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Exact Travelling Wave Solutions Of The Benjamin-Bona-Mahony-Burgers Type (BBMB) Nonlinear Pseudoparabolic Equations By Using The (G'/G) Expansion Method

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Abstract: *In this paper, we consider some nonlinear pseudoparabolic Benjamin-Bona-Mahony-Burger(BBMB) equations by using the (G'/G) expansion method with the aid of computer algebraic system Maple. These equations are of a class of nonlinear pseudoparabolic or Sobolev-type equations* $u_t - \Delta u_t - \alpha \Delta u = f\left(x, u, \nabla u\right)$ *,α is a fixed positive constant, arising from mathematical physics. The method is straightforward and concise, and it be also applied to other nonlinear pseudoparabolic equations.*

Keywords: *The (G'/G) expansion method; Travelling wave solution; Nonlinear pseudoparabolic equation; Benjamin-Bona-Mahony-Burger-type(BBMB) equation.*

(G'/G) Açılım Metoduyla Lineer Olmayan Pseudoparabolik Tipte Benjamin-Bona-Mahony-Burgers Denkleminin Yürüyen Tam Dalga Çözümleri

Öz: *Bu makalede Maple yardımıyla (G'/G) açılım metodu kullanarak lineer olmayan pseudoparabolik Benjamin-Bona-Mahony-Burger(BBMB) denklemlerini ele aldık. Bu denklemler matematiksel fizikte ortaya çıkan, α pozitif bir sabit pozitif bir katsayı olmak üzere* $u_t - \Delta u_t - \alpha \Delta u = f\left(x, u, \nabla u\right)$ *şeklinde lineer olmayan pseudoparabolik veya Sobolev-tipi sınıfındandır. Metod açık anlaşılır ve öz bir metod ve aynı zamanda diğer lineer olmayan pseudoparabolik denklemlere de uygulanabilir bir metoddur.*

Anahtar Kelimeler: *The (G'/G) açılım metodu; Yürüyen dalga çözümü; Lineer olmayan pseudoparabolik denklem; Benjamin-Bona-Mahony-Burger-type(BBMB) denklem.*

INTRODUCTION

Nonlinear partial differential equations arise in a large number of physics, mathematics and engineering problems. In the soliton theory, the study of exact solutions to these nonlinear equations plays a very germane role, as they provide much information about the physical models they describe. Various powerful methods have been employed to construct exact travelling wave solutions to nonlinear partial differential equations. These methods include the inverse scattering transform[1], the Backlund transform[2,3], the Darboux transform[4], the Hirota bilinear method[5], the tanh-function method[6,7], the sine-cosine method[8], the exp-function method[9], the generalized Riccati equation[10], the homogeneous balance method[11], the first integral method[12,13], the (G′/G) expansion method[14,15], and the modified simple equation method[16,18].
The objective of this paper is to use a powerful method called the (G′/G) expansion method to obtain travelling wave solution for the a class of nonlinear pseudoparabolic equations. The method, first introduced by Wang and Zhang[19], has been widely used to obtain exact solutions of nonlinear equations[20,25].
Equations with one-time derivative appearing in the highest order term are called pseudoparabolic and arise in many areas of mathematics and physics. They have been used, for instance, for fluid flow in fissured rock, consolidation of clay, shear in second-order fluids, thermodynamics and propagation of long waves of small amplitude. For more details, we refer reader to [26,30] and references therein.
An important special case of pseudoparabolic-type equations is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + u_x + uu_x = 0 \qquad (1)$$

where $u(x,t)$ represents the fluid velocity in the horizantal direction x and α are positive constants.
A generalized Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + \left(g(u)\right)_x = 0 \qquad (2)$$

