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Solutions of The Rational Difference Equations $x_{n+1} = \frac{x_{n-3}}{1+x_n x_{n-1} x_{n-2}}$

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Abstract: *In this paper the solutions of the following difference equation is examined,*

$$x_{n+1} = \frac{x_{n-3}}{1+x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,... \tag{1}$$

where the initial conditions are positive real numbers.

Keywords: *Difference Equation, Period Four Solution*

$x_{n+1} = \frac{x_{n-3}}{1+x_n x_{n-1} x_{n-2}}$ Rasyonel Fark Denkleminin Çözümleri

Öz: *Bu çalışmada aşağıdaki fark denkleminin çözümleri incelenmiştir,*

$$x_{n+1} = \frac{x_{n-3}}{1+x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,... \tag{1}$$

Burada başlangıç şartları pozitif reel sayılardır.

Anahtar Kelimeler: *Fark Denklemi, Dört Periyotlu Çözüm*

INTRODUCTION

Recently there has been a lot of interest in studying the periodic nature of non-linear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, [1-25].
Cinar, studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}$$

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}$$

for *n=0,1,…,* in [2,3,4], respectively.

In [18] Stevic assumed that $\beta = 1$ and solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n} \text{ for } n = 0,1,2, \ldots$$

where $x_{-1}, x_0 \in (0,\infty)$. Also**,** this results was generalized to the equation of the following form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \text{ for } n = 0,1,2, \ldots$$

where $x_{-1}, x_0 \in (0,\infty)$.

Simsek et. al., studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1} x_{n-3}}$$

for *n=0,1,…,* in [19,20,21] respectively.

In this paper we investigated the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad \text{n=0,1,2,...} \qquad (1)$$

where $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\infty)$.

MAİN RESULT

Let $\bar{x}$ be the unique positive equilibrum of Eq. (1), then clearly

$$\bar{x}=\frac{\bar{x}}{1+\bar{x}\bar{x}\bar{x}}\Rightarrow \bar{x}+\bar{x}^{4}=\bar{x}\Rightarrow \bar{x}^{4}=0\Rightarrow \bar{x}=0$$

We can obtain $\bar{x}=0$.

Theorem 1. Consider the difference equation (1). Then the following statements are true.

a) The sequences $(x_{4n-3}),\ (x_{4n-2}),\ (x_{4n-1}),$ and (x_{4n}) are decreasing and there exist $p.q.r.s\geq 0$ such that

$$\lim_{n\to\infty} x_{4n-3}=p,\ \lim_{n\to\infty} x_{4n-2}=q,\ \lim_{n\to\infty} x_{4n-1}=r \text{ and } \lim_{n\to\infty} x_{4n}=s.$$

b) $(p,q,r,s,p,q,r,s,\ldots)$ is a solution of equation (1) of period four.

c) $p.q.r.s=0$.

d) If there exist $n_0\in N$ such that $x_n x_{n-1}x_{n-2}\geq x_{n+1}x_n x_{n-1}$ for all $n\geq n_0$, then

$$\lim_{n\to\infty} x_n=0.$$

e) The following formulas hold:

$$x_{4n+1}=x_{-3}\left(1-\frac{x_0x_{-1}x_{-2}}{1+x_0x_{-1}x_{-2}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{i-2}x_{i-1}x_i}\right)$$

$$x_{4n+2}=x_{-2}\left(1-\frac{x_0x_{-1}x_{-3}}{1+x_0x_{-1}x_{-2}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{i-2}x_{i-1}x_i}\right)$$

$$x_{4n+3}=x_{-1}\left(1-\frac{x_0x_{-2}x_{-3}}{1+x_0x_{-1}x_{-2}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{i-2}x_{i-1}x_i}\right)$$

$$x_{4n+4}=x_{0}\left(1-\frac{x_{-1}x_{-2}x_{-3}}{1+x_0x_{-1}x_{-2}}\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{i-2}x_{i-1}x_i}\right).$$

f) If $x_{4n+1}\to p\neq 0,\ x_{4n+2}\to q\neq 0$ and $x_{4n+3}\to r\neq 0$ then $x_{4n+1}\to 0$ as $n\to\infty$.

