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Analytical solutions to the advection-diffusion equation with the Atangana-Baleanu derivative over a finite domain

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Abstract

In this paper, an advection-diffusion equation with Atangana-Baleanu derivative is considered. Cauchy and Dirichlet problems have been described on a finite interval. The main aim is to scrutinize the fundamental solutions for the prescribed problems. The Laplace and the finite sin-Fourier integral transformation techniques are applied to determine the concentration profiles corresponding to the fundamental solutions. Results have been obtained as linear combinations of one or bi-parameter Mittag-Leffler functions. Consequently, the effects of the fractional parameter and drift velocity parameter on the fundamental solutions are interpreted by the help of some illustrative graphics.

Keywords: *Atangana-Baleanu derivative, advection-diffusion equation, Laplace integral transformation, Mittag-Leffler function, fundamental solution.*

Sonlu bir bölge üzerinde Atangana-Baleanu türevli adveksiyon-difüzyon denklemine analitik çözümler

Özet

Bu çalışmada Atangana-Baleanu türevli bir adveksiyon-difüzyon denklemi ele alınmıştır. Cauchy ve Dirichlet problemleri sonlu bir aralıkta tanımlanmıştır. Asıl amaç, belirlenen problemler için temel çözümleri irdelemektir. Temel çözümlere karşılık gelen konsantrasyon profillerini belirlemek için Laplace ve sonlu sin-Fourier integral dönüşüm teknikleri uygulanmıştır. Sonuçlar, bir veya iki parametreli Mittag-

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Leffler fonksiyonlarının lineer kombinasyonları olarak elde edilmiştir. Sonuç olarak, kesirli parametrenin ve sürüklenme hızı parametresinin çözümler üzerindeki etkileri bazı açıklayıcı grafikler yardımıyla yorumlanmıştır.

Anahtar Kelimeler: *Atangana-Baleanu türevi, adveksiyon-difüzyon denklemi, Laplace integral dönüşümü, Mittag-Leffler fonksiyonu, temel çözüm.*

1. Introduction

Solute dispersion under the combined effects of diffusion and advection in heterogeneous porous medium is modelled by parabolic advection-diffusion equations (ADE). Many harmful effects on humans and environment such as atmospheric pollutions, contaminated flows in groundwater aquifers, chemical and migration of contaminations in the seawaters and river systems, tracer dispersion in a porous medium are modeled by such type of equations [1-4].

In the recent decades, fractional calculus has been a powerful tool to describe many complex dynamics such as glassy and porous media, dielectric materials, polymers, biological systems. Riemann-Liouville (RL) and Caputo fractional operators which give more realistic description to many physical phenomena than the integer order derivatives are two foremost and commonly used definitions of fractional calculus [5-9]. It is well-known that these operators have non-local description by including non-singular power-law function and hence many computational difficulties make it impossible to find the analytical solutions of fractional order models. Therefore, most of the studies related to fractional order models are focused on the developing of numerical methods [10-15].

The time-nonlocal generalization of Fick's law leads to the time-fractional ADE in terms of Caputo derivative. An extensive survey of the mathematical and physical background of the fractional ADE can be found in [16]. The analytical solutions of one-dimensional fractional advection-diffusion equation have been obtained in terms of H-function [17, 18]. The fundamental solutions to time-fractional advection-diffusion equation have been analyzed in different domains by using integral transform techniques [19-22].

Even though the concepts of RL and Caputo fractional derivatives are advantageous for describing the hereditary and memory features naturally arising in real-world problems, both the derivatives cannot specify the complete memory in the systems and also it is commonly difficult to obtain the analytical solutions because of the singular power kernels in their definitions. Furthermore, many dissipative physical processes such as diffusion, heat transfer, and stress-strain relations cannot be accurately obeyed to the power-law function. All these inadequacies resulting from the natural definitions of conventional fractional derivatives have led to the emergence of new fractional derivatives with non-singular kernels.

In this sense, Caputo and Fabrizio [23] have proposed a fractional derivative with exponential kernel for modelling of relaxation phenomena in a dissipative system. After that Losada and Nieto [24] have defined fractional integral of this new derivative. By using the integral definition, Caputo and Fabrizio [25] studied on some constitutive relations related to fractional diffusion equation. Notice that Caputo-Fabrizio (CF) derivative has been considered only as a formal mathematical definition without any physical background at the beginning. However, Hristov [26] have demonstrated that this definition is naturally arising from a fundamental relation between the flux and the gradient of exponential decay function.