has been considered and a set of new solitons, kinks, antikinks, compactons and Wadati solitons have been derived using by the classical Lie method, where á is a positive constant, $\beta \in R$, and $g(u)$ is a C²-smooth nonlinear function. Equation (2) with the dissipative term αu_{xx} arises in the phenomena for both the bore propagation and the water waves. Peregrine and Benjamin, Bona, Mahony have proposed Eq.(2) with the parameters $g(u) = uu_x, \alpha = 0, \beta = 1$ [31,32]. Moreover, Benjamin, Bona, Mahony have proposed Eq.(2) as an alternative regularized long-wave equation with the same parameters. Tari and Ganji implemented variational iteration and homotopy perturbation methods obtaining approximate explicit solutions for Eq.(2) with $g(u) = \frac{u^2}{2}$ [33]. In addition, for $g(u) = uu_x$, $g(u) = \frac{u^2}{2}$ and $g(u) = \frac{u^3}{2}$ Akçağıl and Gözükızıl obtain some exact soltions by using tanh method[34].
As stated before, pseudoparabolic-type equation arise in many areas of mathematics and physics to describe many physical phenomena. In recent years considerable attention has been paid to the study of

pseudoparabolic -type equations. In this paper, (G′/G) expansion method is used to find the solutions for the pseudoparabolic-type equations stated above.

The main ideas are that the travelling wave solutions of nonlinear equation can be expressed by a polynomial in (G′/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation: $G'' + \lambda G' + \mu G = 0$, where $\xi = x - ct$ and λ, μ, c are constants. The degree of this polynomial can be determined by considering the homogenous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear equations. The coefficients of the polynomial λ, μ and c can be obtained by solving a set of algebraic equations resulting from the process of using the proposed method. Moreover, the travelling wave solutions obtained via this method are expressed by the hyperbolic functions, the trigonometric functions and the rational functions.

DESCRIPTION OF THE (G'/G) EXPANSION METHOD

In this section, we describe the (G′/G) expansion method for finding travelling wave solutions of nonlinear partial differential equations. Suppose that a nonlinear partial differential equation(PDE), say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{xt}, u_{xx}, u_{tt}, \ldots) = 0 \tag{3}$$

where u(x,t) is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which highest order derivatives and nonlinear terms are involved.

The summary of the (G′/G) expansion method, can be presented in the following six steps:

Step 1: To find the travelling wave solutions of Eq.(3) we introduce the wave variable

$$u(x,t) = U(\xi),\ \xi = x - ct \tag{4}$$

where the constant c is generally termed the wave velocity. Substituting Eq.(4) into Eq.(3), we obtain the following ordinary differential equations(ODE) in î (which illustrates a principal advantage of a travelling wave solution, i.e., a PDE is reduced to an ODE).

$$P(U, cU', U', cU'', c^2U'', U'', \ldots) = 0 \tag{5}$$

Step 2: If necessary we integrate Eq.(5) as many times as possible and set the constants of integration to be zero for simplicity.

Step 3: We suppose the solution of nonlinear partial differential equation can be expressed by a polynomial in (G′/G) as

$$u(\xi) = \sum_{i=0}^{M} a_i \left(\frac{G'}{G}\right)^i \tag{6}$$

where G=G(î) satisfies the second-order linear ordinary differential equation

$$G''+\lambda G'+\mu G=0 \tag{7}$$

where $G'=\frac{dG}{d\xi}, G''=\frac{d^2G}{d\xi^2}$, and a_i, λ and μ are real constants with $a_m \neq 0$. Here the prime denotes the derivative with respect to î. Using the general solutions of Eq.(7), we have

$$\left(\frac{G'}{G}\right)=\begin{cases} -\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right), \lambda^2-4\mu>0 \\ -\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right), \lambda^2-4\mu<0 \\ -\frac{\lambda}{2}+\left(\frac{c_2}{c_1+c_2\xi}\right), \lambda^2-4\mu=0 \end{cases} \tag{8}$$

where c_1 and c_2 are arbitrary constants.

Step 4:The positive integer m can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq.(5) as follows: if we define the degree of $u(\xi)$ as $D\left[u(\xi)\right]=m$, then the degree of other expressions is defined by

$$D\left[\frac{d^q u}{d\xi^q}\right]=M+q,$$

$$D\left[u^r\left(\frac{d^q u}{d\xi^q}\right)^s\right]=s(M+q)+Mr,$$

Therefore, we can get the value of m in Eq.(2.4).

Step 5: Substituting Eq.(6) into Eq.(5) using general solutions of Eq.(7) and collecting all terms with the same order of (G′/G) together, then setting each coefficient of this polynomial to zero yield a set of algebraic equations for a_i, c, λ and μ .

