Araştırma Makalesi *BAUN Fen Bil. Enst. Dergisi, 20(2), 437-459, (2018)*

DOI: 10.25092/baunfbed.485589 *J. BAUN Inst. Sci. Technol., 20(2), 437-459, (2018)*

The special gaps of Arf numerical semigroups with small multiplicity

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Geliş Tarihi (Recived Date): 02.08.2018
Kabul Tarihi (Accepted Date): 06.10.2018

Abstract

In this study, we deal with the concept of special gap of a numerical semigroup which is used to find the set of all numerical semigroups containing a given numerical semigroup. We will find the specific gaps of Arf numerical semigroups with small multiplicity. We also find all Arf numerical semigroups containing a given Arf numerical semigroup with small multiplicity.

Keywords: *Numerical semigroups, Arf numerical semigroup, special gaps.*

Küçük katlılıklı Arf sayısal yarıgrupların özel boşlukları

Özet

Bu çalışmada, verilen bir sayısal yarıgrubu kapsayan tüm sayısal yarıgrupların bulunması probleminde kullanılan bir sayısal yarıgrubun özel boşlukları kavramı ile ilgileniyoruz. Küçük katlılıklı Arf sayısal yarıgruplarının özel boşluklarını bulacağız. Verilen küçük katlılıklı Arf sayısal yarıgrubu kapsayan tüm Arf sayısal yarıgruplarını bulacağız.

Anahtar Kelimeler: *Sayısal yarıgrup, Arf sayısal yarıgrup, özel başluklar.*

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1. Introduction

The concepts and bases of this work are summarized as follows: A *numerical semigroup* is a subset of the set of nonnegative integers (denoted here by $\mathbb{N}$) closed under addition, containing the zero element and with finite complement in $\mathbb{N}$. Every numerical semigroup S is finitely generated, that is, $S=\langle u_1,u_2,...,u_p\rangle = u_1\mathbb{N}+u_2\mathbb{N}+...+u_p\mathbb{N}$, where $u_1,u_2,...,u_p$ are positive integers such that $\gcd(u_1,u_2,...,u_p)=1$ (see, for instance, [2, 3, 7]). The last condition is equivalent to saying that S has finite complement in $\mathbb{N}$ (where gcd is the abbreviation for the greatest common divisor).

The set $A=\{u_1,u_2,...,u_p\}$ is said to be the *minimal system of generators* for S if $S=\langle A\rangle$, but $S\neq\langle A\rangle\setminus\{u_i\}$ for any i. The cardinality of the minimal system of generators of S is known as the *embedding dimension* of S, and it is denoted by $e(S)$. The smallest positive element of S is the *multiplicity* of S, and it is denoted by $m(\mathrm{S})$. It is known that $e(S)\leq m(S)$. S is said to be a numerical semigroup of *maximal embedding dimension* if $e(S)=m(S)$.

The greatest integer not in S is known as the Frobenius number of S, though in the literature it is sometimes replaced by the *conductor* of S, which is the least integer x such that $x+n\in S$ for all $n\in\mathbb{N}$. The Frobenius number of S is denoted here by $F(S)$ and it is conductor of S minus one. The conductor of S is denoted by $C(\mathrm{S})=\mathrm{C}$. If S is different from $\mathbb{N}$, it is common to denote the elements of S that are less than or equal to C by $s_0=0,s_1,...,s_{n-1},s_n=C$ with $s_{i-1}<s_i$ si for $1\leq i\leq n=n(S)$, and write

$$S=\{s_0=0,s_1,...,s_{n-1},s_n=C,\rightarrow\}$$

where "$\rightarrow$" means that every integer greater than C belongs to the set. The elements $s_0=0,s_1,...,s_{n-1}$ are called the *small elements* of S. Note that the first nonzero small element $s_1=m(S)$ is the multiplicity of S and $n=n(S)=\#(S\cap\{0,1,...,F(S)\})$ is the number of small elements of S ("$\#A$" denotes the cardinality of the set A). Those positive integers which do not belong to S are called gaps of S. The set of all gaps of S is denoted by $H(\mathrm{S})$. And the number of gaps of S is called the genus of S and it is denoted by $g(S)$. The largest gap of S is the Frobenius number $F(S)$ of S if S is different from $\mathbb{N}$.

If S is a numerical semigroup and $a\in S\setminus\{0\}$, the *Apery set of* S with *respect* to a is the set $Ap(S,a)=\{s\in S: s-a\notin S\}$. It is easy to see that $Ap(S,a)=\{\mathrm{w}_0=0,w_1,...,w_{a-1}\}$, where w_i is the least element of S such that $w_i\equiv i(\bmod a)$. Moreover, $(Ap(S,a)\setminus\{0\})\cup\{a\}$ generates S and $\max(Ap(S,m))=F(S)+m$ for any $a\in S\setminus\{0\}$ [7]. Thus if S is a numerical semigroup with multiplicity m, then S is of *maximal embedding dimension* if only if $(Ap(S,\mathrm{m})\setminus\{0\})\cup\{\mathrm{m}\}$ is the minimal system of generators for S.

Let S is a numerical semigroup. If $x \notin S$ and $x+s \in S$ for all $s \in S \setminus \{0\}$, then the integer x is called a pseudo-Frobenius number of S. The set pseudo-Frobenius number of S is denoted by $PF(S)$. There is the relation $a \leq_s b$ if $b-a \in S$ on the set of integers. And this relation is an order relation. The set of pseudo-Frobenius number of the numerical semigroup S can also be obtained by this relation and Apery set of the numerical semigroup S: $PF(S)=\{w-u : w \in Maximals \leq_s Ap(S,u)\}$ [5].

2. Arf numerical semigroups

A numerical semigroup S is called *Arf* if $x+y-z \in S$ for all $x,y,z \in S$, where $x \geq y \geq z$. This definition was first given by C. Arf [1] in 1949, and therefore the condition in this definition is known as the *Arf condition*. If $x \geq y \geq z$ and $x \geq C(S)$ for all $x,y,z \in S$, then $x+y-z \in C(S)$ and thus $x+y-z \in S$. In order check whether a numerical semigroup is Arf or not, it is enough to check the Arf condition only for the small elements. There are many equivalent conditions to the Arf condition, one of which states that a numerical semigroup is Arf if and only if $2x-y \in S$ for all $x,y \in S$, where $x \geq y$ [4].

Any Arf numerical semigroup is of maximal embedding dimension. Thus, if S is an Arf numerical semigroup with multiplicity m, then S is minimally generated by $(Ap(S,m)\setminus\{0\}) \cup \{m\}$. The largest element of the set $Ap(S,m)$ is $C(S)+m-1$.

The only numerical semigroup with multiplicity one is $\mathbb{N}$, which is trivially Arf.

Every numerical semigroup with multiplicity 2 is an Arf numerical semigroup and if S is a numerical semigroup with conductor $C(S)=C$, then $S=\langle 2, C+1 \rangle$. The following Propositions of Garcia-Sanchez, Heredia, Karakaş and Rosales [4] will be crucial for this study.

2.1 Proposition [[4], Proposition 17] Let C be an integer such that $C \geq 3$ and $C \not\equiv 1 \pmod 3$. Then there is exactly one Arf numerical semigroup S with multiplicity 3 and conductor C given by

i. $S=\langle 3, C+1, C+2 \rangle$ if $C \equiv 0 \pmod 3$,

ii. $S=\langle 3, C, C+2 \rangle$ if $C \equiv 2 \pmod 3$.

2.2 Proposition [[4], Proposition 18] Let S be an Arf numerical semigroup with multiplicity 4 and conductor C.

i. if $C \equiv 0 \pmod 4$, then $S=\langle 4, 4k+2, \mathrm{C}+1, \mathrm{C}+3 \rangle$ for some $1 \leq k \leq \frac{C}{4}$,

ii. if $C \equiv 2 \pmod 4$, then $S=\langle 4, 4k+2, \mathrm{C}+1, \mathrm{C}+3 \rangle$ for some $1 \leq k \leq \frac{C-2}{4}$,

iii. if $C \equiv 3 \pmod 4$, then $S=\langle 4, \mathrm{C}, \mathrm{C}+2, \mathrm{C}+3 \rangle$.

2.3 Proposition [[4], Proposition 20] Let S be an Arf numerical semigroup with multiplicity 5 and conductor C.

i. If $C \equiv 0 (\mathrm{mod}\, 5)$, then S is equal to one of

$a)$ $S = \langle 5, C-2, C+1, C+2, C+4 \rangle$,

$b)$ $S = \langle 5, C+1, C+2, C+3, C+4 \rangle$.

ii. If $C \equiv 2 (\mathrm{mod}\, 5)$, then $S = \langle 5, C, C+1, C+2, C+4 \rangle$,

iii. If $C \equiv 3 (\mathrm{mod}\, 5)$, then $S = \langle 5, C, C+1, C+3, C+4 \rangle$,

iv. If $C \equiv 4 (\mathrm{mod}\, 5)$, then S is equal to one of

$a)$ $S = \langle 5, C-2, C, C+2, C+4 \rangle$

$b)$ $S = \langle 5, C, C+2, C+3, C+4 \rangle$.

