Qualitative Studies
ISSN: 1308 724X (NWSAQS)
ID: 2018.13.2.E0041

Status : Original Study
Received: January 2018
Accepted: April 2018

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DOI	http://dx.doi.org/10.12739/NWSA.2018.13.2.E0041
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ON ORDER STATISTICS FROM DISCRETE VARIABLES

ABSTRACT

In this study, joint *pf* and *df* of any *p* order statistics of *innid* discrete random variables are expressed in several form of integral. Also, expressions connecting distributions of order statistics of *innid* discrete random variables to that of order statistics of *iid* discrete random variables are obtained. Finally, some results related to *pf* and *df* of the order statistics are given.

Keywords: Order Statistics, Permanent, Distribution Function, Probability Function, Discrete Random Variable

1. INTRODUCTION

The joint probability density function (*pdf*) and marginal *pdf* of order statistics of independent but not necessarily identically distributed (*innid*) random variables was derived by Vaughan and Venables [22] by means of permanents. In addition, Balakrishnan [3], and Bapat and Beg [8] obtained the joint *pdf* and distribution function (*df*) of order statistics of *innid* random variables by means of permanents. In the first of two papers, Balasubmanian et al. [5] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, ..., X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian et al. [6] generalized their previous results (1991) to the case of the joint distribution function of several order statistics. Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West [10]. Using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n+1)$ was obtained by Childs and Balakrishnan [11] by adding another independent random variable to the original *n* variables $X_1, X_2, ..., X_n$. Also, Balasubramanian, et al., [7] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In a paper published in 1991, Beg [9] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer et al. [13] derived the expressions for the distribution and density functions by Ryser's method and the distribution of maxima and minima based on permanents. A multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution was defined by Corley [12]. Guilbaud [17] expressed the probability of the functions of *innid* random vectors as a linear combination of probabilities of the functions of independent and identically distributed (*iid*) random

How to Cite:
Güngör, M. and Bulut, Y., (2018). On Order Statistics from Discrete Variables, **Qualitative Studies (NWSAQS)**, 13(2):49-56, DOI: 10.12739/NWSA.2018.13.2.E0041.

vectors and thus also for order statistics of random variables. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller [16].

Several identities and recurrence relations for *pdf* and *df* of order statistics of *iid* random variables were established by numerous authors including Arnold et al. [1], Balasubramanian and Beg [4], David [14], and Reiss [21]. Furthermore, Arnold, et al., [1], David [14], Gan and Bain [15], and Khatri [18] obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Balakrishnan [2] showed that several relations and identities that have been derived for order statistics from continuous distributions also hold for the discrete case. In a paper published in 1986, Nagaraja [19] explored the behavior of higher order conditional probabilities of order statistics in a attempt to understand the structure of discrete order statistics. Later, Nagaraja [20] considered some results on order statistics of a random sample taken from a discrete population.

In general, the distribution theory for order statistics is complex when the parent distribution is discrete. In this study, distributions of order statistics of *innid* discrete random variables are obtained.

As far as we know, these approaches have not been considered in the framework of order statistics from *innid* discrete random variables.

From now on, subscripts and superscripts are defined in first place in which they are used and these definitions will be valid unless they are redefined.

If $\mathrm{a}_1,\mathrm{a}_2,\ldots$ are defined as column vectors, then matrix obtained by taking m_1 copies of a_1, m_2 copies of a_2,… can be denoted as $[\underset{m_1}{\mathrm{a}_1}\ \underset{m_2}{\mathrm{a}_2}\ \ldots]$ and *per*A denotes permanent of a square matrix A, which is defined as similar to determinants except that all terms in expansion have a positive sign.

Let $X_1, X_2,\ldots, X_n$ be *innid* discrete random variables and $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be order statistics obtained by arranging the n X_i's in increasing order of magnitude.