In order to compare the effects of Caputo and Caputo - Fabrizio derivatives on an advection partial differential equation, Baleanu et al. [27] gave a detailed analysis by using some iterative techniques. Rubbab et al. [28] studied the analytical solutions of the Dirichlet problem for an ADE with CF derivative. Hristov [29] gave the real physical relationship between the Cattaneo model for flux relaxation mdelled by Jeffrey’s exponential kernel and the heat diffusion with CF derivative. Singh et al. [30] studied the existence and uniqueness of an epidemiological model for computer viruses with the CF derivative. Yavuz and Evirgen [31] researched the optimal solution trajectories for optimization problem modelled in terms of CF operator.

As a similar manner, Atangana and Baleanu [32] proposed a derivative in sense of RL and Caputo definitions with non-singular Mittag-Leffler function as a memory kernel. It should be noted that the Atangana-Baleanu (AB) derivative is also interpreted as a filter regulator, similar to the CF derivative as well as obeying the derivative properties [33]. The Laplace transformation of AB derivative that is applied to the present study requires physically interpretable integer order initial conditions. It is a remarkable advantageous to model various physical processes in the nature. Yavuz et al. [34] compared approximate-analytical solutions of some types of time-fractional partial differential equations with singular and non-singular kernels by using combined Laplace perturbations method. Alqahtani [35] studied the groundwater model in terms of AB derivative in the subsurface formation known as the unconfined aquifer.

This study addresses the advection diffusion equation with and its initial and boundary value problems, when the Atangana-Baleanu derivative is used. It is organized as follows. Section 2 gives some basic definitions and properties belong to AB operator and Laplace and finite-Fourier transforms. Section 3 is devoted to the Cauchy and Dirichlet problems formulated on a finite interval. Finally, concluding remarks are given in Section 4.

2 Preliminaries

This section will provide some fundamental definitions and properties for the AB derivative.
Let $(a,b)\subset\mathbb{R}$ and let u be a function of the Hilbert space $L^2(a,b)$. u' denotes the derivative of u as distribution on (a,b).

Definition 2.1: The Sobolev space of order 1 in (a,b) is defined as

$$H^1(a,b)=\{u\in L^2(a,b)\,\big|\, u'\in L^2(a,b)\}.$$

Definition 2.2 : Let $\alpha \in (0,1)$ and a function $u \in H^1(a,b),\ b > a.$ The AB fractional derivative in Caputo sense of order α of u with a based point a is defined as [32]

$$ {}^{ABC}D_t^{\alpha}u(t) = \frac{B(\alpha)}{1-\alpha}\int_{\alpha}^{t} u'(s)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right]ds, \tag{3} $$

where $B(\alpha)$ denotes the normalization function which gives a fine-tuning to the corresponding response of a physical process modelled by AB derivative and has the similar properties as in Caputo and Fabrizio case, and is defined as

$$ B(\alpha) = 1-\alpha + \frac{\alpha}{\Gamma(\alpha)}. $$

$E_{\alpha,\beta}(z)$ is the well-known bi-parametric Mittag-Leffler function such that $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$ is defined as by the following series expansion [5]

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},\ (\alpha,\beta > 0). \tag{4} $$

The Mittag-Leffler kernel in AB derivative has a combined form of the exponential and power laws. Therefore, it gives a probabilistic transition between the power-law and the stretched exponential as a waiting time distribution.

Definition 2.3: Let $\alpha \in (0,1)$ and a function $u \in H^1(a,b),\ b > a.$ The AB derivative in the Riemann-Liouville sense of order α of u is defined as [32]

$$ {}^{ABR}D_t^{\alpha}u(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{\alpha}^{t} u(s)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right]ds. \tag{5} $$

Theorem 2.4: The Laplace transform of AB derivative in Riemann-Liouville and Caputo sense are given respectively as [32]

$$ \mathcal{L}\left\{ {}_0^{ABR}D_t^{\alpha}\left\{f(t)\right\}\right\}(s) = \frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\left\{f(t)\right\}(s)}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}, \tag{6} $$

$$ \mathcal{L}\left\{ {}_0^{ABC}D_t^{\alpha}\left\{f(t)\right\}\right\}(s) = \frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\left\{f(t)\right\}(s) - s^{\alpha-1}f(0)}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}. \tag{7} $$