Proof. a) Firstly, we consider the equation (1). From this equation we obtain

$$x_{n+1}(1+x_n x_{n-1}x_{n-2})=x_{n-3}.$$

If $x_n,x_{n-1},x_{n-2}\in(0,+\infty)$, then $(1+x_n x_{n-1}x_{n-2})\in(1,+\infty)$. Since $x_{n+1}<x_{n-3},\ n\in N$, we obtain that $\lim_{n\to\infty} x_{4n-3}=p,\ \lim_{n\to\infty} x_{4n-2}=q,\ \lim_{n\to\infty} x_{4n-1}=r$ and $\lim_{n\to\infty} x_{4n}=s.$

b) $(p,q,r,s,p,q,r,s,\ldots)$ is a solution of equation (1) of period four.

c) In view of the equation (1), we obtain

$$x_{4n+1}=\frac{x_{4n-3}}{1+x_{4n}x_{4n-1}x_{4n-2}}.$$

Taking limit as $n\to\infty$ on both sides of the above equality, we get

$$\lim_{n\to\infty} x_{4n+1}=\lim_{n\to\infty}\frac{x_{4n-3}}{1+x_{4n}x_{4n-1}x_{4n-2}}.$$

Then

$$p=\frac{p}{1+s.r.q}\Rightarrow p+p.q.r.s=p\Rightarrow p.q.r.s=0.$$

d) If there exist $n_0 \in N$ such that $x_n x_{n-1} x_{n-2} \geq x_{n+1} x_n x_{n-1}$ for all $n \geq n_0$, then $p \leq q \leq r \leq s \leq p$. Since $p.q.r.s = 0$ we obtain the result.

e) Subracting x_{n-3} from the left and right-hand sides of equation (1) we obtain

$$x_{n+1} - x_{n-3} = \frac{1}{1 + x_n x_{n-1} x_{n-2}} (x_n - x_{n-4})$$

and the following formula

$$n \geq 1 \text{ for } \left\{ x_n - x_{n-4} = (x_1 - x_{-3}) \prod_{i=1}^{n-1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right. \quad (2)$$

holds. Replacing *n* by *4j* in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+1} - x_{-3} = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \ldots). \quad (3)$$

Also, replacing *n* by *4j*+1 in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+2} - x_{-2} = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \ldots). \quad (4)$$

Also, replacing *n* by *4j*+2 in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+3} - x_{-1} = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \ldots). \quad (5)$$

Also, replacing *n* by *4j*+2 in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+4} - x_0 = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \ldots). \quad (6)$$

From the formulas above, we obtain

$$x_{4n+1} = x_{-3} \left(1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \quad (7)$$

$$x_{4n+2} = x_{-2} \left(1 - \frac{x_0 x_{-1} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \quad (8)$$

$$x_{4n+3} = x_{-1} \left(1 - \frac{x_0 x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \quad (9)$$

$$x_{4n+4} = x_0 \left(1 - \frac{x_{-1} x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right). \quad (10)$$

f) Suppose that $p = q = r = s = 0$. By **e)** we have

$$\lim_{n \to \infty} x_{4n+1} = \lim_{n \to \infty} x_{-3} \left(1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right)$$

$$p = x_{-3} \left(1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right)$$

$$p = 0 \Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i}. \quad (11)$$

Similarly,

$$q = x_{-2} \left(1 - \frac{x_0 x_{-1} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right)$$

$$q = 0 \Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i}. \quad (12)$$

Similarly,

$$r = x_{-1}\left(1-\frac{x_0x_{-2}x_{-3}}{1+x_0x_{-1}x_{-2}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j+2}\frac{1}{1+x_{i-2}x_{i-1}x_i}\right)$$

$$r = 0 \Rightarrow \frac{1+x_0x_{-1}x_{-2}}{x_0x_{-2}x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+2}\frac{1}{1+x_{i-2}x_{i-1}x_i}. \quad (13)$$