2.4 Proposition [[4], Proposition 22] Let S be an Arf numerical semigroup with multiplicity 6 and conductor C. We get

i. If $C \equiv 0 (\mathrm{mod}\, 6)$, then S is equal to one of

$a) \langle 6, C+1, C+2, C+3, C+4, C+5 \rangle$,

$b) \langle 6, 6\mathrm{t}+2, 6\mathrm{t}+4, C+1, C+3, C+5 \rangle$,

$c) \langle 6, 6\mathrm{t}+3, C+1, C+2, C+4, C+5 \rangle$,

$d) \langle 6, 6\mathrm{t}+4, 6\mathrm{t}+8, C+1, C+3, C+5 \rangle$,

for some $1 \le t \le \frac{C}{6} - 1$.

ii. If $C \equiv 2 (\mathrm{mod}\, 6)$, then S is equal to one of

$a) \langle 6, 6\mathrm{u}+2, 6\mathrm{u}+4, C+1, C+3, C+5 \rangle$,

$b) \langle 6, 6v+3, \mathrm{C}, C+2, C+3, C+5 \rangle$,

$c) \langle 6, 6v+4, 6v+8, C+1, C+3, C+5 \rangle$,

for some $1 \le u \le \frac{C-2}{6}$ and $1 \le v \le \frac{C-2}{6} - 1$.

iii. If $C \equiv 3 (\mathrm{mod}\, 6)$, then

$\langle 6, 6\mathrm{u}+3, C+1, C+2, C+4, C+5 \rangle$,

for some $1 \le u \le \frac{C-3}{6}$.

iv. If $C \equiv 4 (\mathrm{mod}\, 6)$, then S is equal to one of

$a) \langle 6, 6\mathrm{u}+2, 6\mathrm{u}+4, C+1, C+3, C+5 \rangle$,

$b) \langle 6, 6u+4, 6\mathrm{u}+8, C+1, C+3, C+5 \rangle$,

for some $1 \le u \le \frac{C-4}{6}$.

v. If $C \equiv 5 (\mathrm{mod}\, 6)$, then

$a) \langle 6, \mathrm{C}, \mathrm{C}+2, C+3, C+4, C+5 \rangle$,

$b) \langle 6, 6u+3, C, C+2, C+3, C+5 \rangle$, for some $1 \le u \le \frac{C-5}{6}$.

3. The special gaps of Arf numerical semigroups

We describe the concept of special gap used to find all numerical semigroups that contain a given numerical semigroup. Let S be a numerical semigroup. The set special gaps of S denoted by $SH(S)=\{x \in PF(S) : 2x \in S\}$. It is clearly that $SH(S) \subset H(S)$. And if $x \in SH(S)$, then $\{x\} \cup S$ is again a numerical semigroup [6]. Previous studies have shown that $\{C-1\} \cup S$ is an Arf semigroup for all Arf semigroup with conductor C [4]. This study, we will find special gaps of some Arf numerical semigroups, and we will determine which $\{x\} \cup S$ semigroup is Arf and in what form they are.

3.1 Lemma If S is an Arf numerical semigroup with multiplicity $m > 1$, then

i. $Maximals \le_s Ap(S,m) = Ap(S,m) \setminus \{0\}$,

ii. $PF(S) = \{w - m : w \in Ap(S,m) \setminus \{0\}\}$.

3.2 Remark It is clearly that if $S = \langle 2, C+1 \rangle$ is the Arf numerical semigroup with conductor $C(S) = C$ and multiplicity 2, then $SH(S) = \{\mathrm{C}-1\}$ and $\{\mathrm{C}-1\} \cup S = \langle 2, \mathrm{C}-1 \rangle$. This numerical semigroup obtained is again an Arf semigroup with multiplicity 2.

3.3 Theorem Let S be a numerical semigroup, as in 2.1 Proposition.

i. If $C \equiv 0 \pmod 3$ and $S = \langle 3, \mathrm{C}+1, \mathrm{C}+2 \rangle$, then

$$SH(S) = \begin{cases} \{C-1\} & for\ \ C = 3 \\ \{C-2, C-1\} & for\ \ C > 3 \end{cases} = \begin{cases} \{2\} & for\ \ C = 3 \\ \{C-2, C-1\} & for\ \ C > 3 \end{cases}$$

ii. If $C \equiv 2 \pmod 3$ and $S = \langle 3, \mathrm{C}, \mathrm{C}+2 \rangle$, then

$$SH(S) = \begin{cases} \{C-1\} & for\ \ C = 5 \\ \{C-3, C-1\} & for\ \ C > 5 \end{cases} = \begin{cases} \{4\} & for\ \ C = 5 \\ \{C-3, C-1\} & for\ \ C > 5 \end{cases}.$$

Proof. We first note that all the semigroups in the 2.1 Proposition are Arf numerical semigroups.

i. If $C \equiv 0 \pmod 3$ and $S = \langle 3, \mathrm{C}+1, \mathrm{C}+2 \rangle$ and $C > 3$, then $Ap(S,3) = \{0, \mathrm{C}+1, C+2\}$ by 2.1 Proposition. Therefore $PF(S) = \{C-2, C-1\}$ by 3.1 Lemma. And

$$C \ge 6$$
$$2C \ge C+6$$
$$2(C-2) \ge C+2$$
$$2(C-1) \ge C+4$$
$$2(C-2), 2(C-1) \in S$$
$$C-2, C-1 \in SH(S).$$

Thus $SH(S)=\{C-1,C-2\}$. Moreover, for $C=3$ we have $S=\langle 3,4,5\rangle$ and $SH(S)=\{2\}$ by the definition. Hence

$$SH(S)=\begin{cases}\{2\} & for\ \ C=3\\ \{C-2,C-1\} & for\ \ C>3.\end{cases}$$

ii. If $C\equiv 0(\mathrm{mod}\,3)$ and $S=\langle 3,\mathrm{C},\mathrm{C}+2\rangle$, then for $C>5$ we get $Ap(S,3)=\{0,\mathrm{C},C+2\}$. Therefore $PF(S)=\{C-1,C-3\}$ by 3.1 Lemma. And

$$C\geq 8$$
$$2C\geq C+8$$
$$2(C-3)\geq C+2$$
$$2(C-1)\geq C+6$$
$$2(C-3),2(C-1)\in S$$
$$C-3,C-1\in SH(S).$$

Thus $SH(S)=\{C-3,C-1\}$. Moreover, for $C=5$ we have $S=\langle 3,5,7\rangle$ and $SH(S)=\{4\}$ by the definition. Hence

$$SH(S)=\begin{cases}\{4\} & for\ \ C=5\\ \{C-3,C-1\} & for\ \ C>5\end{cases}.$$

3.4 Corollary Let S be a numerical semigroup, as in 2.1 Proposition.

i. If $C\equiv 0(\mathrm{mod}\,3)$ and $S=\langle 3,\mathrm{C}+1,\mathrm{C}+2\rangle$, then $\{\mathrm{C}-1\}\cup S=\langle 3,C-1,\mathrm{C}+1\rangle$ is an Arf numerical semigroup as in 2.1 Proposition[ii] for $C>3$. Specially, for $C=3$, $\{2\}\cup S=\langle 2,3\rangle$ is an Arf numerical semigroup with multiplicity two.

ii. If $C\equiv 2(\mathrm{mod}\,3)$ and $S=\langle 3,\mathrm{C},\mathrm{C}+2\rangle$, then $\{\mathrm{C}-1\}\cup S=\langle 3,C-1,\mathrm{C}\rangle$ is an Arf numerical semigroup as in 2.1 Proposition[i].