Let F_i and f_i be *df* and *pf* of X_i (i=1,2,…,n), respectively. Moreover, $X^s_{1:n}, X^s_{2:n},\ldots, X^s_{n:n}$ are order statistics of *iid* discrete random variables with *pf* f^s and *df* F^s, respectively, defined by

$$f^s = \frac{1}{n_s}\sum_{i\in s} f_i \tag{1}$$

and

$$F^s = \frac{1}{n_s}\sum_{i\in s} F_i\ . \tag{2}$$

Here, s is a subset of the integers {1, 2,…, n} with $n_s \ge 1$ elements.

In this study, *df* and *pf* of $X_{r_1:n}, X_{r_2:n},\ldots, X_{r_p:n}$ ($0 = r_0 < r_1 < r_2 < \ldots < r_p < r_{p+1} = n+1$, p=1, 2,…, n) are given. For notational convenience we write $\sum\sum$, $\sum_{m_p,k_p,\dots,m_1,k_1}$ and $\sum_{z_p,\dots,z_2,z_1}$ instead of

$\sum_{\kappa=1}^{n}(-1)^{n-\kappa}\frac{\kappa^n}{n!}\sum_{n_s=\kappa}$, $\sum_{k_1=0}^{r_1-1}\sum_{m_1=0}^{r_2-r_1-1}\sum_{k_2=0}^{r_2-r_1-1}\sum_{m_2=0}^{r_3-r_2-1}\cdots\sum_{k_p=0}^{r_p-r_{p-1}-1}\sum_{m_p=0}^{n-r_p}$ and $\sum_{z_1=0}^{x_1}\sum_{z_2=z_1}^{x_2}\sum_{z_3=z_2}^{x_3}\cdots\sum_{z_p=z_{p-1}}^{x_p}$ in

the expressions below, respectively $(x_i = 0, 1, 2, \dots)$ $(z_0 = 0)$.

2. RESEARCH SIGNIFICANCE

In general, the distribution theory for order statistics is complex when the parent distribution is discrete. In this study, distributions of order statistics of *innid* discrete random variables are obtained. As far as we know, these approaches have not been considered in the framework of order statistics from *innid* discrete random variables. From now on, subscripts and superscripts are defined in first place in which they are used and these definitions will be valid unless they are redefined.

3. DISTRIBUTIONS OF ORDER STATISTICS FROM DISCRETE VARIABLES

In this section, three expressions related to *pf* of $X_{r_1:n}, X_{r_2:n},\dots, X_{r_p:n}$ are given.

Consider

$$\{X_{r_1:n}=x_1, X_{r_2:n}=x_2,\dots, X_{r_p:n}=x_p\},\quad x_1<x_2<\dots<x_p. \tag{3}$$

Probability function of (3) can be written as

$$f_{r_1,r_2,\dots,r_p:n}(x_1,x_2,\dots,x_p)=P\{X_{r_1:n}=x_1, X_{r_2:n}=x_2,\dots, X_{r_p:n}=x_p\}.$$

Joint *pf* of order statistics of *innid* discrete random variables can be expressed in form of integral by permanent as follows.

$$f_{r_1,r_2,\dots,r_p:n}(x_1,x_2,\dots,x_p)=$$

$$\mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)}\int_{F_{\varsigma_4^{(1)}}(x_2-)}^{F_{\varsigma_4^{(1)}}(x_2)}\cdots\int_{F_{\varsigma_{2p}^{(1)}}(x_p-)}^{F_{\varsigma_{2p}^{(1)}}(x_p)}\left(\prod_{w=1}^{p+1} per[\underset{r_w-r_{w-1}-1}{\mathrm{v}^{(w)}-\mathrm{v}^{(w-1)}}][\varsigma_{2w-1}/.)\right)\prod_{t=1}^{p} per[\underset{1}{\mathrm{dv}^{(t)}}][\varsigma_{2t}/.), \tag{4}$$

where $\mathrm{D}=\prod_{w=1}^{p+1}[(r_w-r_{w-1}-1)!]^{-1}$, $\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}$ denotes sum over $\bigcup_{\ell=1}^{2p}\varsigma_\ell$ for which

