Solutions of The Rational Difference Equations

\[ x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,\ldots \]  \hspace{1cm} (1)

Abstract: In this paper the solutions of the following difference equation is examined,

\[ x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,\ldots \]  \hspace{1cm} (1)

where the initial conditions are positive real numbers.

Keywords: Difference Equation, Period Four Solution

Anahtar Kelimeler: Fark Denklemi, Dört Periyotlu Çözüm

Öz: Bu çalışmada aşağıdaki fark denkleminin çözümleri incelenmiştir,

\[ x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,\ldots \]  \hspace{1cm} (1)

Burada başlangıç şartları pozitif reel sayılardır.
INTRODUCTION

Recently there has been a lot of interest in studying the periodic nature of non-linear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, [1-25].

Cinar, studied the following problems with positive initial values

\[ x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}} \]
\[ x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}} \]
\[ x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}} \]

for \( n=0,1,\ldots \), in [2,3,4], respectively.

In [18] Stevic assumed that \( \beta = 1 \) and solved the following problem

\[ x_{n+1} = \frac{x_{n-1}}{1 + x_n} \] for \( n=0,1,2,\ldots \)

where \( x_{-1}, x_0 \in (0,\infty) \). Also, this results was generalized to the equation of the following form:

\[ x_{n+1} = \frac{x_{n-1}}{g(x_n)} \] for \( n=0,1,2,\ldots \)

where \( x_{-1}, x_0 \in (0,\infty) \).

Simsek et. al., studied the following problems with positive initial values

\[ x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}} \]
\[ x_{n+1} = \frac{x_{n-3}}{1 + x_{n-2}} \]
\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1} x_{n-3}} \]

for \( n=0,1,\ldots \), in [19,20,21] respectively.

In this paper we investigated the following nonlinear difference equation

\[ x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}} \] for \( n=0,1,2,\ldots \) \hspace{1cm} (1)

where \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\infty) \).
**MAIN RESULT**

Let $\bar{x}$ be the unique positive equilibrium of Eq. (1), then clearly

$$
\bar{x} = \frac{\bar{x}}{1 + \bar{x} \bar{x}} \Rightarrow \bar{x} + \bar{x} = \bar{x} \Rightarrow \bar{x} = 0
$$

We can obtain $\bar{x} = 0$.

**Theorem 1.** Consider the difference equation (1). Then the following statements are true.

a) The sequences $(x_{4n-3})$, $(x_{4n-2})$, $(x_{4n-1})$, and $(x_{4n})$ are decreasing and there exist $p,q,r,s \geq 0$ such that

$$
\lim_{n \to \infty} x_{4n-3} = p, \quad \lim_{n \to \infty} x_{4n-2} = q, \quad \lim_{n \to \infty} x_{4n-1} = r \quad \text{and} \quad \lim_{n \to \infty} x_{4n} = s.
$$

b) $(p,q,r,s,p,q,r,s,...)$ is a solution of equation (1) of period four.

c) $p.q.r.s = 0$.

d) If there exist $n_0 \in \mathbb{N}$ such that $x_{n_0}x_{n_0-1}x_{n_0-2} \geq x_{n_0+1}x_{n_0+2}x_{n_0+3}$ for all $n \geq n_0$, then

$$
\lim_{n \to \infty} x_n = 0.
$$

e) The following formulas hold:

$$
\begin{align*}
x_{4n+1} &= x_{n-3} \left( 1 - \frac{x_0 x_{n-1} x_{n-2}}{1 + x_0 x_{n-1} x_{n-2}} \sum_{j=0}^{n-1} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right) \\
x_{4n+2} &= x_{n-2} \left( 1 - \frac{x_0 x_{n-1} x_{n-2}}{1 + x_0 x_{n-1} x_{n-2}} \sum_{j=0}^{n-2} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right) \\
x_{4n+3} &= x_{n-1} \left( 1 - \frac{x_0 x_{n-2} x_{n-3}}{1 + x_0 x_{n-2} x_{n-3}} \sum_{j=0}^{n-3} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right) \\
x_{4n+4} &= x_0 \left( 1 - \frac{x_1 x_{n-2} x_{n-3}}{1 + x_1 x_{n-2} x_{n-3}} \sum_{j=0}^{n-4} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right).
\end{align*}
$$

f) If $x_{4n+1} \to p \neq 0$, $x_{4n+2} \to q \neq 0$ and $x_{4n+3} \to r \neq 0$ then $x_{4n+4} \to 0$ as $n \to \infty$.