Step 6:Substitute a_i, c, λ and μ obtained in step 5 and the general solutions of Eq.(7) into Eq.(6). Next, depending on the sign of discriminant $\left(\lambda^2 - 4\mu\right)$, we can obtain the explicit solution of Eq.(3) immediately.

BENJAMIN-BONA-MAHONY-BURGERS (BBMB) EQUATION

The Benjamin-Bona-Mahony-Burgers (BBMPB) equation is given by

$$u_t - u_{xxt} - \alpha u_{xx} + u_x + u u_x = 0 \tag{9}$$

where α is a positive constant. Using the wave variable $\xi = x - ct$ in Eq.(9), then integrating this equation and considering the integration constant to be zero, we obtain

$$\left(1-c\right)U + cU'' - \alpha U' + \frac{U^2}{2} = 0 \tag{10}$$

According to step 4, balancing U^2 and U'' gives N=2. Therefore, the solutions of Eq.(10) can be written in the form

$$U = a_0 + a_1 \frac{G'}{G} + a_2 \left(\frac{G'}{G}\right)^2 \tag{11}$$

where a_0, a_1 and a_2 are constants which are unknowns to be determined later. By Eq.(7) we derive

$$\begin{cases} U' = -2a_2\left(\frac{G'}{G}\right)^3 - \left(2a_2\lambda + a_1\right)\left(\frac{G'}{G}\right)^2 - \left(2a_2\mu + a_1\lambda\right)\left(\frac{G'}{G}\right) - a_1\mu, \\ U'' = 6a_2\left(\frac{G'}{G}\right)^4 + \left(10a_2\lambda + 2a_1\right)\left(\frac{G'}{G}\right)^3 + \left(8a_2\mu + 4a_2\lambda^2 + 3a_1\lambda\right)\left(\frac{G'}{G}\right)^2 \\ + \left(6a_2\lambda\mu + a_1\lambda^2 + 2a_1\mu\right)\left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu. \end{cases} \tag{12}$$

Substituting Eq.(11) and its derivatives Eq.s(12) into Eq.s(10) and equating each coefficient of (G′/G) to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, a_2, λ and c. Solving this system using Maple, we obtain

Set 1. $c=\frac{1}{2}+\frac{\sqrt{25+24\alpha^2}}{10}$, $\lambda=\mp\sqrt{4\mu+\frac{c-1}{6c}}$, $a_2=-12c$, $a_1=-12c\lambda-\frac{12\alpha}{5}$, $a_0=-12c\mu+c-1-\frac{6\alpha\lambda}{5}$;

Set 2. $c=\frac{1}{2}+\frac{\sqrt{25+24\alpha^2}}{10}$, $\lambda=\mp\sqrt{4\mu+\frac{c-1}{6c}}$, $a_2=-12c$, $a_1=-12c\lambda-\frac{12\alpha}{5}$, $a_0=-12c\mu+c-1-\frac{6\alpha\lambda}{5}$;

Set 3. $c=\frac{1}{2}+\frac{\sqrt{25-24\alpha^2}}{10}$, $\lambda=\mp\sqrt{4\mu+\frac{1-c}{6c}}$, $a_2=-12c$, $a_1=-12c\lambda-\frac{12\alpha}{5}$, $a_0=-12c\mu+c-1-\frac{6\alpha\lambda}{5}$;

Set 4. $c=\frac{1}{2}-\frac{\sqrt{25-24\alpha^2}}{10}$, $\lambda=\mp\sqrt{4\mu+\frac{1-c}{6c}}$, $a_2=-12c$, $a_1=-12c\lambda-\frac{12\alpha}{5}$, $a_0=-12c\mu+c-1-\frac{6\alpha\lambda}{5}$;

Set 5. $c=0$, $\lambda=\mp\sqrt{4\mu+\frac{1}{\alpha^2}}$, $a_2=0$, $a_1=-2\alpha$, $a_0=-1-\alpha\lambda$;