3.5 Theorem Let S be a numerical semigroup, as in 2.2 Proposition.

i. If $S=\langle 4,4k+2,\mathrm{C}+1,\mathrm{C}+3\rangle$ when $C\equiv 0(\mathrm{mod}\,4)$ for some $1\leq k\leq\frac{C}{4}$, then

$$SH(S)=\begin{cases}\{4k-2,C-1\} & for\ \ C=4\\ \{4k-2,C-3,C-1\} & for\ \ C>4\end{cases}=\begin{cases}\{2,3\} & for\ \ C=4\\ \{4k-2,C-3,C-1\} & for\ \ C>4\end{cases}.$$

ii. If $S=\langle 4,4k+2,\mathrm{C}+1,\mathrm{C}+3\rangle$ when $C\equiv 2(\mathrm{mod}\,4)$ for some $1\leq k\leq\frac{C-2}{4}$, then $SH(S)=\{4k-2,C-3,C-1\}$.

iii. If $S=\langle 4,\mathrm{C},\mathrm{C}+2,\mathrm{C}+3\rangle$ and $C\equiv 3(\mathrm{mod}\,4)$, then

$$SH(S)=\begin{cases}\{C-2,C-1\} & for\ \ C=7\\ \{C-4,C-2,C-1\} & for\ \ C>7\end{cases}=\begin{cases}\{5,6\} & for\ \ C=7\\ \{C-4,C-2,C-1\} & for\ \ C>7\end{cases}.$$

Proof. We first note that all the semigroups in the 2.2 Proposition are Arf numerical semigroups.

i. If $S=\langle 4,4k+2,\mathrm{C}+1,\mathrm{C}+3\rangle$ when $C\equiv 0(\mathrm{mod}\,4)$ for some $1\le k\le \frac{C}{4}$, then $Ap(S,4)=\{0,4k+2,C+1,\mathrm{C}+3\}$ by 2.2 Proposition. From 3.1 Lemma, $PF(S)=\{4k-2,\mathrm{C}-3,\mathrm{C}-1\}$. Hence we get

$$C\ge 4$$
$$2C\ge C+4$$
$$2(C-1)\ge C+5$$
$$2(C-1)\in S$$
$$C-1\in SH(S)$$

And if $C=4$, then $\mathrm{C}-3=1$ and $2\cdot 1=2\notin S$ and $2\notin SH(S)$. But if $C>4$, then

$$C\ge 8$$
$$2C\ge C+8$$
$$2(C-3)\ge C+2$$
$$2(C-3)\in S$$
$$C-3\in SH(S)$$

and since $2(4k-2)=4(2k-1)\in S$, $4(2k-1)\in SH(S)$. Thus

$$SH(S)=\begin{cases}\{2,3\} & for\ C=4\\ \{4k-2,C-3,C-1\} & for\ C>4\end{cases}$$

ii. If $S=\langle 4,4k+2,\mathrm{C}+1,\mathrm{C}+3\rangle$ when $C\equiv 2(\mathrm{mod}\,4)$ for some $1\le k\le \frac{C-2}{4}$, then $Ap(S,4)=\{0,4k+2,C+1,\mathrm{C}+3\}$ by 2.2 Proposition. Thus $PF(S)=\{4k-2,\mathrm{C}-3,\mathrm{C}-1\}$ by Lemma 3.1. Hence we get

$$C\ge 6$$
$$2C\ge C+6$$
$$2(C-3)\ge C$$
$$2(C-1)\ge C+4$$
$$2(C-3),2(C-1)\in S$$
$$C-3,C-1\in SH(S)$$

and since $2(4k-2)=4(2k-1)\in S$, $4(2k-1)\in SH(S)$. Thus
$\mathrm{SH}(S)=\{4k-2,\mathrm{C}-3,\mathrm{C}-1\}$

iii. If $S=\langle 4,\mathrm{C},\mathrm{C}+2,\mathrm{C}+3\rangle$ and $C\equiv 3(\mathrm{mod}\,4)$, then $Ap(S,4)=\{0,\mathrm{C},C+2,\mathrm{C}+3\}$ by 2.2 Proposition. Thus $PF(S)=\{\mathrm{C}-4,\mathrm{C}-2,\mathrm{C}-1\}$ by Lemma 3.1. Hence we get

$$C\ge 7$$
$$2C\ge C+7$$
$$2(C-1)\ge C+5$$
$$2(C-2)\ge C+3$$
$$2(C-1),2(C-2)\in S$$
$$C-2,\mathrm{C}-1\in SH(S).$$

And if $C=7$, then $\mathrm{C}-4=3$ and $2\cdot 3=6\notin S$ and $6\notin SH(S)$. But if $C>7$, then

$$\begin{aligned} C &\geq 11 \\ 2C &\geq C+11 \\ 2(C-4) &\geq C+3 \\ 2(C-4) &\in S \\ C-4 &\in SH(S). \end{aligned}$$

Thus

$$SH(S)=\begin{cases} \{5,6\} & \text{for } C=7 \\ \{C-4, C-2, C-1\} & \text{for } C>7. \end{cases}$$

3.6 Corollary Let S be a numerical semigroup, as in 2.2 Proposition

i. If $S=\langle 4, 4k+2, C+1, C+3\rangle$ and $C\equiv 0 (\mathrm{mod}\, 4)$ for some $1\leq k\leq \frac{C}{4}$, then $(4k-2)\cup S$ and $(C-1)\cup S$ are Arf numerical semigroups. So

$$(4k-2)\cup S=\begin{cases} \langle 2, C+1\rangle & \text{for } k=1 \\ \langle 4, 4k-2, C+1, C+3\rangle & \text{for } k>1 \end{cases}$$

is an Arf numerical semigroups as with multiplicity 2 or as in 2.2 Proposition[i]

$$(C-1)\cup S=\begin{cases} \langle 4, 4k+2, C-1, C+1\rangle & \text{for } k\neq \frac{C}{4} \\ \langle 4, C-1, C+1, C+2\rangle & \text{for } k=\frac{C}{4} \end{cases}$$

is an Arf numerical semigroups as in 2.2 Proposition[ii] or in 2.2 Proposition[iii]. Specially, if $C=4$, then $(C-1)\cup S=\langle 4, 4k+2, C-1, C+1\rangle=\langle 3,4,5\rangle$ is an Arf numerical semigroup as in 2.1 Proposition[i].

ii. If $S=\langle 4, 4k+2, C+1, C+3\rangle$ and $C\equiv 2 (\mathrm{mod}\, 4)$ for some $1\leq k\leq \frac{C-2}{4}$, then $(4k-2)\cup S$ and $(C-1)\cup S$ are Arf numerical semigroups. Thus,

$$(4k-2)\cup S=\begin{cases} \langle 2, C+1\rangle & \text{for } k=1 \\ \langle 4, 4k-2, C+1, C+3\rangle & \text{for } k\neq 1 \end{cases}$$

is an Arf numerical semigroups with multiplicity 2 or as in 2.2 Proposition[ii]. And $(C-1)\cup S=\langle 4, 4k+2, C-1, C+1\rangle$ is an Arf numerical semigroup as in 2.2 Proposition[i].

iii. If $S=\langle 4, C, C+2, C+3\rangle$ and $C\equiv 3 (\mathrm{mod}\, 4)$, then $\{C-1\}\cup S=\langle 4, C-1, C, C+2\rangle$ is an Arf numerical semigroups as in 2.2 Proposition[ii].

3.7 Theorem Let S be a numerical semigroup, as in 2.3 Proposition.

i. If $C\equiv 0 (\mathrm{mod}\, 5)$ and $S=\langle 5, C-2, C+1, C+2, C+4\rangle$, then

$$SH(S)=\begin{cases} \{6,7,9\} & \text{for } C=10 \\ \{C-7, C-4, C-3, C-1\} & \text{for } C>10 \end{cases}.$$

ii. If $C \equiv 0 (\bmod 5)$ and $S = \langle 5, C+1, C+2, C+3, C+4 \rangle$, then

$$SH(S) = \begin{cases} \{3,4\} & \text{for } C = 5 \\ \{C-4, C-3, C-2, C-1\} & \text{for } C > 5 \end{cases}.$$

iii. If $C \equiv 2 (\bmod 5)$ and $S = \langle 5, C, C+1, C+2, C+4 \rangle$, then

$$SH(S) = \begin{cases} \{4,6\} & \text{for } C = 7 \\ \{C-5, C-4, C-3, C-1\} & \text{for } C > 7 \end{cases}.$$

iv. If $C \equiv 3 (\bmod 5)$ and $S = \langle 5, C, C+1, C+3, C+4 \rangle$, then

$$SH(S) = \begin{cases} \{4,6,7\} & \text{for } C = 8 \\ \{C-5, C-4, C-2, C-1\} & \text{for } C > 8 \end{cases}.$$

v. If $C \equiv 4 (\bmod 5)$ and $S = \langle 5, C-2, C, C+2, C+4 \rangle$, then

$$SH(S) = \begin{cases} \{6,8\} & \text{for } C = 9 \\ \{C-7, C-5, C-3, C-1\} & \text{for } C > 9 \end{cases}.$$

vi. If $C \equiv 4 (\bmod 5)$ and $S = \langle 5, C, C+2, C+3, C+4 \rangle$, then

$$SH(S) = SH(S) = \begin{cases} \{6,7,8\} & \text{for } C = 9 \\ \{C-5, C-3, C-2, C-1\} & \text{for } C > 9 \end{cases}.$$

Proof. We first note that all the semigroups in the 2.3 Proposition are Arf numerical semigroups.

