2k)} = a_2, \ldots, \lim_{n \to \infty} x_{(2k+2)n-(k-1)} = a_{k-1},
\]
\[
\lim_{n \to \infty} x_{(2k+2)n-(k)} = a_k, \quad \lim_{n \to \infty} x_{(2k+2)n-(k+1)} = a_{k+1}, \ldots, \lim_{n \to \infty} x_{(2k+2)n} = a_{2k+2}.
\]

b) \((a_1, a_2, \ldots, a_{2k+2}, a_1, a_2, \ldots, a_{2k+2})\) is a solution of equation (1) of period \(2k+2\).

c) In view of the equation (1), we obtain
\[
x_{(2k+2)n+1} = \frac{x_{(2k+2)n-(2k+1)}}{1 + x_{(2k+2)n-k}}.
\]

Taking limit as \( n \to \infty \) on both sides of the above equality, we get
\[
\lim_{n \to \infty} x_{(2k+2)n+1} = \lim_{n \to \infty} \frac{x_{(2k+2)n-(2k+1)}}{1 + x_{(2k+2)n-k}}.
\]

Then
\[
\lim_{n \to \infty} x_{(2k+2)n+1} \lim_{n \to \infty} x_{(2k+2)n-k} = 0 \text{ or } a_1 a_k = 0.
\]

Similarly,
\[
\lim_{n \to \infty} x_{(2k+2)n-k} \lim_{n \to \infty} x_{(2k+2)n+2k+2} = 0 \text{ or } a_{k-1} a_{2k+2} = 0.
\]

d) If there exist \( n_0 \in N \) such that \( x_{n-k} \geq x_{n+1} \) for all \( n \geq n_0 \), then \( a_1 \leq \ldots \leq a_k, \ldots, a_{k-1} \leq \ldots \leq a_{2k+2} \leq a_{k-1} \). Since \( a_1 a_k = 0, \ldots, a_{k-1} a_{2k+2} = 0 \) we obtain the result.

e) Subtracting \( x_{n-(2k+1)} \) from the left and right-hand sides of equation (1) we obtain
\[ x_{n+1} - x_{n-(2k+1)} = \frac{1}{1+x_{n-k}}(x_{n-k} - x_{n-(3k+2)}) \]

and the following formula

\[
\begin{align*}
&\left\{ \begin{array}{l}
\frac{x}{(k+1)n-(k+1)^2-1} - \frac{x}{(k+1)n-(k+2)^2-2} = (x_1 - x_{-(2k+1)}) \\
\frac{x}{(k+1)n-(k+1)^2-2} - \frac{x}{(k+1)n-(k+2)^2-2} = (x_{k+1} - x_{-(2k+1)}) \\
\end{array} \right. \\
&n \geq k+1 \text{ for } \quad \vdots \\
&n -(k+1) \sum_{j=0}^{n} \frac{1}{1+x_{-(k+1)i-j}} \\
\end{align*}
\]

holds. Replacing \( n \) by \( 2j \) in (2) and summing from \( j = 0 \) to \( j = n \) we obtain

\[
x_{(2k+2)n+1} - x_{-(2k+1)} = (x_1 - x_{-(2k+1)}) \sum_{j=0}^{2j} \prod_{i=1}^{j} \frac{1}{1+x_{-(k+1)i-j}} \quad (n = 0, 1, 2, \ldots)
\]

\[
\vdots
\]

\[
x_{(2k+2)n+k+1} - x_{-(k+1)} = (x_{k+1} - x_{-(k+1)}) \sum_{j=0}^{2j} \prod_{i=1}^{j} \frac{1}{1+x_{-(k+1)i-j}} \quad (n = 0, 1, 2, \ldots)
\]

Also, replacing \( n \) by \( 2j+1 \) in (2) and summing from \( j = 0 \) to \( j = n \) we obtain

\[
x_{(2k+2)n+k+2} - x_{-(k)} = (x_1 - x_{-(2k+1)}) \sum_{j=0}^{2j+1} \prod_{i=1}^{j} \frac{1}{1+x_{-(k+1)i-j}} \quad (n = 0, 1, 2, \ldots)
\]

\[
\vdots
\]

\[
x_{(2k+2)n+2k+2} - x_{0} = (x_{k+1} - x_{-(k+1)}) \sum_{j=0}^{2j+1} \prod_{i=1}^{j} \frac{1}{1+x_{-(k+1)i-j}} \quad (n = 0, 1, 2, \ldots)
\]