$$\varsigma_\upsilon\cap\varsigma_\vartheta=\phi \text{ for } \upsilon\neq\vartheta,\quad \bigcup_{\ell=1}^{2p+1}\varsigma_\ell=\{1,2,\dots,n\},$$

$$v^{(t)}_{\varsigma^{(i_w)}_{2w-1}}=\left[v^{(t)}_{\varsigma^{(1)}_{2t}}-F_{\varsigma^{(1)}_{2t}}(x_t-)\right]\frac{f_{\varsigma^{(i_w)}_{2w-1}}(x_t)}{f_{\varsigma^{(1)}_{2t}}(x_t)}+F_{\varsigma^{(i_w)}_{2w-1}}(x_t-) \text{ and}$$

$$\varsigma_\ell=\begin{cases}\{\varsigma_\ell^{(1)}\}, & \text{if } \ell \text{ even}\\ \{\varsigma_\ell^{(1)},\varsigma_\ell^{(2)},\dots,\varsigma_\ell^{(r_{\frac{\ell+1}{2}}-r_{\frac{\ell-1}{2}}-1)}\}, & \text{if } \ell \text{ odd.}\end{cases}$$

In (4), $\mathrm{v}^{(w)}=(v_1^{(w)},v_2^{(w)},\dots,v_n^{(w)})'$, $\mathrm{dv}^{(w)}=(dv_1^{(w)},dv_2^{(w)},\dots,dv_n^{(w)})'$, $\mathrm{v}^{(0)}=0=(0,0,\dots,0)'$, and $\mathrm{v}^{(p+1)}=1=(1,1,\dots,1)'$ are column vectors. $\mathrm{A}[\varsigma_\ell/.)$ is matrix obtained from A by taking rows whose indices are in ς_ℓ.

Using expansion of permanent in (4), we get

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\mathrm{D}\sum_P \int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \int_{F_{i_{r_2}}(x_2-)}^{F_{i_{r_2}}(x_2)} \cdots \int_{F_{i_{r_p}}(x_p-)}^{F_{i_{r_p}}(x_p)} \left(\prod_{w=1}^{p+1}\prod_{l=r_{w-1}+1}^{r_w-1}[v_{i_l}^{(w)}-v_{i_l}^{(w-1)}]\right)\prod_{t=1}^{p} dv_{i_{r_t}}^{(t)} \quad (5)$$

where $\sum_P$ denotes sum over all $n!$ permutations $(i_1,i_2,\ldots,i_n)$ of (1,2,…,n) and $v_{i_l}^{(w)}=[v_{i_{r_w}}^{(w)}-F_{i_{r_w}}(x_w-)]\dfrac{f_{i_l}(x_w)}{f_{i_{r_w}}(x_w)}+F_{i_l}(x_w-)$.

Furthermore, using expansion of permanent in (4), (4) can also be written as follows.

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}} \int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} \int_{F_{\varsigma_4^{(1)}}(x_2-)}^{F_{\varsigma_4^{(1)}}(x_2)} \cdots \int_{F_{\varsigma_{2p}^{(1)}}(x_p-)}^{F_{\varsigma_{2p}^{(1)}}(x_p)} \prod_{w=1}^{p+1}\prod_{i_w=1}^{r_w-r_{w-1}-1}[v_{\varsigma_{2w-1}^{(i_w)}}^{(w)}-v_{\varsigma_{2w-1}^{(i_w)}}^{(w-1)}]\prod_{t=1}^{p} dv_{\varsigma_{2t}^{(1)}}^{(t)}, \quad (6)$$

where $v_{\varsigma_{2w-1}^{(i_w)}}^{(t)}=\left[v_{\varsigma_{2t}^{(1)}}^{(t)}-F_{\varsigma_{2t}^{(1)}}(x_t-)\right]\dfrac{f_{\varsigma_{2w-1}^{(i_w)}}(x_t)}{f_{\varsigma_{2t}^{(1)}}(x_t)}+F_{\varsigma_{2w-1}^{(i_w)}}(x_t-).$

If $x_1=x_2=\ldots=x_p=x$, it should be written $\iint\ldots\int$ instead of $\int$ in (1) , where $\iint\ldots\int$ is to be carried out over region:

$F_{\varsigma_2^{(1)}}(x-)\le v_{\varsigma_2^{(1)}}^{(1)}\le v_{\varsigma_4^{(1)}}^{(2)}\le\ldots\le v_{\varsigma_{2p}^{(1)}}^{(p)}\le F_{\varsigma_{2p}^{(1)}}(x)$, $F_{\varsigma_2^{(1)}}(x-)\le v_{\varsigma_2^{(1)}}^{(1)}\le F_{\varsigma_2^{(1)}}(x)$,
$F_{\varsigma_4^{(1)}}(x-)\le v_{\varsigma_4^{(1)}}^{(2)}\le F_{\varsigma_4^{(1)}}(x)$, . . . , $F_{\varsigma_{2p}^{(1)}}(x-)\le v_{\varsigma_{2p}^{(1)}}^{(p)}\le F_{\varsigma_{2p}^{(1)}}(x)$.

Moreover, if $x_1\le x_2\le\ldots\le x_p$, it should be written $\iint\ldots\int$ instead of $\int$ in (1) , where $\iint\ldots\int$ is to be carried out over region:

$v_{\varsigma_2^{(1)}}^{(1)}\le v_{\varsigma_4^{(1)}}^{(2)}\le\ldots\le v_{\varsigma_{2p}^{(1)}}^{(p)}$, $F_{\varsigma_2^{(1)}}(x_1-)\le v_{\varsigma_2^{(1)}}^{(1)}\le F_{\varsigma_2^{(1)}}(x_1)$, $F_{\varsigma_4^{(1)}}(x_2-)\le v_{\varsigma_4^{(1)}}^{(2)}\le F_{\varsigma_4^{(1)}}(x_2)$, . . . ,
$F_{\varsigma_{2p}^{(1)}}(x_p-)\le v_{\varsigma_{2p}^{(1)}}^{(p)}\le F_{\varsigma_{2p}^{(1)}}(x_p)$.

In general, the identities in (4)-(6) for *pf* and *df* of order statistics of *innid* discrete random variables are complicated.

However, *df* and *pf* of order statistics of *innid* discrete random variables can be obtained easily from the identities in the following theorems.

Furthermore, the following theorems connect *pf* and *df* of order statistics of *innid* discrete random variables to that of order statistics of *iid* discrete random variables using (1) and (2).

Theorem 3.1.

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum\sum n!\mathrm{D}\iint\ldots\int\left(\prod_{w=1}^{p+1}[v^{(s,w)}-v^{(s,w-1)}]^{r_w-r_{w-1}-1}\right)\prod_{t=1}^{p} dv^{(s,t)} \quad (7)$$

$x_1<x_2<\ldots<x_p$, where $v^{(s,0)}=0$, $v^{(s,p+1)}=1$ and $\iint\ldots\int$ is to be carried out over region: $F^s(x_1-)\le v^{(s,1)}\le F^s(x_1)$, $F^s(x_2-)\le v^{(s,2)}\le F^s(x_2)$, . . . , $F^s(x_p-)\le v^{(s,p)}\le F^s(x_p)$.

Proof. Consider

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=P\{X_{r_1:n}=x_1,X_{r_2:n}=x_2,\ldots,X_{r_p:n}=x_p\}$$

$$=\sum\sum P\{X^s_{r_1:n}=x_1,X^s_{r_2:n}=x_2,\ldots,X^s_{r_p:n}=x_p\}\ . \qquad (8)$$

In (4), (5) and (6), it is taken into account (8). Thus, (7) is obtained.

If $x_1=x_2=\ldots=x_p=x$, $\iint\ldots\int$ in (7) is to be carried out over region: $F^s(x-)\le v^{(s,1)}\le v^{(s,2)}\le\ldots\le v^{(s,p)}\le F^s(x)$, $F^s(x-)\le v^{(s,1)}\le F^s(x)$, $F^s(x-)\le v^{(s,2)}\le F^s(x)$, . . ., $F^s(x-)\le v^{(s,p)}\le F^s(x)$.