i. If $S = \langle 5, C-2, C+1, C+2, C+4 \rangle$ when $C \equiv 0 (\bmod 5)$, then $Ap(S,5) = \{0, C-2, C+1, C+2, C+4\}$ by 2.3 Proposition. From 3.1 Lemma, $PF(S) = \{C-7, C-4, C-3, C-1\}$. We get

$$\begin{gathered} C \geq 10 \\ 2C \geq C+10 \\ 2(C-4) \geq C+2 \\ 2(C-3) \geq C+4 \\ 2(C-1) \geq C+8 \\ 2(C-4), 2(C-3), 2(C-1) \in S \\ C-4, C-3, C-1 \in SH(S). \end{gathered}$$

And if $C = 10$, then $C-7 = 3$ and $2 \cdot 3 = 6 \notin S$ and $6 \notin SH(S)$. But if $C > 10$, then

$$\begin{gathered} C \geq 15 \\ 2C \geq C+15 \\ 2(C-7) \geq C+1 \\ 2(C-7) \in S \\ C-7 \in SH(S). \end{gathered}$$

Thus,

$$SH(S) = \begin{cases} \{6,7,9\} & \text{for } C = 10 \\ \{C-7, C-4, C-3, C-1\} & \text{for } C > 10 \end{cases}$$

ii. If $S = \langle 5, C+1, C+2, C+3, C+4 \rangle$ and $C \equiv 0 (\bmod 5)$, then $Ap(S,5) = \{0, C+1, C+2, C+3, C+4\}$ by 2.3 Proposition. Thus $PF(S) = \{C-4, C-3, C-2, C-1\}$ by 3.1 Lemma. Hence we get

$$C \geq 5$$
$$2C \geq C+5$$
$$2(C-2) \geq C+1$$
$$2(C-1) \geq C+3$$
$$2(C-2), 2(C-1) \in S$$
$$C-2, C-1 \in SH(S).$$

And if $C=5$, then $C-4=1$ and $C-3=2$. We have $2 \cdot 1 = 2 \notin S$ and $2 \cdot 2 = 4 \notin S$. Hence $1,2 \notin SH(S)$. But if $C>5$, then

$$C \geq 10$$
$$2C \geq C+10$$
$$2(C-4) \geq C+2$$
$$2(C-3) \geq C+4$$
$$2(C-4), 2(C-3) \in S$$
$$C-4, C-3 \in SH(S).$$

Thus,

$$SH(S) = \begin{cases} \{3,4\} & for \;\; C=5 \\ \{C-4, C-3, C-2, C-1\} & for \;\; C>5. \end{cases}$$

iii. If $C \equiv 2 (\mathrm{mod}\, 5)$ and $S = \langle 5, C, C+1, C+2, C+4 \rangle$, then $Ap(S,5) = \{0, C, C+1, C+2, C+4\}$ by 2.3 Proposition. Hence $PF(S) = \{C-5, C-4, C-3, C-1\}$ by 3.1 Lemma. Hence we get

$$C \geq 7$$
$$2C \geq C+7$$
$$2(C-3) \geq C+1$$
$$2(C-1) \geq C+5$$
$$2(C-3), 2(C-1) \in S$$
$$C-3, C-1 \in SH(S).$$

And if $C=7$, then $C-5=2$ and $C-4=3$. We have $2 \cdot 2 = 4 \notin S$ and $2 \cdot 3 = 6 \notin S$. Hence $2,3 \notin SH(S)$. But if $C>7$, then

$$C \geq 12$$
$$2C \geq C+12$$
$$2(C-5) \geq C+2$$
$$2(C-4) \geq C+4$$
$$2(C-5), 2(C-4) \in S$$
$$C-5, C-4 \in SH(S).$$

Thus,

$$SH(S) = \begin{cases} \{4,6\} & for \;\; C=7 \\ \{C-5, C-4, C-3, C-1\} & for \;\; C>7 \end{cases}.$$

iv. If $C \equiv 3 (\mathrm{mod}\, 5)$ and $S = \langle 5, C, C+1, C+3, C+4 \rangle$, then $Ap(S,5) = \{0, C, C+1, C+3, C+4\}$ by 2.3 Proposition. Thus $PF(S) = \{C-5, C-4, C-2, C-1\}$ by 3.1 Lemma. Hence we get

$$C \geq 8$$
$$2C \geq C+8$$
$$2(C-4) \geq C$$

$$2(C-2) \geq C+4$$
$$2(C-1) \geq C+6$$
$$2(C-4), 2(C-2), 2(C-1) \in S$$
$$C-4, C-2, C-1 \in SH(S)$$

And if $C=8$, then $C-5=3$ and $2 \cdot 3 = 6 \notin S$ and $3 \notin SH(S)$. But if $C>8$, then

$$C \geq 13$$
$$2C \geq C+13$$
$$2(C-5) \geq C+3$$
$$2(C-5) \in S$$
$$C-5 \in SH(S).$$

Thus,

$$SH(S) = \begin{cases} \{4,6,7\} & for \;\; C=8 \\ \{C-5, C-4, C-2, C-1\} & for \;\; C>8 \end{cases}.$$

v. If $C \equiv 4 (\text{mod } 5)$ and $S = \langle 5, C-2, C, C+2, C+4 \rangle$, then $Ap(S,5) = \{0, C-2, C, C+2, C+4\}$ by 2.3 Proposition. Thus $PF(S) = \{C-7, C-5, C-3, C-1\}$ by 3.1 Lemma. Hence we get

$$C \geq 9$$
$$2C \geq C+9$$
$$2(C-3) \geq C+3$$
$$2(C-1) \geq C+7$$
$$2(C-3), 2(C-1) \in S$$
$$C-3, C-1 \in SH(S).$$

And if $C=9$, then $C-7=2$ and $C-5=4$. We have $2 \cdot 2 = 4 \notin S$ and $2 \cdot 4 = 8 \notin S$. Hence $2, 4 \notin SH(S)$. But if $C>9$, then

$$C \geq 14$$
$$2C \geq C+14$$
$$2(C-7) \geq C$$
$$2(C-5) \geq C+4$$
$$2(C-7), 2(C-5) \in S$$
$$C-7, C-5 \in SH(S).$$

Thus,

$$SH(S) = \begin{cases} \{6,8\} & for \;\; C=9 \\ \{C-7, C-5, C-3, C-1\} & for \;\; C>9 \end{cases}.$$

vi. If $C \equiv 4 (\text{mod } 5)$ and $S = \langle 5, C, C+2, C+3, C+4 \rangle$, then $Ap(S,5) = \{0, C, C+2, C+3, C+4\}$ by 2.3 Proposition. Thus $PF(S) = \{C-5, C-3, C-2, C-1\}$ by 3.1 Lemma. Hence we get

$$C \geq 9$$
$$2C \geq C+9$$
$$2(C-3) \geq C+3$$
$$2(C-2) \geq C+5$$
$$2(C-1) \geq C+7$$
$$2(C-3), 2(C-2), 2(C-1) \in S$$
$$C-3, C-2, C-1 \in SH(S).$$

And if $C=9$, then $C-5=4$ and $2\cdot 4=8\notin S$ and $4\notin SH(S)$. But if $C>9$, then

$$C\geq 14$$
$$2C\geq C+14$$
$$2(C-5)\geq C+4$$
$$2(C-5)\in S$$
$$C-5\in SH(S).$$

Thus,

$$SH(S)=\begin{cases}\{6,7,8\} & for\ \ C=9\\ \{C-5,C-3,C-2,C-1\} & for\ \ C>9\end{cases}.$$

3.8 Corollary Let S be a numerical semigroup, as in 2.3 Proposition.

i. If $C\equiv 0(\bmod 5)$ and $S=\langle 5,C-2,C+1,C+2,C+4\rangle$, then $\{C-1\}\cup S=\langle 5,C-2,C-1,C+1,C+2\rangle$ is an Arf numerical semigroup as in 2.3 Proposition[iii].

ii. If $C\equiv 0(\bmod 5)$ and $C>5$ and $S=\langle 5,C+1,C+2,C+3,C+4\rangle$, then $\{C-2\}\cup S=\langle 5,C-2,C+1,C+2,C+4\rangle$ and $\{C-1\}\cup S=\langle 5,C-1,C+1,C+2,C+3\rangle$ are Arf numerical semigroups. However, if $C=5$, then $\{C-2\}\cup S$ and $\{C-1\}\cup S$ are as in 2.1 Proposition[ii] and 2.2 Proposition[i], respectively. If $C>5$, then $\{C-2\}\cup S$ and $\{C-1\}\cup S$ are as in 2.3 Proposition[i-a] and 2.3 Proposition[iv-b], respectively.

iii. If $C\equiv 2(\bmod 5)$ and $S=\langle 5,C,C+1,C+2,C+4\rangle$, then $\{C-1\}\cup S=\langle 5,C-1,C,C+1,C+2\rangle$ is an Arf numerical semigroup as in 2.3 Proposition [i-b].

iv. If $C\equiv 3(\bmod 5)$ and $S=\langle 5,C,C+1,C+3,C+4\rangle$, then $\{C-1\}\cup S=\langle 5,C-1,C,C+1,C+3\rangle$ is an Arf numerical semigroup as in 2.3 Proposition[ii].

v. If $C\equiv 4(\bmod 5)$ and $S=\langle 5,C-2,C,C+2,C+4\rangle$, then $\{C-1\}\cup S=\langle 5,C-2,C-1,C,C+2\rangle$ is an Arf numerical semigroup as in 2.3 Proposition [ii].

vi. If $C\equiv 4(\bmod 5)$ and $S=\langle 5,C,C+2,C+3,C+4\rangle$, then $\{C-2\}\cup S=\langle 5,C-2,C,C+2,C+3\rangle$ and $\{C-1\}\cup S=\langle 5,C-1,C,C+2,C+3\rangle$ are Arf numerical semigroups as in 2.3 Proposition [iv-a] and 2.3 Proposition[iii], respectively.

