Grassmann images of tensor product surfaces in R^4

Eray DEMİRBAŞ[1], Kadri ARSLAN[2,*]

[1]*Uludağ University, Institute of Science, Department of Mathematics, Bursa.*
[2] *Uludağ University, Arts and Science Faculty, Department of Mathematics, Bursa.*

Abstract

Surfaces in 4-dimensional Euclidean space are the generalization of classical surfaces. They are important for construct geometric model of surfaces taking projections of lower dimensional cases. The Grassmann image of surfaces are also important for theoretical physics. In the present study we consider tensor product surfaces in 4-dimensional Euclidean space R^4. We give necessary and sufficient conditions for tensor product surfaces whose Grassmann images lay on the product of two spheres.

Keywords: Tensor product, Grassmann manifold, Grassmann image.

R^4 de tensör çarpım yüzeylerinin Grassmann görüntüleri

Özet

4-boyutlu Öklid uzayındaki yüzeyler klasik yüzeylerin bir genelleştirilmesidir. Daha düşük boyutlu durumlarda yüzeylerin izdüşümü alınarak yüzeylerin geometrik modellemesi de önemlidir. Ayrıca yüzeylerin Grassmann görüntüleri teorik fizikte de önem taşımaktadır. Bu çalışmada 4- boyutlu Öklid uzayında tensör çarpım yüzeylerinin Grassmann görüntüleri ele alınmıştır. Tensör çarpım yüzeylerinin Grassmann görüntüsünün iki kürenin çarpımı olması için gerek ve yeter şartlar verilmiştir.

Anahtar kelimeler: Tensör çarpım, Grassmann manifold, Grassmann görüntü.

Eray DEMİRBAŞ, eray-demirbas@hotmail.com, http://orcid.org/0000-0003-4292-5880
* Kadri ARSLAN, arslan@uludag.edu.tr, http://orcid.org/0000-0002-1440-7050

1. Introduction

Let M and N be two differentiable manifolds and

$$f : M \to E^m, h : N \to E^n$$

two immersions. The tensor product map is defined by

$$(f \otimes h)(p,q) = f(p) \otimes h(q).$$

Necessary and sufficient conditions for $(f \otimes h)$ to be an immersion were obtained in [7]. Further, tensor products of spherical and equivariant immersions were studied in [6] by F. Decruyenaere, F. Dillen, L. Verstraelen and I Mihai. For many immersions f, h which are not transversal, the tensor product $(f \otimes h)$ is still worthwhile to be investigated and in many cases still produces an immersion. Tensor product surfaces of Euclidean plane curves were studied in [4] and [5] by the Ion Mihai and B. Rouxel see also [8] and [2].

Let be M^n a regular submanifold in R^{n+d} given with the isometric immersion

$$f : D \subset R^n \to M^n \subset R^{n+d},$$

one can take some unit normal vector n(x) at each point $x \in M^n$ and make it a begin at a fixed point o in R^{n+d}. When x varies in M^n, the end point of n(x) describes some submanifold in the sphere $x \in S^{n+d-1}$. This submanifold is called the *spherical image of the* M^n. Instead of n_x one can take the normal space N_x of dimension d. Since N_x depends on x, then it is possible to say that N_x is a space function on M^n. Draw a d-dimensional space N through the fixed point o in R^{n+d} such that N is parallel to N_x. Consequently, N belongs to set of all d-dimensional planes which pass through the fixed point o in R^{n+d}. The set of all d-dimensional planes that pass through the origin $o \in R^{n+d}$ compose a *Grassman manifold* $G(n, n+d)$.

Consequently, the d-dimensional plane in Euclidean space R^{n+d} which passes through the fixed point o in R^{n+d} is called the point in a Grassmann manifold. Denote a point in $G(n, n+d)$ by p. Now, one consider the correspondence $\Psi : x \to p$. Since the image of Ψ is located in a Grassmann manifold $G(n, n+d)$, Ψ is naturally called a *Grassmann mapping* (in analogy to spherical and the image $\Psi(M^n)$ is called the *Grassmann image* of the submanifold M^n. Other names such as *generalized image* or *tangentially image* are used [1]. This paper is organized as follows: In section 2 we give some basic concepts of the Grassmann manifold $G(2,4)$. Using the Plücker relations it can be seen that the Grassmann manifold $G(2,4)$ is isometric to product manifold of two unit sphere S_1^2 and S_2^2 (as a smooth manifold). In Section 3 we consider the Grassmann image of surface M^2 in R^4. We also give the parametrization of the image of the Grassmann mapping

$\Psi : M^2 \to G(2,4)$

for a given surface M^2 in R^4. In the final section we obtained some original results of the tensor product surfaces R^4. Furthermore, we give the necessary and sufficient conditions for the product surfaces whose Grassmann images lay on the product of two spheres.