Similarly,

$$s = x_0\left(1-\frac{x_{-1}x_{-2}x_{-3}}{1+x_0x_{-1}x_{-2}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j+3}\frac{1}{1+x_{i-2}x_{i-1}x_i}\right)$$

$$s = 0 \Rightarrow \frac{1+x_0x_{-1}x_{-2}}{x_{-1}x_{-2}x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+3}\frac{1}{1+x_{i-2}x_{i-1}x_i}. \quad (14)$$

From the equations (11) and (12),

$$\frac{1+x_0x_{-1}x_{-2}}{x_0x_{-1}x_{-2}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1+x_{i-2}x_{i-1}x_i} > \frac{1+x_0x_{-1}x_{-2}}{x_0x_{-1}x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{i-2}x_{i-1}x_i} \quad (15)$$

thus, $x_{-3} > x_{-2}$.

From the equations (12) and (13),

$$\frac{1+x_0x_{-1}x_{-2}}{x_0x_{-1}x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{i-2}x_{i-1}x_i} > \frac{1+x_0x_{-1}x_{-2}}{x_0x_{-2}x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+2}\frac{1}{1+x_{i-2}x_{i-1}x_i} \quad (16)$$

thus, $x_{-2} > x_{-1}$.

From the equations (13) and (14),

$$\frac{1+x_0x_{-1}x_{-2}}{x_0x_{-2}x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+2}\frac{1}{1+x_{i-2}x_{i-1}x_i} > \frac{1+x_0x_{-1}x_{-2}}{x_{-1}x_{-2}x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+3}\frac{1}{1+x_{i-2}x_{i-1}x_i} \quad (17)$$

thus, $x_{-1} > x_0$.

From here we obtain $x_{-3} > x_{-2} > x_{-1} > x_0$. We arrive at a contradiction which completes the proof of theorem.

EXAMPLES

Example 1: If the initial conditions are selected as follows:

$x[-3]=2;x[-2]=3;x[-1]=4;x[0]=5;$

The following solutions are obtained:

x(n)={ 0.0327869, 1.81188, 3.08397, 4.22581, 0.0013321, 1.78096, 3.05336, 4.19542, 0.000055937, 1.77969, 3.05208, 4.19414, $2.35212\text{x}10^{-6}$, 1.77963, 3.05203, 4.19409, $9.89108\text{x}10^{-8}$, 1.77963, 3.05203, 4.19409, $4.15939\text{x}10^{-9}$, 1.77963, 3.05203, 4.19409, $1.7491\text{x}10^{-10}$, 1.77963, 3.05203, 4.19409, $7.35532\text{x}10^{-12}$, 1.77963, 3.05203, 4.19409, $3.09306\text{x}10^{-13}$,1.77963,3.05203, 4.19409,...}

The graph of the solutions is given below.