3.9 Theorem Let S be a numerical semigroup as in 2.4 Proposition[i]. For some $1\leq t\leq \frac{C}{6}-1$ and $C\equiv 0(\bmod 6)$.

i. If $S=\langle 6,C+1,C+2,C+3,C+4,C+5\rangle$, then

$$SH(S)=\begin{cases}\{3,4,5\} & for\ \ C=6\\ \{C-5,C-4,C-3,C-2,C-1\} & for\ \ C>6\end{cases}$$

ii. If $S=\langle 6,6t+2,6t+4,C+1,C+3,C+5\rangle$, then

$$SH(S)=\begin{cases}\{4,C-5,C-3,C-1\} & \text{for } t=1\\ \{6t-4,6t-2,C-5,C-3,C-1\} & \text{for } t>1\end{cases}$$

iii. If $S=\langle 6,6t+3,C+1,C+2,C+4,C+5\rangle$, then

$SH(S)=\{6t-3,C-5,C-4,C-2,C-1\}$

iv. If $S=\langle 6,6t+4,6t+8,C+1,C+3,C+5\rangle$, then

$$SH(S)=\begin{cases}\{8,C-5,C-3,C-1\} & \text{for } t=1\\ \{6t-2,6t+2,C-5,C-3,C-1\} & \text{for } t>1\end{cases}$$

Proof. We first note that all the semigroups in the 2.4 Proposition are Arf numerical semigroups.

i. If $C\equiv 0(\bmod 6)$ and $S=\langle 6,C+1,C+2,C+3,C+4,C+5\rangle$, then $Ap(S,6)=\{0,C+1,C+2,C+3,C+4,C+5\}$ by 2.4 Proposition. Thus $PF(S)=\{C-5,C-4,C-3,C-2,C-1\}$ by 3.1 Lemma. Hence we get

$$C\geq 6$$
$$2C\geq C+6$$
$$2(C-3)\geq C$$
$$2(C-2)\geq C+2$$
$$2(C-1)\geq C+4$$
$$2(C-3),2(C-2),2(C-1)\in S$$
$$C-3,C-2,C-1\in SH(S).$$

And if $C=6$, then $C-5=1$ and $C-4=2$. We have $2\cdot 1=2\notin S$ and $2\cdot 2=4\notin S$. Hence $1,2\notin SH(S)$. But if $C>6$, then

$$C\geq 12$$
$$2C\geq C+12$$
$$2(C-5)\geq C+2$$
$$2(C-4)\geq C+4$$
$$2(C-5),2(C-4)\in S$$
$$C-5,C-4\in SH(S).$$

Thus,

$$SH(S)=\begin{cases}\{3,4,5\} & \text{for } C=6\\ \{C-5,C-4,C-3,C-2,C-1\} & \text{for } C>6\end{cases}.$$

ii. If $C\equiv 0(\bmod 6)$ and $S=\langle 6,6t+2,6t+4,C+1,C+3,C+5\rangle$, then $Ap(S,6)=\{0,6t+2,6t+4,C+1,C+3,C+5\}$ by 2.4 Proposition. Thus $PF(S)=\{6t-4,6t-2,C-5,C-3,C-1\}$ by 3.1 Lemma. Hence we get if $C>6$ and $t=1$, then $6t-4=2$ and $2.2=4\notin S$ and $6t-4=2\notin SH(S)$. If $C>6$ and $t>1$, then $6t-4>2$ and $2(6t-4)=6(2t-2)+4=6k+4\in S(\exists k\in\mathbb{N},k=2t-2)$ and $6t-4\in SH(S)$. Similarly $2(6t-2)=6(2t-1)+2=6m+2\in S(\exists m\in\mathbb{N},m=2t-1)$ and $6t-2\in SH(S)$. Moreover,

$$C\geq 12$$
$$2C\geq C+12$$
$$2(C-5)\geq C+2$$
$$2(C-3)\geq C+6$$

$$2(C-1)\geq C+10$$
$$2(C-5),2(C-3),2(\mathrm{C}-1)\in S$$
$$C-5,C-3,C-1\in SH(S).$$

Thus $SH(S)=\begin{cases}\{4,C-5,C-3,C-1\} & t=1\\ \{6t-4,6t-2,C-5,\mathrm{C}-3,\mathrm{C}-1\} & t>1\end{cases}$

iii. If $S=\langle 6,6\mathrm{t}+3,\mathrm{C}+1,C+2,C+4,C+5\rangle$ when $C\equiv 0(\mathrm{mod}\,6)$ and, then $Ap(S,6)=\{0,6\mathrm{t}+3,\mathrm{C}+1,\mathrm{C}+2,C+4,C+5\}$ by 2.4 Proposition. Thus $PF(S)=\{6\mathrm{t}-3,\mathrm{C}-5,\mathrm{C}-4,\mathrm{C}-2,C-1\}$ by 3.1 Lemma. Hence we get that if $t\geq 1$, then $6t-3\geq 3$ and $2\cdot(6t-3)=6\cdot(2t-1)=6k\in S(\exists k\in\mathbb{N},\mathrm{k}=2t-1)$ and $6t-3\in SH(S)$. Also

$$C\geq 12$$
$$2C\geq C+12$$
$$2(C-5)\geq C+2$$
$$2(C-4)\geq C+4$$
$$2(C-2)\geq C+8$$
$$2(C-1)\geq C+10$$
$$2(C-5),2(C-4),2(C-2),2(\mathrm{C}-1)\in S$$
$$C-5,C-4,C-2,C-1\in SH(S)$$

Thus $SH(S)=\{6t-3,C-5,C-4,C-2,C-1\}$.

iv. If $C\equiv 0(\mathrm{mod}\,6)$ and $S=\langle 6,6\mathrm{t}+4,6\mathrm{t}+8,C+1,C+3,C+5\rangle$, then $Ap(S,6)=\{0,6\mathrm{t}+4,6\mathrm{t}+8,\mathrm{C}+1,C+3,C+5\}$. Since $6t+4$, $6t+8$, $C+1$, $C+3$ $C+5$ are also in minimal system of generators of S, $6\mathrm{t}+4,6\mathrm{t}+8,\mathrm{C}+1,C+3,C+5\in Maximals\leq_s Ap(S,6)$. Thus $PF(S)=\{6\mathrm{t}-2,6\mathrm{t}+2,\mathrm{C}-5,\mathrm{C}-3,C-1\}$. Hence we get that if $t=1$, then $6t-2=4$ and $6t+2=8$. So $2\cdot 4=8\notin S$ and $2\cdot 8=16\in S$. We have that $4\notin SH(\mathrm{S})$ and $8\in SH(\mathrm{S})$. If $t>1$, then $6t-2>4$ and $2(6t-2)=6(2t-2)+8=6k+8\in S(\exists k\in\mathbb{N},k=2t-2)$ and $6t-2\in SH(S)$. Similarly $2(6t+2)=6(2t)+4=6m+4\in S(\exists m\in\mathbb{N},\mathrm{m}=2t)$ and $6t+2\in SH(S)$. Also

$$C\geq 12$$
$$2C\geq C+12$$
$$2(C-5)\geq C+2$$
$$2(C-3)\geq C+6$$
$$2(C-1)\geq C+10$$
$$2(C-5),2(C-3),2(\mathrm{C}-1)\in S$$
$$C-5,C-3,C-1\in SH(S).$$

Thus $SH(S)=\begin{cases}\{8=6t+2,C-5,C-3,C-1\} & t=1\\ \{6t-2,6t+2,C-5,\mathrm{C}-3,\mathrm{C}-1\} & t>1\end{cases}$

3.10 Corollary Let S be a numerical semigroup, as in 2.4 Proposition[i]. For some $1\leq t\leq \frac{C}{6}-1$ and $C\equiv 0(\mathrm{mod}\,6)$.