Using these values in Eq.(11) when $\lambda^2-4\mu>0$, we obtain the hyperbolic solutions respectively:

$$u_1(x,t)=\frac{3(c-1)}{2}-\frac{6\alpha}{5}\sqrt{\frac{c-1}{6c}}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}\right)$$
$$-\frac{c-1}{2}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}\right)^2, \tag{13}$$

where $\xi=x-\left(\frac{1}{2}+\frac{\sqrt{25+24\alpha^2}}{10}\right)t$ and $c=\frac{1}{2}+\frac{\sqrt{25+24\alpha^2}}{10}$,

$$u_2(x,t)=\frac{3(c-1)}{2}-\frac{6\alpha}{5}\sqrt{\frac{c-1}{6c}}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}\right)$$
$$-\frac{c-1}{2}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi\right)}\right)^2, \tag{14}$$

where $\xi=x-\left(\frac{1}{2}-\frac{\sqrt{25+24\alpha^2}}{10}\right)t$ and $c=\frac{1}{2}-\frac{\sqrt{25+24\alpha^2}}{10}$,

$$u_3(x,t)=\frac{c-1}{2}-\frac{6\alpha}{5}\sqrt{\frac{1-c}{6c}}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}\right)$$
$$+\frac{c-1}{2}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}\right)^2, \tag{15}$$

where $\xi = x - \left(\frac{1}{2} + \frac{\sqrt{25-24\alpha^2}}{10}\right)t$ and $c = \frac{1}{2} + \frac{\sqrt{25-24\alpha^2}}{10}$,

$$u_4(x,t) = \frac{c-1}{2} - \frac{6\alpha}{5}\sqrt{\frac{1-c}{6c}}\left(\frac{c_1 \sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}{c_1 \cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}\right)$$
$$+\frac{c-1}{2}\left(\frac{c_1 \sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}{c_1 \cosh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi\right)}\right)^2, \quad (16)$$

where $\xi = x - \left(\frac{1}{2} - \frac{\sqrt{25-24\alpha^2}}{10}\right)t$ and $c = \frac{1}{2} - \frac{\sqrt{25-24\alpha^2}}{10}$,

$$u_5(x,t) = -1 - \left(\frac{c_1 \sinh\left(\frac{x}{2\alpha}\right) + c_2 \cosh\left(\frac{x}{2\alpha}\right)}{c_1 \cosh\left(\frac{x}{2\alpha}\right) + c_2 \sinh\left(\frac{x}{2\alpha}\right)}\right), \quad (17)$$

In particular, if we take $c_1 \neq 0, c_2^2 < c_1^2$, then Eq.s(13)-(17) lead the formal solitary wave solutions to Eq.(9) as

$$u_1(x,t) = c - 1 - \frac{6\alpha}{5}\sqrt{\frac{c-1}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi + \xi_0\right) + \frac{c-1}{2}\sec h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi + \xi_0\right), \quad (18)$$

where $\xi = x - \left(\frac{1}{2} + \frac{\sqrt{25+24\alpha^2}}{10}\right)t$ and $c = \frac{1}{2} + \frac{\sqrt{25+24\alpha^2}}{10}$,

$$u_2(x,t) = c - 1 - \frac{6\alpha}{5}\sqrt{\frac{c-1}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi + \xi_0\right) + \frac{c-1}{2}\sec h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{6c}}\xi + \xi_0\right), \quad (19)$$

where $\xi = x - \left(\frac{1}{2} - \frac{\sqrt{25+24\alpha^2}}{10}\right)t$ and $c = \frac{1}{2} - \frac{\sqrt{25+24\alpha^2}}{10}$,

$$u_3(x,t) = c - 1 - \frac{6\alpha}{5}\sqrt{\frac{1-c}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi + \xi_0\right) - \frac{c-1}{2}\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi + \xi_0\right), \tag{20}$$

where $\xi = x - \left(\frac{1}{2} + \frac{\sqrt{25-24\alpha^2}}{10}\right)t$ and $c = \frac{1}{2} + \frac{\sqrt{25-24\alpha^2}}{10}$,

$$u_4(x,t) = c - 1 - \frac{6\alpha}{5}\sqrt{\frac{1-c}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi + \xi_0\right) - \frac{c-1}{2}\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{1-c}{6c}}\xi + \xi_0\right), \tag{21}$$

where $\xi = x - \left(\frac{1}{2} - \frac{\sqrt{25-24\alpha^2}}{10}\right)t$ and $c = \frac{1}{2} - \frac{\sqrt{25-24\alpha^2}}{10}$,

$$u_5(x,t) = -1 - \tanh\left(\frac{x}{2\alpha} + \xi_0\right), \tag{22}$$

where $\xi_0 = \tanh^{-1}\left(\frac{c_2}{c_1}\right)$.