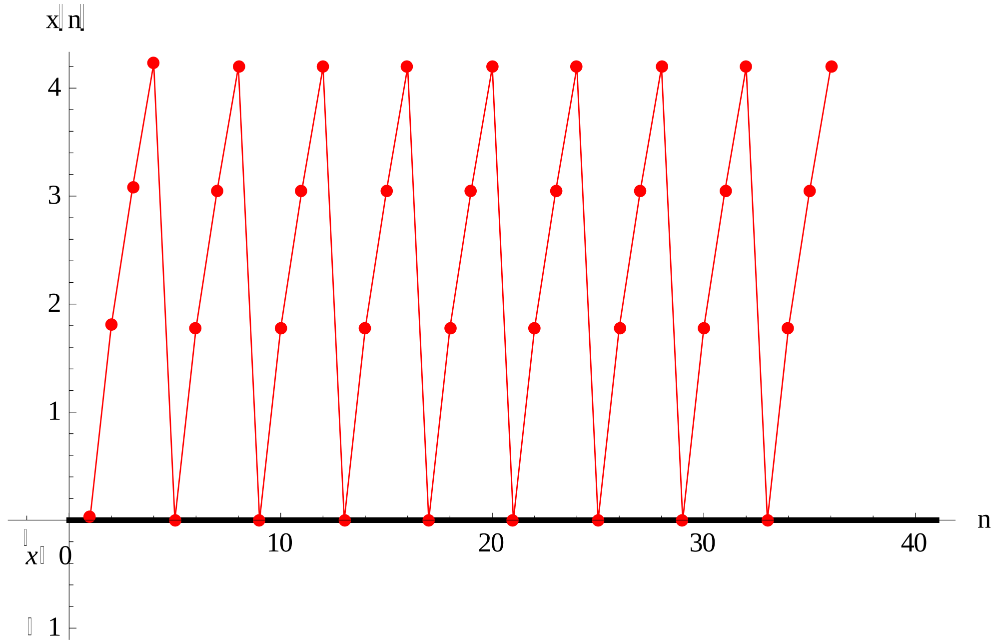

Figure 3.1. x(n) graph of the solutions

Example 2: If the initial conditions are selected as follows:

$$x[-3]=5;x[-2]=4;x[-1]=3;x[0]=2;$$

The following solutions are obtained:

x(n)={0.2, 1.81818, 1.73684, 1.22581, 0.0410596, 1.67202, 1.60202, 1.10435, 0.0103735, 1.64189, 1.57245, 1.07554, 0.00274663, 1.63429, 1.56489, 1.06804, 0.000736065, 1.63229, 1.56289, 1.06604, 0.000197891, 1.63175,1.56235, 1.0655, 0.0000532489, 1.6316, 1.5622, 1.06536, 0.0000143316, 1.63156,1.56217, 1.06532,3.85751x10^{-6}, 1.63155, 1.56216, 1.06531, … }

The graph of the solutions is given below.