2. Grassmann manifold G(2,4)

Let x_i be Cartesian coordinates in R^4. The 2 dimensional plane N can be represented by a pair ξ, η of mutually orthogonal unit vectors in this plane. Consider the ordinary bivector

$$p = \xi \wedge \eta \tag{1}$$

with *Plücker coordinates*

$$p_{ij} = \begin{vmatrix} \xi_i & \xi_j \\ \eta_i & \eta_j \end{vmatrix}, \ p_{ij} = -p_{ji}, 1 \le i < j \le 4 \tag{2}$$

where ξ_i and η_i are components of ξ and η respectively.

There are six Plücker coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$. It is easy to check that p_{ij} stay unchanged under a rotation of the basis ξ, η in N. The set of Plücker coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ forms the radius vector p of the corresponding point of a manifold $G(2,4)$ embedded in 6-dimensional Euclidean space R^6. The coordinates of the point p of $G(2,4)$ satisfy the two equations (*Plücker relations*) (see [3]);

(F1) $\langle p, p \rangle = \sum_{i \prec j} p^2_{ij} = 1 \tag{3}$

(F2) $p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23} = 0. \tag{4}$

Consider a bivector

$$q = \tau \wedge \upsilon \tag{5}$$

defined by the plane $N^\perp$ complementary to p, such that $\xi, \eta, \tau, \upsilon$ form a positive oriented basis in R^4. So the scalar product $\langle p, q \rangle = 0$. Consequently, the components q_{ij} of the q can be represented in terms of p_{ij}. Namely,

$q_{12} = p_{34}$

$q_{13} = -p_{24}$

$q_{14} = p_{23}$

$q_{23} = p_{14}$ (6)

$q_{24} = -p_{13}$

$q_{34} = p_{12}$

(See, [1]). The end point of the vector p with components $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ attached to the origin determined the point in R^6. Various points obtained in that way and the coordinates of which satisfy (3) and (4) are located in some algebraic *four-dimensional submanifold* in R^6. That submanifold is the Grassmann manifold $G(2,4)$ immersed in R^6.

Denote by $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}$ the Cartesian coordinates in R^6 from the gradients of the Plücker relations one can determine two normal of the submanifold $G(2,4)$:

$$\frac{1}{2} gradF_1 = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = p$$

$$\frac{1}{2} gradF_2 = (p_{34}, -p_{24}, p_{23}, p_{14}, -p_{13}, p_{12}) = q.$$

Since $\langle p, q \rangle = 0$, these normals are mutually orthogonal and unit, because $\langle p, p \rangle = \langle q, q \rangle = 1$. Equation (3) means that $G(2,4)$ located in the 5-dimensional unit sphere S^5, also q is the normal to S^5. The metric of $G(2,4)$ is induced by the embedding in R^6 (See, [1]).

From the Plücker relations (3) and (4) we can see that the Grassmann manifold $G(2,4)$ is isometric to product manifold of two unit spheres S_1^2 and S_2^2 (as a smooth manifold), i.e., $G(2,4) \equiv S_1^2 \times S_2^2$. The unit normal vectors of the unit sphere S_1^2 and S_2^2 as

$\xi_1 = p_{12} + p_{34},$ $\eta_1 = p_{12} - p_{34}$

$\xi_2 = p_{13} + p_{42},$ $\eta_2 = p_{13} - p_{42}$ (7)

$\xi_3 = p_{14} + p_{23},$ $\eta_3 = p_{14} - p_{23}$

and using the Plücker relations we get

$$\sum_1^3 \xi_i^2 = \sum_1^3 \eta_i^2 = 1$$

One can consider a standard immersion of the product $S_1^2 \times S_2^2 \subset R^6$;

$$x_1^2 + x_2^2 + x_3^2 = 1$$
$$x_4^2 + x_5^2 + x_6^2 = 1.$$

(8)

Consequently this product can be parametrized in the following way;

$$\begin{aligned}
x_1 &= \cos u_1, & x_4 &= \cos u_3 \\
x_2 &= \sin u_1 \sin u_2, & x_5 &= \sin u_3 \sin u_4 \\
x_3 &= \sin u_1 \cos u_2, & x_6 &= \sin u_3 \cos u_4
\end{aligned}$$

(9)

(see [3]). Hence, for a spherical surface S in R^3 the position vector (x_1, x_2, x_3) can be considered as a normal vector of the surface. Therefore

$$\begin{aligned}
\xi_1 &= \cos u_1, & \eta_1 &= \cos u_3 \\
\xi_2 &= \sin u_1 \sin u_2, & \eta_2 &= \sin u_3 \sin u_4 \\
\xi_3 &= \sin u_1 \cos u_2, & \eta_3 &= \sin u_3 \cos u_4
\end{aligned}$$

(10)

holds.