**Proof.**
a) Firstly, we consider the equation (1). From this equation we obtain

$$
x_{n+1}(1 + x_n x_{n-1} x_{n-2}) = x_{n-3}.
$$

If $x_n, x_{n-1}, x_{n-2} \in (0, +\infty)$, then $(1 + x_n x_{n-1} x_{n-2}) \in (1, +\infty)$. Since $x_{n+1} < x_{n-3}$, $n \in \mathbb{N}$, we obtain that

$$
\lim_{n \to \infty} x_{4n-3} = p, \quad \lim_{n \to \infty} x_{4n-2} = q, \quad \lim_{n \to \infty} x_{4n-1} = r \quad \text{and} \quad \lim_{n \to \infty} x_{4n} = s.
$$

b) $(p,q,r,s,p,q,r,s,...)$ is a solution of equation (1) of period four.

c) In view of the equation (1), we obtain

$$
x_{4n+1} = \frac{x_{4n-3}}{1 + x_0 x_{4n-1} x_{4n-2}}.
$$

Taking limit as $n \to \infty$ on both sides of the above equality, we get

$$
\lim_{n \to \infty} x_{4n+1} = \lim_{n \to \infty} \frac{x_{4n-3}}{1 + x_0 x_{4n-1} x_{4n-2}}.
$$

Then

$$
p = \frac{p}{1 + s r q} \Rightarrow p + p q r s = p \Rightarrow p q r s = 0.
$$
d) If there exist \( n_0 \in \mathbb{N} \) such that \( x_n x_{n-1} x_{n-2} \geq x_{n+1} x_{n-1} x_{n-2} \) for all \( n \geq n_0 \), then \( p \leq q \leq r \leq s \leq p \). Since \( p, q, r, s = 0 \) we obtain the result.

e) Subtracting \( x_{n-3} \) from the left and right-hand sides of equation (1) we obtain
\[
x_{n+1} - x_{n-3} = \frac{1}{1 + x_n x_{n-1} x_{n-2}} (x_n - x_{n-4})
\]
and the following formula
\[
n \geq 1 \text{ for } x_n - x_{n-4} = (x_1 - x_{-3}) \prod_{i=1}^{n-1} \frac{1}{1 + x_{j-2} x_{j-1} x_j}
\]
holds. Replacing \( n \) by \( 4j \) in (2) and summing from \( j = 0 \) to \( j = n \) we obtain
\[
x_{4n+1} - x_{-3} = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \quad (n = 0, 1, 2, \ldots).
\]
Also, replacing \( n \) by \( 4j+1 \) in (2) and summing from \( j = 0 \) to \( j = n \) we obtain
\[
x_{4n+2} - x_{-2} = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \quad (n = 0, 1, 2, \ldots).
\]
Also, replacing \( n \) by \( 4j+2 \) in (2) and summing from \( j = 0 \) to \( j = n \) we obtain
\[
x_{4n+3} - x_{-1} = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \quad (n = 0, 1, 2, \ldots).
\]
Also, replacing \( n \) by \( 4j+2 \) in (2) and summing from \( j = 0 \) to \( j = n \) we obtain
\[
x_{4n+4} - x_0 = (x_1 - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \quad (n = 0, 1, 2, \ldots).
\]
From the formulas above, we obtain
\[
x_{4n+1} = x_3 \left( 1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right)
\]
\[
x_{4n+2} = x_2 \left( 1 - \frac{x_0 x_{-1} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right)
\]
\[
x_{4n+3} = x_{-1} \left( 1 - \frac{x_0 x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right)
\]
\[
x_{4n+4} = x_0 \left( 1 - \frac{x_{-1} x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{n} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right)
\]
f) Suppose that \( p = q = r = s = 0 \). By e) we have
\[
\lim_{n \to \infty} x_{4n+1} = \lim_{n \to \infty} x_3 \left( 1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right)
\]
\[
p = x_3 \left( 1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \prod_{j=0}^{\infty} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right)
\]
\[
p = 0 \Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j} \frac{1}{1 + x_{j-2} x_{j-1} x_j}.
\]
Similarly,
\[
q = x_2 \left( 1 - \frac{x_0 x_{-1} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{j-2} x_{j-1} x_j} \right)
\]
\[
q = 0 \Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{j-2} x_{j-1} x_j}.
\]
Similarly,