i. If $S=\langle 6,C+1,C+2,C+3,C+4,C+5\rangle$ and $C=6$, then $\{3\}\cup S=\langle 3,7,8\rangle$ and $\{4\}\cup S=\langle 4,6,7,9\rangle$ and $\{5\}\cup S=\langle 5,6,7,8,9\rangle$ are Arf numerical semigroups as in 2.1 Proposition[i] and 2.2 Proposition[ii] and 2.3 Proposition[i-b].
If $S=\langle 6,C+1,C+2,C+3,C+4,C+5\rangle$ and $C>6$, then $\{C-3\}\cup S=\langle 6,C-3,C+1,C+2,C+4,C+5\rangle$ and $\{C-2\}\cup S=\langle 6,\mathrm{C}-2,\mathrm{C}+1,C+2,C+3,C+5\rangle$ and $\{C-1\}\cup S=\langle 6,C-1,C+1,C+2,C+3,C+4\rangle$ are Arf numerical semigroups as in 2.4 Proposition[i-c] and 2.4 Proposition[i-d] and 2.4 Proposition[v-a], respectively.

ii. If $S=\langle 6,6\mathrm{t}+2,6\mathrm{t}+4,C+1,C+3,C+5\rangle$, then $\{C-1\}\cup S=\langle 6,6t+2,6t+4,C-1,C+1,C+3\rangle$ is an Arf numerical semigroups as in 2.4 Proposition[iv-a].
If $t=1$, then $\{4\}\cup S=\langle 4,6,C+1,C+3\rangle$ is an Arf numerical semigroups as in 2.2 Proposition[i].
If $t>1$, then $\{6\mathrm{t}-2\}\cup S=\langle 6,6t-2,6t+2,C+1,C+3,C+5\rangle$ is an Arf numerical semigroups as in 2.4 Proposition[i-d]

iii. If $S=\langle 6,6\mathrm{t}+3,C+1,\mathrm{C}+2,C+4,C+5\rangle$, then

$$\{6\mathrm{t}-3\}\cup S=\begin{cases}\langle 3,C+1,C+2\rangle & \textit{for } t=1\\ \langle 6,6t-3,C+1,\mathrm{C}+2,C+4,C+5\rangle & \textit{for } t>1\end{cases}$$

is an Arf numerical semigroup. When $t=1$, it is an Arf numerical semigroups as in Proposition 1[i]. When $t>1$, it is an Arf numerical semigroups as in 2.4 Proposition[i-c]. $\{C-1\}\cup S=\langle 6,6t+3,C-1,C+1,\mathrm{C}+2,C+4\rangle$ is an Arf numerical semigroups. 2.4 Proposition[v-b]

iv. If $S=\langle 6,6\mathrm{t}+4,6\mathrm{t}+8,\mathrm{C}+1,C+3,C+5\rangle$, then $\{6\mathrm{t}+2\}\cup S=\langle 6,6t+2,6\mathrm{t}+4,\mathrm{C}+1,C+3,C+5\rangle$ and $\{C-1\}\cup S=\langle 6,6t+4,6\mathrm{t}+8,C-1,C+1,C+3\rangle$ are Arf numerical semigroups as in 2.4 Proposition[i-b] and Proposition1[iv-b], respectively.

3.11 Theorem Let S be a numerical semigroup, as in 2.4 Proposition[ii]. For some $1\le u\le \frac{C-2}{6}$ and $1\le v\le \frac{C-2}{6}-1$ and $C\equiv 2(\mathrm{mod}\,6)$.

i. If $S=\langle 6,6u+2,6u+4,C+1,C+3,C+5\rangle$, then

$$SH(S)=\begin{cases}\{6u-2=4,C-5,C-3,C-1\} & \textit{for } u=1\\ \{6u-4,6u-2,C-5,\mathrm{C}-3,\mathrm{C}-1\} & \textit{for } u>1\end{cases}$$

ii. If $S=\langle 6,6v+3,\mathrm{C},C+2,C+3,C+5\rangle$, then

$$SH(\mathrm{S})=\{6v-3,\mathrm{C}-6,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-1\}$$

iii. If $S=\langle 6,6v+4,6v+8,C+1,C+3,C+5\rangle$, then

$$SH(S)=\begin{cases}\{6\mathrm{v}+2=8,C-5,C-3,C-1\} & \textit{for } v=1\\ \{6v-2,6v+2,C-5,\mathrm{C}-3,\mathrm{C}-1\} & \textit{for } v>1\end{cases}.$$

Proof. We first note that all the semigroups in the 2.4 Proposition are Arf numerical semigroups. For some $1 \leq u \leq \frac{C-2}{6}$ and $1 \leq v \leq \frac{C-2}{6} - 1$ and $C \equiv 2 (\text{mod}\, 6)$.

i. If $S = \langle 6, 6u+2, 6u+4, C+1, C+3, C+5 \rangle$ when $C \equiv 2 (\text{mod}\, 6)$ then $Ap(S,6) = \{0, 6u+2, 6u+4, C+1, C+3, C+5\}$ by 2.4 Proposition. Thus $PF(S) = \{6u-4, 6u-2, C-5, C-3, C-1\}$ by 3.1 Lemma. Hence we get that if $u=1$, then $2(6 \cdot 1 - 4) = 4 \notin S$ and $u \neq 1$, then $2(6u-4) = 6(2u-2)+4 \in S$. Hence if $u \neq 1$, then $6u-4 \in SH(S)$. Similarly, $2(6u-2) = 6(2u-1)+2 = 6k+2 \in S (\exists k \in \mathbb{N}, k = 2u-1)$ and $6u-2 \in SH(S)$. Moreover

$$C \geq 8$$
$$2C \geq C+8$$
$$2(C-5) \geq C-2$$
$$2(C-3) \geq C+2$$
$$2(C-1) \geq C+6$$
$$2(C-5), 2(C-3), 2(C-1) \in S$$
$$C-5, C-3, C-1 \in SH(S)$$

Thus, $SH(S) = \begin{cases} \{6u-2=4, C-5, C-3, C-1\} & for\ \ u=1 \\ \{6u-4, 6u-2, C-5, C-3, C-1\} & for\ \ u>1 \end{cases}$.

ii. If $C \equiv 2 (\text{mod}\, 6)$ and $S = \langle 6, 6v+3, C, C+2, C+3, C+5 \rangle$, then $Ap(S,6) = \{0, 6v+3, C, C+2, C+3, C+5\}$ by 2.4 Proposition. Thus $PF(S) = \{6v-3, C-6, C-4, C-3, C-1\}$ by 3.1 Lemma. Since $2(6v-3) = 6(2v-1) = 6k \in S$ $(\exists k \in \mathbb{N},\ k = 2v-1)$, we get $6v-3 \in SH(S)$ and

$$C \geq 14$$
$$2C \geq C+14$$
$$2(C-6) \geq C+2$$
$$2(C-4) \geq C+6$$
$$2(C-3) \geq C+8$$
$$2(C-1) \geq C+12$$
$$2(C-6), 2(C-4), 2(C-3), 2(C-1) \in S$$
$$C-6, C-4, C-3, C-1 \in SH(S)$$

And $SH(\mathrm{S}) = \{6v-3, C-6, C-4, C-3, C-1\}$.

iii. If $C \equiv 2 (\text{mod}\, 6)$ and $S = \langle 6, 6v+4, 6v+8, C+1, C+3, C+5 \rangle$, then $Ap(S,6) = \{0, 6v+4, 6v+8, C+1, C+3, C+5\}$ by 2.4 Proposition. Thus $PF(S) = \{6v-2, 6v+2, C-5, C-3, C-1\}$ by 3.1 Lemma. Hence we get that if $v=1$, then $2(6 \cdot 1 - 2) = 8 \notin S$ and $v \neq 1$, then $2(6v-2) = 6(2v-1)+2 = 6n+2 \in S\ (\exists k \in \mathbb{N},\ n = 2v-1)$. Hence if $v \neq 1$, then $6v-2 \in SH(S)$. Similarly, $2(6v+2) = 6(2v)+4 = 6m+4 \in S\ (\exists k \in \mathbb{N},\ m = 2v)$ and $6v+2 \in SH(S)$. Moreover

$$C \geq 14$$
$$2C \geq C+14$$

$$2(C-5) \geq C+4$$
$$2(C-3) \geq C+8$$
$$2(C-1) \geq C+12$$
$$2(C-5), 2(C-3), 2(C-1) \in S$$
$$C-5, C-3, C-1 \in SH(S)$$

Thus $SH(S)=\begin{cases} \{6v+2, C-5, C-3, C-1\} & for\ v=1 \\ \{6v-2, 6v+2, C-5, C-3, C-1\} & for\ v>1 \end{cases}$.

3.12 Corollary Let S be a numerical semigroup, as in 2.4 Proposition[ii].

i. If $S=\langle 6, 6u+2, 6u+4, C+1, C+3, C+5\rangle$, then $\{6u-2\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups and $\{C-5\}\cup S$ is an Arf numerical semigroup for $u=\dfrac{C-2}{6}$.