Now, we obtained of the above formulas,

\[
\begin{align*}
x_{(2k+2)n+1} &= x_{-(2k+1)} \left[ 1 - \frac{x_{-(k+1)}}{1+x_{-(k+1)}} \sum_{j=0}^{n} \prod_{i=1}^{j} \frac{1}{1+x_{-(k+1)i-j}} \right] \\
\vdots
\end{align*}
\]

\[
\begin{align*}
x_{(2k+2)n+k+1} &= x_{-(k+1)} \left[ 1 - \frac{x_{-(k+1)}}{1+x_{-(k+1)}} \sum_{j=0}^{n} \prod_{i=1}^{j} \frac{1}{1+x_{-(k+1)i-j}} \right]
\end{align*}
\]
\[
x_{(2k+2)n+k+2} = x_{-k}\left(1 - \frac{x_{-(2k+1)}}{1 + x_{-k}} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}} \right)
\]

\[
x_{(2k+2)n+k+2} = x_{0}\left(1 - \frac{x_{-(k+1)}}{1 + x_{0}} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i}} \right)
\]

f) Suppose that \(a_1 = a_{k+2} = 0\). By e) we have

\[
\lim_{n \to \infty} x_{(2k+2)n+1} = \lim_{n \to \infty} x_{-(2k+1)}\left(1 - \frac{x_{-k}}{1 + x_{-k}} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}} \right)
\]

\[
a_1 = x_{-(2k+1)}\left(1 - \frac{x_{-k}}{1 + x_{-k}} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}} \right)
\]

\[
a_1 = 0 \implies \frac{1 + x_{-k}}{x_{-k}} = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}}
\]

Similarly,

\[
\lim_{n \to \infty} x_{(2k+2)n+k+2} = \lim_{n \to \infty} x_{-k}\left(1 - \frac{x_{-(2k+1)}}{1 + x_{-k}} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}} \right)
\]

\[
a_{k+2} = x_{-k}\left(1 - \frac{x_{-(2k+1)}}{1 + x_{-k}} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}} \right)
\]

\[
a_{k+2} = 0 \implies \frac{1 + x_{-k}}{x_{-(2k+1)}} = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}}
\]

From the equations (7) and (8),

\[
\frac{1 + x_{-k}}{x_{-k}} = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}} > \frac{1 + x_{-k}}{x_{-(2k+1)}} = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{n} 1}{1 + x_{(k+1)i-k}}
\]

thus, \(x_{-(2k+1)} > x_{-k}\).

Suppose that \(a_{k+1} = a_{2k+2} = 0\). From the equation (10) in e) follows, Proof of the equation (9) is similar and will be omitted.
\[ \frac{1+x_0}{x_{-0}} = \sum_{j=0}^{\infty} \prod_{i=1}^{2j} \frac{1}{1+x_{(k+1)i-k}} > \frac{1+x_{-0}}{x_{-(k+1)}} = \sum_{j=0}^{\infty} \prod_{i=1}^{2j+1} \frac{1}{1+x_{(k+1)i-k}} \]  

(10)

thus, \( x_{-(k+1)} > x_0 \).

From here we obtain \( x_{-(2k+1)} > x_{-2k} > \ldots > x_{-1} > x_0 \). We arrive at a contradiction which completes the proof of theorem.

3. EXAMPLES

Example 3.1: Consider the following equation \( x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}} \) which is special case of \( k = 1 \).

If the initial conditions are selected as follows:

\[ x[-3]=2; x[-2]=3; x[-1]=4; x[0]=5; \]

The following solutions are obtained:

\[ x(n)=\{ 0.0327869, 1.81188, 3.08397, 4.22581, 0.0013321, 1.78096, 3.05336, 4.19542, 0.000055937, 1.77969, 3.05208, 4.19414, 2.35212x10^{-6}, 1.77963, 3.05203, 4.19409, 9.89108x10^{-8}, 1.77963, 3.05203, 4.19409, 4.15939x10^{-9}, 1.77963, 3.05203, 4.19409, 1.7491x10^{-10}, 1.77963, 3.05203, 4.19409, 7.35532x10^{-12}, 1.77963, 3.05203, 4.19409, 3.09306x10^{-13}, 1.77963, 3.05203, 4.19409, \ldots \} \]

The graph of the solutions is given below.

![Figure 3.1. x(n) graph of the solutions.](image-url)
Example 3.2: Consider the following equation 
\[ x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}} \]
which is special case of \( k = 1 \).

If the initial conditions are selected as follows:
\[ x[-3] = 2; x[-2] = 0.1; x[-1] = 0.01; x[0] = 0.001; \]

The following solutions are obtained:
\[ x(n) = \{2, 0.099998, 0.009998, 0.000998004, 2, 0.099996, 0.00999601, 0.000996013, 1.99999, 0.999994, 0.00999401, 0.000994027, 1.99999, 0.999992, 0.00999203, 0.000992044, 1.99999, 0.99999, 0.00999005, 0.000990066, 1.99999, 0.9999881, 0.00998807, 0.000988093, 1.99999, 0.9999861, 0.0099861, 0.000986123, 1.99999, 0.9999841, 0.00998413, 0.000984159, 1.99999, 0.9999822, 0.00998216, 0.000982198, \ldots \} \]

The graph of the solutions is given below.