\[
\begin{align*}
    r &= x_{-1} \left( 1 - \frac{x_0 x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \right) \sum_{j=0}^{\infty} \prod_{i=1}^{j} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} \\
    r &= 0 \Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j+2} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} .
\end{align*}
\]

(13)

Similarly,

\[
\begin{align*}
    s &= x_0 \left( 1 - \frac{x_{-1} x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \right) \sum_{j=0}^{\infty} \prod_{i=1}^{j+2} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} \\
    s &= 0 \Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_{-1} x_{-2} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j+3} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} .
\end{align*}
\]

(14)

From the equations (11) and (12),

\[
\frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} > \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j+1} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} .
\]

(15)

thus, \( x_{-3} > x_{-2} \).

From the equations (12) and (13),

\[
\frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j+1} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} > \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j+2} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} .
\]

(16)

thus, \( x_{-2} > x_{-1} \).

From the equations (13) and (14),

\[
\frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j+2} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} > \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{j+3} \frac{1}{1 + x_{-2-j} x_{-1-j} x_i} .
\]

(17)

thus, \( x_{-1} > x_0 \).

From here we obtain \( x_{-3} > x_{-2} > x_{-1} > x_0 \). We arrive at a contradiction which completes the proof of theorem.

**EXAMPLES**

**Example 1:** If the initial conditions are selected as follows:

\[
x[-3]=2; x[-2]=3; x[-1]=4; x[0]=5;
\]

The following solutions are obtained:

\[
x(n) = \{ 0.0327869, 1.81188, 3.08397, 4.22581, 0.0013321, 1.78096, 3.05336, 4.19542, 0.000055937, 1.77969, 3.05208, 4.19414, 2.35212 \times 10^{-6}, 1.77963, 3.05203, 4.19409, 9.89108 \times 10^{-8}, 1.77963, 3.05203, 4.19409, 4.15939 \times 10^{-9}, 1.77963, 3.05203, 4.19409, 1.7491 \times 10^{-10}, 1.77963, 3.05203, 4.19409, 3.09306 \times 10^{-13}, 1.77963, 3.05203, 4.19409, \ldots \}
\]

The graph of the solutions is given below.
Example 2: If the initial conditions are selected as follows:

\[ x[-3]=5; x[-2]=4; x[-1]=3; x[0]=2; \]

The following solutions are obtained:

\[ x(n) = \{0.2, 1.81818, 1.73684, 1.22581, 0.0410596, 1.67202, 1.60202, 1.10435, 0.0410596, 1.64189, 1.57245, 1.0755, 0.0027463, 1.63429, 1.56489, 1.06804, 0.000736065, 1.63229, 1.56289, 1.06604, 0.000197891, 1.63175, 1.56235, 1.0655, 0.0000532489, 1.6316, 1.5622, 1.06536, 0.0000143316, 1.63156, 1.56217, 1.06531, 10^{-6}, 1.63155, 1.56216, 1.06531, \ldots \} \]

The graph of the solutions is given below.
Example 3: If the initial conditions are selected as follows:

\[ x[-3]=2; x[-2]=0.1; x[-1]=0.01; x[0]=0.001; \]

The following solutions are obtained:

\[ x(n)\{2, 0.099998, 0.00999, 0.000998004, 2, 0.099996, 0.00999601, 0.000996013, 1.99999, 0.99994, 0.00999401, 0.000994027, 1.9999, 0.99992, 0.00999203, 0.000992044, 1.9999, 0.9999, 0.00999005, 0.00099900166, 1.9999, 0.999981, 0.00998807, 0.000988093, 1.9999, 0.9999861, 0.009986123, 1.99998, 0.9999841, 0.00998413, 0.000984159, 1.99998, 0.9999822, 0.00998216, 0.000982198, ... \}

The graph of the solutions is given below.
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