3. Grassmann image of a surface

Let $M^2 \subset R^4$ be a regular curve given with a regular patch

$$x : D \subset R^2 \to R^4.$$

The tangent space $T_x M^2$ of M^2 at point $x(u, v)$ is spanned by the vectors

$$x_u = \frac{\partial x}{\partial u}, \quad x_v = \frac{\partial x}{\partial v}.$$

(11)

Using the Gram Schmidt orthonormalization process to the vectors x_u, x_v we obtain the following orthonormal vectors

$$e_1 = \frac{x_u}{\sqrt{g_{11}}}$$

$$e_2 = \frac{1}{\sqrt{g_{11}}\sqrt{g}}(g_{11}x_v - g_{12}x_u)$$

where g_{ij} denotes the metric tensor of M^2 and $g = \det(g_{ij})$.

Let N be a plane parallel to N^x of M^2. If the normal plane N^x is spanned by the orthonormal vectors n_1, n_2 then the bivector p is of the form $p = n_1 \wedge n_2$.
Since $n_1(u, v), n_2(u, v)$ are vector functions of u and v then the image of a Grassmann mapping

$$\Psi : M^n \mapsto G(2,4); \; x(u,v) \mapsto \Psi(x(u,v)) = p(u,v)$$

is in general some two dimensional surface Γ^2 which is located in the submanifold $G(2,4)$. The unit bivector

$$q = \frac{x_u \wedge x_v}{\sqrt{g}} \tag{12}$$

is the complement of p. Further, the normal space of the Grassmann submanifold $G(2,4)$ (of the surface M^2) at point p is spanned by

$$q = \frac{x_u \wedge x_v}{\sqrt{g}}, p = n_1 \wedge n_2. \tag{13}$$

So, the image of the Grassmann mapping $\Psi : M^2 \to G(2,4)$ has the parametrization

$$x(u,v) \mapsto \Psi(x(u,v)) = p(u,v) = \begin{pmatrix} p_{12} \\ p_{13} \\ p_{14} \\ p_{23} \\ p_{24} \\ p_{34} \end{pmatrix} = \begin{pmatrix} q_{34} \\ -q_{34} \\ q_{23} \\ q_{14} \\ -q_{13} \\ q_{12} \end{pmatrix}. \tag{14}$$

3. Results

In the following section, we will consider the tensor product immersions, actually surfaces in R^4, which are obtained from two Euclidean plane curves. We recall definitions and results of [7]. Let $c_1 : R \to R^2$ and $c_2 : R \to R^2$ be two Euclidean curves. Put $c_1(u) = (\lambda(u), \delta(u))$ and $c_2(v) = (\alpha(v), \beta(v))$ then their tensor product surface is given by patch

$$x = c_1 \otimes c_2 : R^2 \to R^4;$$
$$x(u,v) = (\alpha(v)\lambda(u), \beta(v)\lambda(u), \alpha(v)\delta(u), \beta(v)\delta(u)) \tag{15}$$

(see [8] and [6]). If we take c_1 as an unit plane circle centered at 0 and $c_2(v) = (\alpha(v), \beta(v))$ is an Euclidean plane curve. Then the surface patch becomes

$$M^2 : x(u,v) = (\alpha(v)\cos u, \beta(v)\cos u, \alpha(v)\sin u, \beta(v)\sin u). \tag{16}$$

The tangent space of M^2 is spanned by

$$x_u = (-\alpha \sin u, -\beta \sin u, \alpha \cos u, \beta \cos u),$$
$$x_v = (\alpha' \cos u, \beta' \cos u, \alpha' \sin u, \beta' \sin u).$$

Hence the coefficients of first fundamental form of the surface are

$$g_{11} = \langle x_u, x_u \rangle = \alpha^2 + \beta^2$$
$$g_{12} = \langle x_u, x_v \rangle = 0 \tag{17}$$
$$g_{22} = \langle x_v, x_v \rangle = (\alpha')^2 + (\beta')^2.$$

We obtain the following result.