The finite sin-Fourier transform is defined in the domain $0 \le x \le L$ as [8]:

$$\mathrm{F}\left\{f(x)\right\}=\tilde{f}(\xi_k)=\int_0^L f(x)\sin(\xi_k x)dx \tag{8}$$

with its inverse transform:

$$\mathrm{F}^{-1}\left\{\tilde{f}(\xi_k)\right\}=f(x)=\frac{2}{L}\sum_{k=1}^{\infty}\tilde{f}(\xi_k)\sin(\xi_k x) \tag{9}$$

where $\xi_k=\dfrac{k\pi}{L},\ k=1,2,3,....$ The finite sin-Fourier transform of the second order derivative of a given function can be calculated by the following property [8]:

$$\mathrm{F}\left\{\frac{d^2 f(x)}{dx^2}\right\}=-\xi_k{}^2\tilde{f}(\xi_k)+\xi_k\left[f(0)-(-1)^k f(L)\right]. \tag{10}$$

It is well known fact that boundaries of the corresponding domain of a boundary-value problem determine the integral transformation techniques to be applied.

3 Fundamental solutions

In this section, we investigate the fundamental solutions to the Cauchy and Dirichlet problems based upon the ADE with AB derivative detailed in the below sub-sections:

3.1 Cauchy problem

The ADE with regard to AB derivative on a finite interval is defined by

$${}_0^{ABC}D_t^{\alpha}c(x,t)=a\frac{\partial^2 c(x,t)}{\partial x^2}-\upsilon\frac{\partial c(x,t)}{\partial x},\quad x\in(0,L),\ t>0, \tag{11}$$

for $\alpha\in(0,1)$, diffusion coefficient $a>0$ and the velocity quantity $\upsilon>0$.

We consider Eq. (11) corresponding to the following initial condition

$$t=0:\ \ c(x,t)=\delta(x-\zeta),\ \zeta\in(0,L) \tag{12}$$

such that Dirac delta function is rather preferred into the description of initial-boundary value problems because of its useful integral properties. In addition, we suppose the homogeneous boundary conditions at the ends of the segment are considered.

$$\begin{aligned}&x=0:\ \ c(x,t)=0,\\&x=L:\ \ c(x,t)=0.\end{aligned} \tag{13}$$

To simplify the problem (11)-(13), we use an auxiliary function

$$c(x,t)=\exp\left(\frac{\upsilon x}{2a}\right)u(x,t), \tag{14}$$

which reduces the main problem into the following form:

$${}_{0}^{ABC}D_t^{\alpha}u(x,t)=a\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\upsilon^2}{4a}u(x,t), \tag{15}$$

$$u(x,0)=\delta(x-\zeta)\exp\left(-\frac{\upsilon\zeta}{2a}\right), \tag{16}$$

$$u(0,t)=u(L,t)=0. \tag{17}$$

While obtaining the initial condition (16), we have used the shifting property $f(x)\delta(x-\zeta)=f(\zeta)\delta(x-\zeta)$ that can be implemented from the point of distribution theory.

Applying the Laplace transform with respect to time t and finite sin-Fourier transform with respect to the spatial coordinate x with the initial and boundary conditions gives:

$$\bar{u}^{*}(\xi_k,s)=\frac{\gamma}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}\frac{s^{\alpha-1}}{s^{\alpha}+\frac{\alpha\gamma\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}}\sin(\xi_k\zeta)\exp\left(-\frac{\upsilon\zeta}{2a}\right)\ ,\ \gamma=\frac{1}{1-\alpha}. \tag{18}$$

Here, s represents the Laplace transform variable and $\xi_k=\frac{k\pi}{L}$. By inverting the integral transforms, we obtain:

$$u(x,t)=\frac{2}{L}\exp\left(-\frac{\upsilon\zeta}{2a}\right)\sum_{k=1}^{\infty}\frac{\gamma}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}E_{\alpha}\left(-\frac{\alpha\gamma\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}t^{\alpha}\right)\sin(\xi_k\zeta)\sin(\xi_k x), \tag{19}$$

where E_{α} is the Mittag-Leffler function with one-parameter that has the following useful Laplace transform formula [8]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+b}\right\}=E_{\alpha}(-bt^{\alpha}). \tag{20}$$