Moreover, if $x_1\le x_2\le\ldots\le x_p$, $\iint\ldots\int$ in (7) is to be carried out over region: $v^{(s,1)}\le v^{(s,2)}\le\ldots\le v^{(s,p)}$, $F^s(x_1-)\le v^{(s,1)}\le F^s(x_1)$, $F^s(x_2-)\le v^{(s,2)}\le F^s(x_2)$, . . ., $F^s(x_p-)\le v^{(s,p)}\le F^s(x_p)$.

Theorem 3.2.

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum\sum n!\mathrm{D}\int_0^{F^s(x_1)}\int_{v^{(s,1)}}^{F^s(x_2)}\cdots\int_{v^{(s,p-1)}}^{F^s(x_p)}\left(\prod_{w=1}^{p+1}[v^{(s,w)}-v^{(s,w-1)}]^{r_w-r_{w-1}-1}\right)\prod_{t=1}^{p}dv^{(s,t)} \qquad (9)$$

Proof. It can be written

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{z_p,\ldots,z_2,z_1} f_{r_1,r_2,\ldots,r_p:n}(z_1,z_2,\ldots,z_p)\ .$$

The above identity can be expressed as

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=$$

$$\sum_{z_p,\ldots,z_2,z_1}\sum\sum n!\mathrm{D}\int_{F^s(z_1-)}^{F^s(z_1)}\int_{F^s(z_2-)}^{F^s(z_2)}\cdots\int_{F^s(z_p-)}^{F^s(z_p)}\left(\prod_{w=1}^{p+1}[v^{(s,w)}-v^{(s,w-1)}]^{r_w-r_{w-1}-1}\right)\prod_{t=1}^{p}dv^{(s,t)}$$

Thus, the proof is completed.

4. RESULTS

In this section, results related to *pf* and *df* of $X^s_{r_1:n},X^s_{r_2:n},\ldots,X^s_{r_p:n}$ are given. We express following result for *pf* of *r*th order statistic of *innid* discrete random variables.

Result 4.1.

$$f_{r_1:n}(x_1)=\sum\sum\frac{n!}{(r_1-1)!(n-r_1)!}\int_{F^s(x_1-)}^{F^s(x_1)}[v^{(s,1)}]^{r_1-1}[1-v^{(s,1)}]^{n-r_1}dv^{(s,1)} \qquad (10)$$

Proof. In (7), if $p=1$, (10) is obtained.

Specially, in (10), by taking $n=2$, $p=1$ and $r_1=2$, the following identity is obtained.

$$f_{2:2}(x_1)=\sum\sum 2\int_{F^S(x_1-)}^{F^S(x_1)}v^{(s,1)}dv^{(s,1)}=\sum\sum\left(v^{(s,1)}\right)^2\Big|_{F^S(x_1-)}^{F^S(x_1)}=\sum\sum[\left(F^S(x_1)\right)^2-\left(F^S(x_1-)\right)^2]$$

$$=-\frac{1}{2}\sum_{n_S=1}[\left(F^S(x_1)\right)^2-\left(F^S(x_1-)\right)^2]+2\sum_{n_S=2}[\left(F^S(x_1)\right)^2-\left(F^S(x_1-)\right)^2]$$

$$=-\frac{1}{2}\{[\left(F_1(x_1)\right)^2-\left(F_1(x_1-)\right)^2]+[\left(F_2(x_1)\right)^2-\left(F_2(x_1-)\right)^2\}$$

$$+2\{\left(\frac{F_1(x_1)+F_2(x_1)}{2}\right)^2-\left(\frac{F_1(x_1-)+F_2(x_1-)}{2}\right)^2\}$$

$$=-\frac{1}{2}\{[F_1(x_1)-F_1(x_1-)]f_1(x_1)+[F_2(x_1)-F_2(x_1-)]f_2(x_1)\}$$

$$+\frac{1}{2}\{[F_1(x_1)+F_2(x_1)+F_1(x_1-)+F_2(x_1-)][f_1(x_1)+f_2(x_1)]\}$$

$$=\frac{1}{2}\{F_1(x_1)f_2(x_1)+F_2(x_1)f_1(x_1)+F_1(x_1-)f_2(x_1)+F_2(x_1-)f_1(x_1)\}$$