Example 3.3: Consider the following equation 
\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}} \]
which is special case of \( k = 2 \).

If the initial conditions are selected as follows:
\[ x[-5] = 2; x[-4] = 3; x[-3] = 4; x[-2] = 5; x[-1] = 6; x[0] = 7; \]

The following solutions are obtained:
\[ x(n) = \{ 0.333333, 0.428571, 0.53.75, 4, 2, 4.66667, 0.0701754, 0.0824176, 0.0882353, 3.5041, 3.8802, 4.28829, 0.0155804, 0.0168881, 0.016685, 3.45034, 3.81576, 4.21791, 0.00350093, \ldots \} \]
0.00350685, 0.00319765, 3.4383, 4.2044, 0.0007888, 0.000730224, 0.000614404, 3.43559, 3.79965, 4.20189, 0.000177834, 0.000152141, 0.000118112, 3.43498, 3.79907,…}

The graph of the solutions is given below.

**Figure 3.3.** x(n) graph of the solutions

**Example 3.4:** Consider the following equation \( x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}} \) which is special case of \( k = 2 \).

If the initial conditions are selected as follows:

\( x[-5] = 0.1; x[-4] = 0.01; x[-3] = 0.001; x[-2] = 2; x[-1] = 4; x[0] = 0.000001 \)

The following solutions are obtained:

\( x(n) = \{ 0.0333333, 0.002, 0.000999999, 1.93548, 3.99202, 9.99001\times10^{-7}, 0.0113553, 0.00040064, 0.000999998, 1.91375, 3.99042, 9.98003\times10^{-7}, 0.00389714, 0.0000802818, 0.000999997, 1.90632, 3.9901, 9.97006\times10^{-7}, 0.00134092, 0.0000160882, 0.000999996, 1.90377, 3.99003, 9.9601\times10^{-7}, 0.000461785, 3.22407\times10^{-6}, 0.000999995, 1.90289, 3.99002, 9.95015\times10^{-7}, 0.000159078, 6.46104\times10^{-7}, 0.000999994, 1.90259, 3.99002, 9.94021\times10^{-7}, …} \)

The graph of the solutions is given below.
Figure 3.4. $x(n)$ graph of the solutions.

REFERENCES:


[2] Cinar C., On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}$, Appl. Math. Comp., 158 (3), (2004), 809–812.

[3] Cinar C., On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}$, Appl. Math. Comp., 158 (3), (2004), 793–797.

[4] Cinar C., On the positive solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}$, Appl. Math. Comp., 156 (3), (2004), 587–590.

[5] Elabbasy E. M., El-Metwally H., Elsayed E. M., On the difference equation $x_{n+1} = \alpha x_n - \frac{b x_n}{cx_n - dx_{n-1}}$, Advances in Difference Equation, (2006), 1-10.


\[ x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k-1} x_{n-i}} \], J. Conc. Appl. Math., 5(2), (2007), 101-113.


\[ x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\chi + x_n} \], Math. Sci. Res. Hot-Line, 4, 2, (2000), 1-11.

[17] Kulenović M.R.S., Ladas G., Sizer W.S., On the recursive sequence \( x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\chi x_n + \delta x_{n-1}} \)

[18] Stevic S., On the recursive sequence \( x_{n+1} = \frac{x_{n-1}}{g(x_n)} \), Taiwanese J. Math., 6, 3, (2002), 405-414.

[19] Şimşek D., Çınar C. and Yalçınkaya İ., On the recursive sequence \( x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}} \), Int. J. Contemp. Math. Sci., 1, 9-12, (2006), 475-480.

[21] Şimşek D., Çınar C., Karataş R., Yağlınkaya İ., "n the recursive sequence \( x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}} \), Int. J. Pure Appl. Math., 28, 1, (2006), 117-124.

[22] Şimşek D., Çınar C., Yağlınkaya İ., On The Recursive Sequence \( x(n+1) = \frac{x[n-(5k+9)]}{1+x(n-4)x(n-9)} \), Taiwanese Journal of Mathematics, Vol. 12, 5, (2008), 1087-1098.


[24] Şimşek D., Eröz M., Solutions of The Rational Difference Equations \( x_{n+1} = \frac{x_{n-3}}{1+x_{n}x_{n-1}x_{n-2}} \), Manas Journal of Engineering, 4, 1, (2016), 12-20.