THE GENERALIZED BENJAMIN-BONA-MAHONY-BURGERS (BBMB) EQUATIONS

Consider the Oskolkov-Benjamin-Bona-Mahony-Burgers(OBBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + \left(g(u)\right)_x = 0 \tag{23}$$

where α is positive and $\beta \in R$.

Case 1. $g(u) = uu_x$

Using the wave variable $\xi = x - ct$ carries (23), then integrating this equation and considering the integration constant to be zero, we obtain

$$(\beta-c)U-\alpha U'+cU''+UU'=0 \tag{24}$$

According to step 4, balancing UU′ and U″ gives N=1. Therefore, the solutions of Eq.(24) can be written in the form

$$U(\xi)=a_0+a_1\left(\frac{G'}{G}\right), \tag{25}$$

where a_0 and a_1 are constants which are unknowns to be determined later. Substituting Eq.(25) and its derivatives into Eq.s(24) and equating each coefficient of (G′/G) to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, λ and c.Solving this system using Maple, we obtain

$$c=\beta, a_1=2c, a_0=\beta\lambda+\alpha;$$

Using these values in Eq.(25) when $\lambda^2-4\mu>0$, we obtain the hyperbolic solution:

$$u_1(x,t)=\alpha+\beta\sqrt{\lambda^2-4\mu}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right) \tag{26}$$

where $\xi=x-ct$.

When $\lambda^2-4\mu<0$, we obtain the trigonometric solution:

$$u_2(x,t)=\alpha+\beta\sqrt{4\mu-\lambda^2}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right) \tag{27}$$

where $\xi=x-ct$.

When $\lambda^2-4\mu=0$, we obtain the rational solution:

$$u_3(x,t)=\alpha+2\beta\left(\frac{c_2}{c_1+c_2\xi}\right), \tag{28}$$

where $\xi = x - ct$.

In particular, if $c_1 \neq 0$ and $c_2 = 0$, then the solutions in Eq.s(26)-(27) become respectively:

$$u_{1,1}(x,t) = \alpha + \beta\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right), \tag{29}$$

where $\xi = x - ct$ and $\lambda^2 - 4\mu > 0$,

$$u_{2,1}(x,t) = \alpha - \beta\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right), \tag{30}$$

where $\xi = x - ct$ and $\lambda^2 - 4\mu < 0$.

The same manner, if $c_1 = 0$ and $c_2 \neq 0$, then the solutions in Eq.s(26)-(28) become respectively:

$$u_{1,2}(x,t) = \alpha + \beta\sqrt{\lambda^2 - 4\mu}\coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right), \tag{31}$$

where $\xi = x - ct$ and $\lambda^2 - 4\mu > 0$,

$$u_{2,2}(x,t) = \alpha + \beta\sqrt{4\mu - \lambda^2}\cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right), \tag{32}$$

where $\xi = x - ct$ and $\lambda^2 - 4\mu < 0$,

$$u_{3,1}(x,t) = \alpha + \frac{2\beta}{x - \beta t}, \tag{33}$$

where $\lambda^2 - 4\mu = 0$.