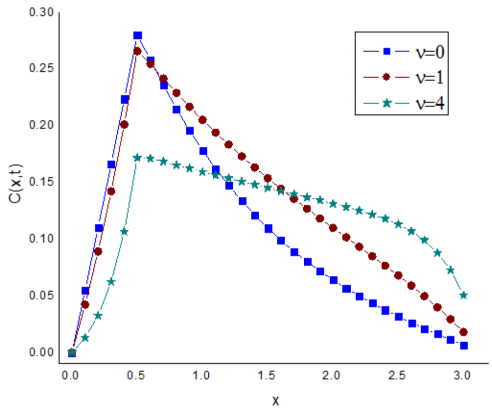
By substituting Eq. (19) into the Eq. (14), we obtain the concentration function $c(x,t)$ as the result of Cauchy problem

$$c(x,t)=\frac{2}{L}\exp\left(\frac{\upsilon(x-\zeta)}{2a}\right)\sum_{k=1}^{\infty}\frac{\gamma}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}E_\alpha\left(-\frac{\alpha\gamma\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}t^\alpha\right)\sin(\xi_k\zeta)\sin(\xi_k x). \quad (21)$$

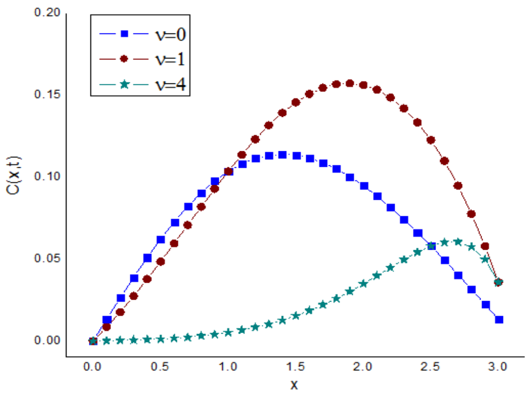
In the case of $\alpha=1$, i.e. $\gamma\to\infty$, the classical solution of advection-diffusion equation solution

$$c(x,t)=\frac{2}{L}\exp\left(\frac{\upsilon(x-\zeta)}{2a}\right)\sum_{k=1}^{\infty}\exp\left(-\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)t\right)\sin(\xi_k\zeta)\sin(\xi_k x). \quad (22)$$

To obtain all the graphics, we assume time a (diffusion coefficient) $=t$ (time) $=1$ for only computational convenience. We first illustrate the concentration profiles, respectively for fractional and integer values of α, in the Figures (1a) and (1b). In both of the graphics, we obtain an inverse correlation between drift parameter and the concentration, i.e. the values of $c(x,t)$ are decreasing while the drift parameter is increasing. In the case of $\alpha=0.5$, sharps in the solutions around the $x=0.5$ have been observed. We must note that this character is surely depends on our arbitrary choices on problem parameters. Another interesting outcome, despite the different drift parameters, concentration profiles reach the maximum values in the case $\alpha=0.5$ around the $x=0.5$. However, this condition changes for $\alpha=1$, the maximum concentration changes with respect to the spatial coordinate x.



(a)



(b)

Figure 1. Dependence of the concentration function as the result of Cauchy problem on drift velocity parameter: (a) $\alpha=0.5, t=1$ and (b) $\alpha=t=1$.

3.2 Dirichlet problem

In this section, we consider two types of boundary-value problems on ADE equipped with AB derivative.

1. Case:

$$ {}^{ABC}_{\ 0}D_t^{\alpha} c(x,t) = a\frac{\partial^2 c(x,t)}{\partial x^2} - \upsilon \frac{\partial c(x,t)}{\partial x}, \quad x \in \left(0, L\right),\ t > 0, \tag{23} $$

where $t > 0$, $a > 0$, $\upsilon > 0$,

with the assumption of homogeneous initial condition

$$ t = 0: \quad c(x,t) = 0, \tag{24} $$

and the Dirichlet boundary conditions at the ends of the segment:

$$ \begin{aligned} x = 0: &\quad c(x,t) = \delta(t), \\ x = L: &\quad c(x,t) = 0. \end{aligned} \tag{25} $$