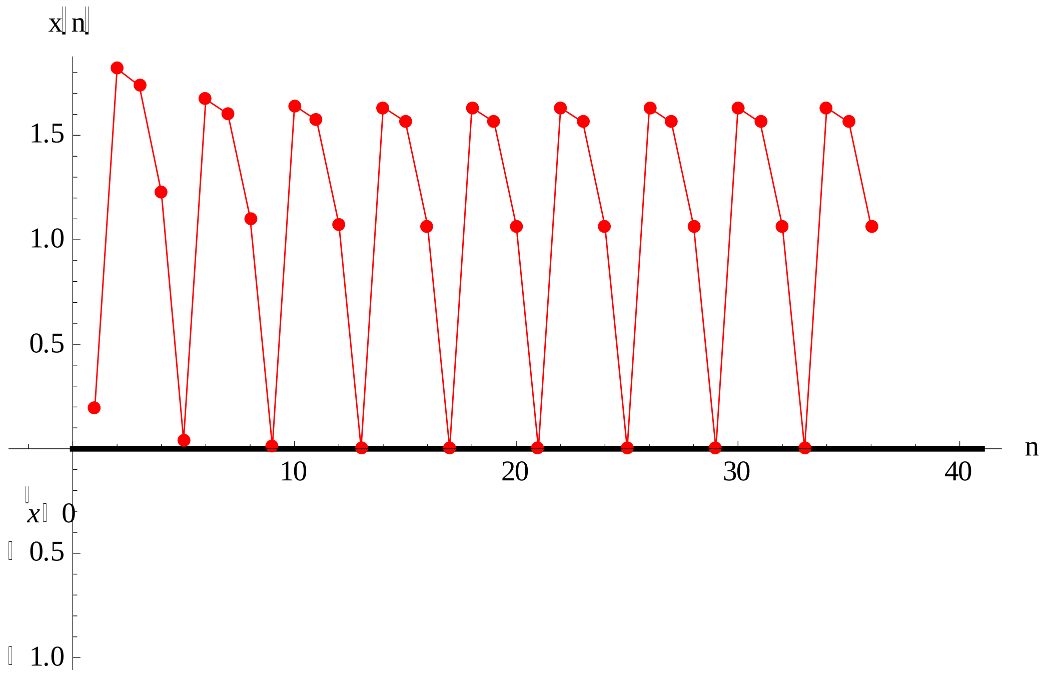

Figure 3.2. x(n) graph of the solutions

Example 3: If the initial conditions are selected as follows:

$$x[-3]=2;x[-2]=0.1;x[-1]=0.01;x[0]=0.001;$$

The following solutions are obtained:

x(n)={2, 0.099998, 0.009998, 0.000998004, 2, 0.099996, 0.00999601, 0.000996013, 1.99999, 0.099994, 0.00999401, 0.000994027, 1.99999, 0.099992, 0.00999203, 0.000992044, 1.99999, 0.09999, 0.00999005, 0.000990066, 1.99999, 0.0999881, 0.00998807, 0.000988093, 1.99999, 0.0999861, 0.0099861, 0.000986123, 1.99998, 0.0999841, 0.00998413, 0.000984159, 1.99998, 0.0999822, 0.00998216, 0.000982198, …}

The graph of the solutions is given below.

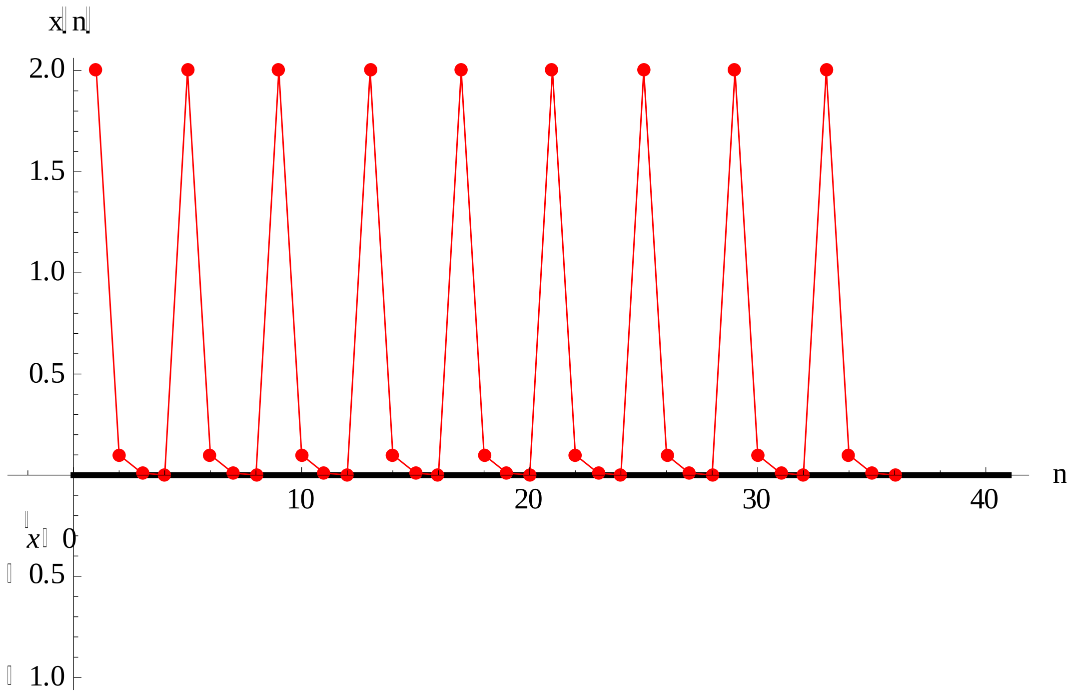

Figure 3.3. x(n) graph of the solutions

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