Case 2. $g(u) = \dfrac{u^2}{2}$

Using the wave variable $\xi = x - ct$, then integrating this equation and considering the integration constant to be zero, we obtain

$$(c - \beta)U + \alpha U' - cU'' - \frac{U^2}{2} = 0 \tag{34}$$

According to step 4, balancing U^2 and U'' gives N=2. Therefore, the solutions of Eq.(34) can be written in the form

$$U(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2, \tag{35}$$

where a_0, a_1 and a_2 are constants which are unknowns to be determined later. Substituting Eq.(35) and its derivatives Eq.s(12) into Eq.s(34) and equating each coefficient of (G′/G) to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, a_2, λ and c.Solving this system using Maple, we obtain

Set 1. $c=\frac{\beta}{2}+\frac{\sqrt{25\beta^2+24\alpha^2}}{10},\ \lambda=\mp\sqrt{4\mu+\frac{c-\beta}{6c}},\ a_2=-12c,\ a_1=-12c\lambda-\frac{12\alpha}{5},$
$a_0=c-12c\mu-\beta-\frac{6\alpha\lambda}{5};$

Set 2. $c=\frac{\beta}{2}-\frac{\sqrt{25\beta^2+24\alpha^2}}{10},\ \lambda=\mp\sqrt{4\mu+\frac{c-\beta}{6c}},\ a_2=-12c,\ a_1=-12c\lambda-\frac{12\alpha}{5},$
$a_0=c-12c\mu-\beta-\frac{6\alpha\lambda}{5};$

Set 3. $c=\frac{\beta}{2}+\frac{\sqrt{25\beta^2-24\alpha^2}}{10},\ \lambda=\mp\sqrt{4\mu+\frac{\beta-c}{6c}},\ a_2=-12c,\ a_1=-12c\lambda-\frac{12\alpha}{5},$
$a_0=c-12c\mu-\beta-\frac{6\alpha\lambda}{5};$

Set 4. $c=\frac{\beta}{2}-\frac{\sqrt{25\beta^2-24\alpha^2}}{10},\ \lambda=\mp\sqrt{4\mu+\frac{\beta-c}{6c}},\ a_2=-12c,\ a_1=-12c\lambda-\frac{12\alpha}{5},$
$a_0=c-12c\mu-\beta-\frac{6\alpha\lambda}{5};$

Set 5. $c=0,\ \lambda=\mp\sqrt{4\mu+\frac{\beta^2}{\alpha^2}},\ a_2=0,\ a_1=-2\alpha,\ a_0=-\beta-\alpha\lambda.$

Using these values in Eq.(35) when $\lambda^2-4\mu>0$, we obtain the hyperbolic solutions respectively:

$$u_1(x,t)=\frac{3(c-\beta)}{2}-\frac{6\alpha}{5}\sqrt{\frac{c-\beta}{6c}}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}\right)$$
$$-\frac{c-\beta}{2}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}\right)^2, \tag{36}$$

where $\xi=x-\left(\frac{\beta}{2}+\frac{\sqrt{25\beta^2+24\alpha^2}}{10}\right)t$ and $c=\frac{\beta}{2}+\frac{\sqrt{25\beta^2+24\alpha^2}}{10}$,

$$u_2(x,t)=\frac{3(c-\beta)}{2}-\frac{6\alpha}{5}\sqrt{\frac{c-\beta}{6c}}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}\right)$$
$$-\frac{c-\beta}{2}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right)}\right)^2, \tag{37}$$

where $\xi=x-\left(\frac{\beta}{2}-\frac{\sqrt{25\beta^2+24\alpha^2}}{10}\right)t$ and $c=\frac{\beta}{2}-\frac{\sqrt{25\beta^2+24\alpha^2}}{10}$,

$$u_3(x,t)=\frac{c-\beta}{2}-\frac{6\alpha}{5}\sqrt{\frac{\beta-c}{6c}}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}\right)$$
$$+\frac{c-\beta}{2}\left(\frac{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}\right)^2, \tag{38}$$

where $\xi = x - \left(\frac{\beta}{2} + \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}\right)t$ and $c = \frac{\beta}{2} + \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}$,

$$u_4(x,t) = \frac{c-\beta}{2} - \frac{6\alpha}{5}\sqrt{\frac{\beta-c}{6c}}\left(\frac{c_1 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}{c_1 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}\right)$$

$$+\frac{c-\beta}{2}\left(\frac{c_1 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}{c_1 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta-c}{6c}}\xi\right)}\right)^2, \tag{39}$$

where $\xi = x - \left(\frac{\beta}{2} - \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}\right)t$ and $c = \frac{\beta}{2} - \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}$,

$$u_5(x,t) = -\beta - \beta\left(\frac{c_1 \sinh\left(\frac{\beta}{2\alpha}x\right) + c_2 \cosh\left(\frac{\beta}{2\alpha}x\right)}{c_1 \cosh\left(\frac{\beta}{2\alpha}x\right) + c_2 \sinh\left(\frac{\beta}{2\alpha}x\right)}\right). \tag{40}$$