$$=\frac{1}{2}\{[2F_1(x_1)-f_1(x_1)]f_2(x_1)+[2F_2(x_1)-f_2(x_1)]f_1(x_1)\}$$

$$=F_1(x_1)f_2(x_1)+F_2(x_1)f_1(x_1)-f_1(x_1)f_2(x_1)\,.$$

In Result 4.2-4.3, *pf* of minimum and maximum order statistics of *innid* discrete random variables are given, respectively.

Result 4.2.

$$f_{1:n}(x_1)=\sum\sum n\int_{F^s(x_1-)}^{F^s(x_1)}[1-v^{(s,1)}]^{n-1}dv^{(s,1)} \tag{11}$$

Proof. In (10), if $r_1=1$, (11) is obtained.

Result 4.3.

$$f_{n:n}(x_1)=\sum\sum n\int_{F^s(x_1-)}^{F^s(x_1)}[v^{(s,1)}]^{n-1}dv^{(s,1)} \tag{12}$$

Proof. In (10), if $r_1=n$, (12) is obtained.

In the following result, we express joint *pf* of $X^s_{1:n}, X^s_{2:n},\ldots,X^s_{p:n}$.

Result 4.4.

$$f_{1,2,\ldots,p:n}(x_1,x_2,\ldots,x_p)=\sum\sum\frac{n!}{(n-p)!}\iint\ldots\int[1-v^{(s,p)}]^{n-p}\prod_{t=1}^{p}dv^{(s,t)} \tag{13}$$

$x_1\le x_2\le\ldots\le x_p$, where $\iint\ldots\int$ is to be carried out over region: $v^{(s,1)}\le v^{(s,2)}\le\ldots\le v^{(s,p)}$, $F^s(x_1-)\le v^{(s,1)}\le F^s(x_1)$, $F^s(x_2-)\le v^{(s,2)}\le F^s(x_2)$, . . . , $F^s(x_p-)\le v^{(s,p)}\le F^s(x_p)$.

Proof. In (8), if $r_1=1, r_2=2,\ldots,r_p=p,$ (13) is obtained.

We now give three results for *df* of single order statistic of *innid* discrete random variables.

Result 4.5.

$$F_{r_1:n}(x_1)=\sum\sum\frac{n!}{(r_1-1)!(n-r_1)!}\int_0^{F^s(x_1)}[v^{(s,1)}]^{r_1-1}[1-v^{(s,1)}]^{n-r_1}dv^{(s,1)} \tag{14}$$

Proof. In (9), if $p=1$, (14) is obtained.

Result 4.6.

$$F_{1:n}(x_1)=\sum\sum n\int_0^{F^s(x_1)}[1-v^{(s,1)}]^{n-1}dv^{(s,1)} \tag{15}$$

Proof. In (14), if $r_1=1$, (15) is obtained.

Result 4.7.

$$F_{n:n}(x_1)=\sum\sum n\int_0^{F^s(x_1)}[v^{(s,1)}]^{n-1}dv^{(s,1)} \tag{16}$$

Proof. In (14), if $r_1=n$, (16) is obtained.

In the following result, we express joint *df* of $X^s_{1:n},X^s_{2:n},\dots,X^s_{p:n}$.

Result 4.8.

$$F_{1,2,\dots,p:n}(x_1,x_2,\dots,x_p)=\sum\sum\frac{n!}{(n-p)!}\int_0^{F^s(x_1)}\int_{v^{(s,1)}}^{F^s(x_2)}\dots\int_{v^{(s,p-1)}}^{F^s(x_p)}[1-v^{(s,p)}]^{n-p}\prod_{t=1}^{p}dv^{(s,t)} \tag{17}$$

Proof. In (9), if $r_1=1,\ r_2=2,\dots,r_p=p$, (17) is obtained.

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