If $u=1$, then $\{6u-2\}\cup S=\{4\}\cup S=\langle 4, 6, C+1, C+3\rangle$ is an Arf numerical semigroup as in 2.2 Proposition[ii].

If $u\neq 1$, then $\{6u-2\}\cup S=\langle 6, 6u-2, 6u+2, C+1, C+3, C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii-c].

If $u=\dfrac{C-2}{6}$, then $\{C-1\}\cup S=\langle 6, C-1, C, C+1, C+2, C+3\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[i-a].

If $u\neq\dfrac{C-2}{6}$, then $\{C-1\}\cup S=\langle 6, 6u+2, 6u+4, C-1, C+1, C+3\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[i-b].

If $C=8$, then $\{C-5\}\cup S=\langle 3, 8, 10\rangle$ is an Arf numerical semigroup as in 2.1 Proposition[ii].

If $C>8$ and $u=\dfrac{C-2}{6}$, then $\{C-5\}\cup S=\langle 6, 6u-3=C-5, C, C+2, C+3, C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii-b].

ii. If $S=\langle 6, 6v+3, C, C+2, C+3, C+5\rangle$, then $\{6v-3\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups.

If $v=1$, then $\{6v-3\}\cup S=\{3\}\cup S=\langle 3, C, C+2\rangle$ is an Arf numerical semigroup as in 2.1 Proposition[ii].

If $v\neq 1$, then $\{6v-3\}\cup S=\langle 6, 6v-3, C, C+2, C+3, C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii-b].

And $\{C-1\}\cup S=\langle 6, 6v+3, C-1, C, C+2, C+3\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[i-b].

iii. If $S=\langle 6, 6v+4, 6v+8, C+1, C+3, C+5\rangle$, then $\{6v+2\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups. And $\{6v+2\}\cup S=\langle 6, 6v+2, 6v+4, C+1, C+3, C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii-a].

$\{C-1\}\cup S=\langle 6,6v+4,6v+8,C-1,C+1,C+3\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[i- d]

3.13 Theorem Let S be a numerical semigroup as in 2.4 Proposition[iii]. For some $1\le u\le \frac{C-3}{6}$ and $C\equiv 3(\bmod 6)$,

If $S=\langle 6,6u+3,C+1,C+2,C+4,C+5\rangle$, then

$$SH(S)=\begin{cases}\{6u-3,C-4,C-2,C-1\} & for\ C=9\\ \{6u-3,C-5,C-4,C-2,C-1\} & for\ C>9\end{cases}.$$

Proof. We first note that all the semigroups in the 2.4 Proposition are Arf numerical semigroups.

If $S=\langle 6,6u+3,C+1,C+2,C+4,C+5\rangle$ when $C\equiv 3(\bmod 6)$ then $Ap(S,6)=\{0,6u+3,C+1,C+2,C+4,C+5\}$ by 2.4 Proposition. Thus $PF(S)=\{6u-3,C-5,C-4,C-2,C-1\}$ by 3.1 Lemma. Since $2(6u-3)=6(2u-1)=6t\in S\left(\exists k\in\mathbb{N},\ t=2u-1\right)$ we get $6u-3\in SH(S)$. Moreover, if $C=9$, then $C-5=4$ and $2\cdot 4=8\notin S$ and $C-5=4\notin SH(S)$. But if $C>9$, then

$$C>9$$
$$2C>C+9$$
$$2(C-5)>C-1$$
$$2(C-4)>C+1$$
$$2(C-2)>C+5$$
$$2(C-1)>C+7$$
$$2(C-5),2(C-4),2(C-2),2(C-1)\in S$$
$$C-5,C-4,C-2,C-1\in SH(S).$$

Hence $SH(S)=\begin{cases}\{3,5,7,8\} & for\ C=9\\ \{6u-3,C-5,C-4,C-2,C-1\} & for\ C\neq 9\end{cases}$.

3.14 Corollary Let S be a numerical semigroup, as in 2.4 Proposition[iii]. Then $\{6u-3\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups.

If $u=1$, then $\{6u-3=3\}\cup S=\langle 3,C+1,C+2\rangle$ is an Arf numerical semigroup as in 2.1 Proposition[i]. If $u\neq 1$, then $\{6u-3\}\cup S=\langle 6,6u-3,C+1,C+2,C+4,C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[iii].

If $u=\frac{C-3}{6}$, then $\{C-1\}\cup S=\langle 6,C-1,C,C+1,C+2,C+4\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii- a].

If $u\neq\frac{C-3}{6}$, then $\{C-1\}\cup S=S=\langle 6,6u+3,C-1,C+1,C+2,C+4\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii- b].

3.15 Theorem Let S be a numerical semigroup, as in 2.4 Proposition[iv]. For some $1 \le u \le \frac{C-4}{6}$ and $C \equiv 4 (\mathrm{mod}\, 6)$.

i. If $S = \langle 6, 6u+2, 6u+4, C+1, C+3, C+5 \rangle$, then

$$SH(S) = \begin{cases} \{6u-2=4, C-5, C-3, C-1\} & for\ u=1 \\ \{6u-4, 6u-2, C-5, \mathrm{C}-3, \mathrm{C}-1\} & for\ u>1 \end{cases}$$

ii. *If* $S = \langle 6, 6u+4, 6u+8, C+1, C+3, C+5 \rangle$, *then*

$$SH(S) = \begin{cases} \{6u+2=8, C-5, C-3, C-1\} & for\ u=1 \\ \{6u-2, 6u+2, C-5, \mathrm{C}-3, \mathrm{C}-1\} & for\ u>1 \end{cases}.$$

Proof. We first note that all the semigroups in the 2.4 Proposition are Arf numerical semigroups.

i. If $S = \langle 6, 6\mathrm{u}+2, 6\mathrm{u}+4, \mathrm{C}+1, C+3, C+5 \rangle$, then $Ap(S,6) = \{0, 6\mathrm{u}+2, 6u+4, \mathrm{C}+1, C+3, C+5\}$ by 2.4 Proposition. Thus $PF(S) = \{6\mathrm{u}-4, 6\mathrm{u}-2, \mathrm{C}-5, \mathrm{C}-3, C-1\}$ by 3.1 Lemma. If $u=1$, then $2(6u-4) = 2(6\cdot 1-4) = 2\cdot 2 = 4 \notin S$ and we get $2 \notin SH(S)$, $2(6u-2) = 2(6\cdot 1-2) = 2\cdot 4 = 8 \in S$ and $4 \in SH(S)$. If $u \ne 1$, then $2(6u-4) = 12\mathrm{u}-8 = 6(2\mathrm{u}-2)+4 = 6s+4 \in \mathrm{S}\left(\exists k \in \mathbb{N},\ s = 2u-2\right)$ and $2(6u-2) = 6(2\mathrm{u}-1)+2 = 6t+2 \in \mathrm{S}\left(\exists k \in \mathbb{N},\ t = 2u-1\right)$ and $6u-4, 6u-2 \in SH(S)$. Moreover

$$C \ge 10$$
$$2C \ge C+10$$
$$2(C-5) \ge C$$
$$2(C-3) \ge C+4$$
$$2(C-1) \ge C+8$$
$$2(C-5), 2(\mathrm{C}-3), 2(\mathrm{C}-1) \in S$$
$$C-5, \mathrm{C}-3, C-1 \in SH(S).$$

Thus $SH(S) = \begin{cases} \{6u-2=4, C-5, C-3, C-1\} & for\ u=1 \\ \{6u-4, 6u-2, C-5, \mathrm{C}-3, \mathrm{C}-1\} & for\ u>1 \end{cases}$.

ii. If $S = \langle 6, 6\mathrm{u}+4, 6\mathrm{u}+8, \mathrm{C}+1, C+3, C+5 \rangle$, then $Ap(S,6) = \{0, 6\mathrm{u}+4, 6u+8, \mathrm{C}+1, C+3, C+5\}$ by 2.4 Proposition. Thus $PF(S) = \{6\mathrm{u}-2, 6\mathrm{u}+2, \mathrm{C}-5, \mathrm{C}-3, C-1\}$ by 3.1 Lemma. If $u=1$, then $2\left(6u-2\right) = 2(6\cdot 1-2) = 2\cdot 4 = 8 \notin S$ and $2\left(6u+2\right) = 2(6\cdot 1+2) = 2\cdot 8 = 16 \in S$. So $4 \notin SH(S)$ and $6u+2 = 8 \in SH(S)$. If $u \ne 1$,then $2(6u-2) = 12\mathrm{u}-4 = 6(2\mathrm{u}-1)+2 = 6t-1 \in \mathrm{S}\left(\exists k \in \mathbb{N},\ t = 2u-1\right)$ and $6u-2 \in SH(S)$, $2(6u+2) = 6(2\mathrm{u})+4 = 6m+4 \in \mathrm{S}\left(\exists k \in \mathbb{N},\ m = 2u\right)$ and $6u+2 \in SH(S)$. Moreover

$$C \ge 10$$
$$2C \ge C+10$$
$$2(C-5) \ge C$$
$$2(C-3) \ge C+4$$

$$2(C-1)\geq C+8$$
$$2(C-5),2(\mathrm{C}-3),2(\mathrm{C}-1)\in S$$
$$C-5,\mathrm{C}-3,C-1\in SH(S)$$

Thus $SH(S)=\begin{cases}\{6u+2=8,C-5,C-3,C-1\} & for\ u=1\\ \{6u-2,6u+2,C-5,\mathrm{C}-3,\mathrm{C}-1\} & for\ u>1\end{cases}$.