In particular, if we take $c_1 \neq 0, c_2^2 < c_1^2$, then Eq.s(36)-(40) lead the formal solitary wave solutions to Eq.(23) as

$$u_1(x,t) = c - \beta - \frac{6\alpha}{5}\sqrt{\frac{c-\beta}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi + \xi_0\right) + \frac{c-\beta}{2}\sec h^2\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi + \xi_0\right), \tag{41}$$

where $\xi = x - \left(\frac{\beta}{2} + \frac{\sqrt{25\beta^2 + 24\alpha^2}}{10}\right)t$ and $c = \frac{\beta}{2} + \frac{\sqrt{25\beta^2 + 24\alpha^2}}{10}$,

$$u_2(x,t) = c - \beta - \frac{6\alpha}{5}\sqrt{\frac{c-\beta}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi + \xi_0\right) + \frac{c-\beta}{2}\sec h^2\left(\frac{1}{2}\sqrt{\frac{c-\beta}{6c}}\xi\right), \tag{42}$$

where $\xi = x - \left(\frac{\beta}{2} - \frac{\sqrt{25\beta^2 + 24\alpha^2}}{10}\right)t$ and $c = \frac{\beta}{2} - \frac{\sqrt{25\beta^2 + 24\alpha^2}}{10}$,

$$u_3(x,t) = c - \beta - \frac{6\alpha}{5}\sqrt{\frac{\beta - c}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{\beta - c}{6c}}\xi + \xi_0\right) - \frac{c-\beta}{2}\sec h^2\left(\frac{1}{2}\sqrt{\frac{\beta - c}{6c}}\xi\right), \quad (43)$$

where $\xi = x - \left(\frac{\beta}{2} + \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}\right)t$ and $c = \frac{\beta}{2} + \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}$,

$$u_4(x,t) = c - \beta - \frac{6\alpha}{5}\sqrt{\frac{\beta - c}{6c}}\tanh\left(\frac{1}{2}\sqrt{\frac{\beta - c}{6c}}\xi + \xi_0\right) - \frac{c-\beta}{2}\sec h^2\left(\frac{1}{2}\sqrt{\frac{\beta - c}{6c}}\xi\right), \quad (44)$$

where $\xi = x - \left(\frac{\beta}{2} - \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}\right)t$ and $c = \frac{\beta}{2} - \frac{\sqrt{25\beta^2 - 24\alpha^2}}{10}$,

$$u_5(x,t) = -\beta - \beta\tanh\left(\frac{\beta}{2\alpha}x + \xi_0\right), \quad (45)$$

where $\xi_0 = \tanh^{-1}\left(\frac{c_2}{c_1}\right)$.

CONCLUSIONS

In this paper, we implemented the (G′/G) expansion method to solve some nonlinear pseudoparabolic Benjamin-Bona-Mahony-Burgers equations and obtained new solutions which could not be obtained in the past. Besides, we have seen that the (G′/G) expansion method is applied successfully and reliable to solve not only for the class of nonlinear evolution equations but also for the a class of nonlinear pseudoparabolic and Sobolev-type equations.

We have seen that three types of travelling wave solutions were successfully found, in terms of hyperbolic, trigonometric and rational functions. It will be more important to seek solutions of higher-order nonlinear equations which can be reduced to ODEs of the order greater than 2. We have noted that this method changes the given difficult problems into simple problems which can be solved easily. The method yields a general solution with free parameters which can be identified by the above conditions in section 2. Moreover, some numerical methods like the Adomian decomposition method and homotopy perturbation method depend on the initial conditions and obtain a solution in a series which converges to the exact solution of the problem. However,it is obtained by the (G′/G) expansion method a general solution without approximation and there is no need to apply the initial and boundary conditions at the outset. The (G′/G) expansion method is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation. The solution procedure can be easily implemented in Mathematica or Maple.

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