Proposition 1. Let M^2 be tensor product surface given with the parametrization (16). Then the Grassmann image Γ^2 of M^2 has the parametrization

$$p(u,v) = \frac{1}{\lambda} \begin{pmatrix} (\alpha\beta' - \alpha'\beta)\cos u \sin u \\ \beta\beta' \\ -\alpha'\beta \sin^2 u - \alpha'\beta \cos^2 u \\ -\alpha\beta' \sin^2 u - \alpha'\beta \cos^2 u \\ \alpha\alpha' \\ (\alpha'\beta - \alpha\beta')\cos u \sin u \end{pmatrix} \tag{18}$$

where $\lambda^2 = g_{11}g_{22} - g_{12}^2$ is the Riemannian metric on M^2.

Proof. This components q_{ij} of the bivector $q = \dfrac{x_u \wedge x_v}{\lambda}$ can be represented in terms of p_{ij} by

$$q_{12} = p_{34} = \frac{(\alpha\beta' - \alpha'\beta)\cos u \sin u}{\lambda}$$

$$q_{13} = -p_{24} = \frac{-\alpha\alpha'}{\lambda}$$

$$q_{14} = p_{23} = \frac{-\alpha\beta' \sin^2 u - \alpha'\beta \cos^2 u}{\lambda} \tag{19}$$

$$q_{23} = p_{14} = \frac{-\alpha'\beta \sin^2 u - \alpha'\beta \cos^2 u}{\lambda}$$

$$q_{24} = -p_{13} = \frac{-\beta\beta'}{\lambda}$$

$$q_{34} = p_{12} = \frac{(\alpha\beta' - \alpha'\beta)\cos u \sin u}{\lambda}$$

So, substituting (18) into (14) we obtain (17).

As a consequence of Proposition 1 we obtain the following result.

Theorem 2. Let M^2 be a tensor product surface given with the parametrization (16). If the Grassmann image Γ^2 has the parametrization (16) then the surface M^2 is a Clifford

torus in R^4.

Proof. Using (7) with (19) one can get

$$\xi_1 = p_{12} + p_{34} = 0,$$
$$\xi_2 = p_{13} + p_{42} = \frac{\beta\beta' - \alpha\alpha'}{\lambda},$$
$$\xi_3 = p_{14} + p_{23} = -\frac{(\alpha'\beta + \alpha\beta')}{\lambda},$$

(20)

and

$$\eta_1 = p_{12} - p_{34} = \frac{(\alpha'\beta - \alpha\beta')\sin(-2u)}{\lambda}$$
$$\eta_2 = p_{13} - p_{42} = \frac{\alpha\alpha' + \beta\beta'}{\lambda}$$
$$\eta_3 = p_{14} - p_{23} = \frac{(\alpha'\beta - \alpha\beta')\cos(-2u)}{\lambda}$$

(21)

For simplicity, if we take

$$\beta\beta' - \alpha\alpha' = \varphi,$$
$$\alpha\alpha' + \beta\beta' = \delta$$
$$-(\alpha'\beta + \alpha\beta') = \theta,$$
$$\alpha'\beta - \alpha\beta' = \phi$$

(22)

then

$$\xi_1 = p_{12} + p_{34} = 0$$
$$\xi_2 = p_{13} + p_{42} = \frac{\phi}{\lambda}$$
$$\xi_3 = p_{14} + p_{23} = \frac{\theta}{\lambda}$$

(23)

and

$$\eta_1 = p_{12} - p_{34} = \frac{\varphi\sin(-2u)}{\lambda}$$
$$\eta_2 = p_{13} - p_{42} = \frac{\delta}{\lambda}$$
$$\eta_3 = p_{14} - p_{23} = \frac{\varphi\cos(-2u)}{\lambda}$$

(24)

holds. For the values
$\phi = $ constant and $\delta = 0$ we obtain a circle on S_2^2. So we get

$$\alpha'\beta - \alpha\beta' = c$$
$$\alpha\alpha' + \beta\beta' = 0.$$

(25)

Consequently these equalities show that the planar curve $c_2(v) = (\alpha(v), \beta(v))$ is a circle given with the parametrization

$$\alpha(v) = r\cos\left(\frac{cv}{r^2}\right),$$
$$\beta(v) = r\sin\left(\frac{cv}{r^2}\right).$$

Hence the tensor product surface M^2 becomes a Clifford torus.

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