3.16 Corollary Let S be a numerical semigroup, as in 2.4 Proposition[iv].

i. If $S=\langle 6,6\mathrm{u}+2,6u+4,C+1,C+3,C+5\rangle$, then $\{6\mathrm{u}-2\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups.

   If $u=1$, then $\{6u-2\}\cup S=\{4\}\cup S=\langle 4,6,C+1,C+3\rangle$ is an Arf numerical semigroup as in 2.2 Proposition[ii]. If $u\neq 1$, then $\{6u-2\}\cup S=\langle 6,6u-2,6u+2,C+1,C+3,C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[iv-b].

   And $\{C-1\}\cup S=\langle 6,6u+2,6u+4,C-1,C+1,C+3\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii-a].

ii. If $S=\langle 6,6u+2,6u+4,C+1,C+3,C+5\rangle$, then $\{6u+2\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups.

   $\{6u+2\}\cup S=\langle 6,6u+2,6u+4,C+1,C+3,C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[iv-a].

   If $u=\dfrac{C-4}{6}$, then $\{C-1\}\cup S=\langle 6,6u+3,6u+4,6u+8,C+1,C+3\rangle=\langle 6,C-1,C,C+1,C+3,C+4\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[iii].

   If $u\neq\dfrac{C-4}{6}$, then $\{C-1\}\cup S=\langle 6,6u+4,6u+8,C-1,C+1,C+3\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[ii-c].

3.17 Theorem Let S be a numerical semigroup, as in 2.4 Proposition[v]. For some $1\leq u\leq\frac{C-5}{6}$ and $C\equiv 5(\mathrm{mod}\,6)$.

i. If $S=\langle 6,\mathrm{C},\mathrm{C}+2,C+3,C+4,C+5\rangle$, then

$$SH(\mathrm{S})=\begin{cases}\{\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-2,\mathrm{C}-1\} & for\ C=11\\ \{\mathrm{C}-6,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-2,\mathrm{C}-1\} & for\ C>11\end{cases}$$

ii. If $S=\langle 6,6u+3,\mathrm{C},C+2,C+3,C+5\rangle$, then

$$SH(\mathrm{S})=\begin{cases}\{6u-3,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-1\} & for\ C=11\\ \{6u-3,\mathrm{C}-6,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-1\} & for\ C>11\end{cases}.$$

Proof. We first note that all the semigroups in the 2.4 Proposition are Arf numerical semigroups.

i. If $S=\langle 6,C,C+2,C+3,C+4,C+5\rangle$, then $Ap(S,6)=\{0,\mathrm{C},C+2,\mathrm{C}+3,C+4,C+5\}$ by 2.4 Proposition. Thus $PF(S)=\{C-6,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-2,C-1\}$ by 3.1 Lemma. Hence we get

$$\begin{gathered} C\geq 11 \\ 2C\geq C+11 \\ 2(C-4)\geq C+3 \\ 2(C-3)\geq C+5 \\ 2(C-2)\geq C+7 \\ 2(C-1)\geq C+9 \\ 2(\mathrm{C}-4),2(\mathrm{C}-3),2(\mathrm{C}-2),2(\mathrm{C}-1)\in S \\ \mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-2,\mathrm{C}-1\in SH(S) \end{gathered}$$

If $C=11$, then $2\cdot(C-6)=2\cdot 5=10\notin S$ and $5\notin SH(S)$. But, If $C>11$, then and $2C>C+11$ and $2(C-6)>C-1$. So $2(C-6)\in S$ and $(C-6)\in SH(S)$.

So $SH(\mathrm{S})=\begin{cases}\{\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-2,\mathrm{C}-1\} & for\ C=11\\ \{\mathrm{C}-6,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-2,\mathrm{C}-1\} & for\ C>11\end{cases}$.

ii. If $C\equiv 5(\mathrm{mod}\,6)$ and $S=\langle 6,6\mathrm{u}+3,\mathrm{C},\mathrm{C}+2,C+3,C+5\rangle$, then $Ap(S,6)=\{0,6\mathrm{u}+3,\mathrm{C},\mathrm{C}+2,C+3,C+5\}$ by 2.4 Proposition. Thus $PF(S)=\{6\mathrm{u}-3,\mathrm{C}-6,\mathrm{C}-4,\mathrm{C}-3,C-1\}$ by 3.1 Lemma. Hence we get, then $2\cdot(6\mathrm{u}-3)=6(u-1)=6n\in S\ (\exists k\in\mathbb{N},\ n=u-1)$ and $6u-3\in SH(S)$. Moreover

$$\begin{gathered} C\geq 11 \\ 2C\geq C+11 \\ 2(C-4)\geq C+3 \\ 2(C-3)\geq C+5 \\ 2(C-1)\geq C+9 \\ 2(\mathrm{C}-4),2(\mathrm{C}-3),2(\mathrm{C}-1)\in S \\ \mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-1\in SH(S) \end{gathered}$$

If $C=11$, then $2(C-6)=2\cdot 5=10\notin S$ and $5\notin SH(S)$. But, If $C>11$, then and $2C>C+11$ and $2(C-6)>C-1$. So $2(C-6)\in S$ and $(C-6)\in SH(S)$.

So $SH(\mathrm{S})=\begin{cases}\{6u-3,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-1\} & for\ C=11\\ \{6u-3,\mathrm{C}-6,\mathrm{C}-4,\mathrm{C}-3,\mathrm{C}-1\} & for\ C>11\end{cases}$.

3.18 Corollary Let S be a numerical semigroup, as in 2.4 Proposition[v].

i. If $S=\langle 6,\mathrm{C},\mathrm{C}+2,C+3,C+4,C+5\rangle$, then $\{C-2\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups.
$\{C-2\}\cup S=\langle 6,C-2,C,C+2,C+3,C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[v-b].
$\{C-1\}\cup S=\langle 6,C-1,C,C+2,C+3,C+4\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[iv-b].

ii. If $S=\langle 6,6u+3,\mathrm{C},C+2,C+3,C+5\rangle$, then $\{6\mathrm{u}-3\}\cup S$ and $\{C-1\}\cup S$ are Arf numerical semigroups.

If $u=1$, then $\{6u-3\}\cup S=\langle 3,C,C+2\rangle$ is an Arf numerical semigroup as in 2.1 Proposition[ii].
If $u>1$, then $\{6u-3\}\cup S=\langle 6,6u-3,C,C+2,C+3,C+5\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[v-b].
And $\{C-1\}\cup S=\langle 6,6u+3,C-1,C,C+2,C+3\rangle$ is an Arf numerical semigroup as in 2.4 Proposition[iii].

3.19 Example 1. Let $S=\langle 5,8,9,11,12\rangle$. S is an Arf numerical semigroup as in 2.3 Proposition(iii). For $\{7\}\cup S=\langle 5,7,8,9,11\rangle$ is an Arf numerical semigroup containing S (the only one that differs in just one element). From $\{7\}\cup S=\langle 5,7,8,9,11\rangle$ we obtain a new Arf semigroup which $\{6\}\cup(\{7\}\cup S)=\langle 5,7,8,9,11\rangle$. By repeating this process we obtain all Arf semigroup containing $\langle 5,8,9,11,12\rangle$, which we draw bellow as a graph in Fig. 1

$S=\langle 5,8,9,11,12\rangle$
$\downarrow\cup\{7\}$
$\langle 5,7,8,9,11\rangle$
$\downarrow\cup\{6\}$
$\langle 5,6,7,8,9\rangle$
$\swarrow\cup\{4\}$ $\searrow\cup\{3\}$
$\langle 4,5,6,7\rangle$ $\langle 3,5,7\rangle$
$\swarrow\cup\{2\}$ $\searrow\cup\{3\}$ $\swarrow\cup\{4\}$
$\langle 2,5\rangle$ $\langle 3,4,5\rangle$
$\searrow\cup\{3\}$ $\swarrow\cup\{2\}$
$\langle 2,3\rangle$
$\downarrow\cup\{1\}$
$\mathbb{N}$

Figure1. Arf semigroup containing $\langle 5,8,9,11,12\rangle$.

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