Similar to the Cauchy problem, by introducing the auxiliary function $u(x,t)$ defined by Eq. (14), Dirichlet problem (23)-(25) reduces to

$$ {}^{ABC}_{\ 0}D_t^{\alpha} u(x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\upsilon^2}{4a} u(x,t), \tag{26} $$

with the initial condition

$$ u(x,0) = 0, \tag{27} $$

and the boundary conditions

$$ \begin{aligned} &u(0,t) = \delta(t), \\ &u(L,t) = 0. \end{aligned} \tag{28} $$

By applying the Laplace and the finite sin-Fourier transform, respectively, we get

$$ \overline{u}^*(\xi_k, s) = a\xi_k \left(\frac{1}{\gamma + a\xi_k^2 + \frac{\upsilon^2}{4a}} + \frac{a\gamma^2}{\left(\gamma + a\xi_k^2 + \frac{\upsilon^2}{4a}\right)^2} \frac{1}{s^{\alpha} + \frac{a\gamma\left(a\xi_k^2 + \frac{\upsilon^2}{4a}\right)}{\gamma + a\xi_k^2 + \frac{\upsilon^2}{4a}}} \right). \tag{29} $$

After taking the inverse integral transforms, the auxiliary function is found in the following form

$$u(x,t)=\frac{2a}{L}\sum_{k=1}^{\infty}\xi_k\left(\frac{\delta(t)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}+\frac{a\gamma^2}{\left(\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}\right)^2}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{a\gamma\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}t^{\alpha}\right)\right)\sin(\xi_k x) \quad (30)$$

where bi-parameter Mittag-Leffler function $E_{\alpha,\alpha}$ has the following commonly used Laplace transform property

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}+b}\right\}=t^{\alpha-1}E_{\alpha,\alpha}(-bt^{\alpha}). \quad (31)$$

Returning to the concentration function $c(x,t)$ given by Eq. (14), the fundamental solution can be obtained:

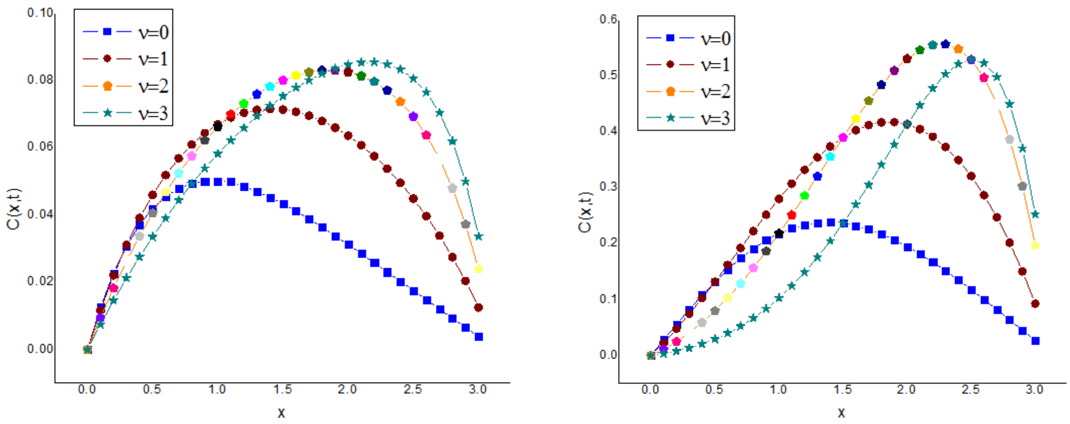
$$c(x,t)=\frac{2}{L}a\exp\left(\frac{\upsilon x}{2a}\right)\sum_{k=1}^{\infty}\left[\frac{\delta(t)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}+\frac{a\gamma^2}{\left(\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}\right)^2}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{a\gamma\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}t^{\alpha}\right)\right]\xi_k\sin(\xi_k x) \quad (32)$$

The first term $\delta(t)$ is a natural result also encountered in the fundamental solutions arising from the ADE model with CF derivative [37]. Moreover, it is a remarkable difference between the fundamental solutions of ADE models with Caputo fractional derivative [19-22], and the CF models [28, 37] with the current AB model for ADE.

By using the definition $\delta(t)=0$ for $t>0$ and taking the limit for $\alpha=1$, i.e. $\gamma\to\infty$, in the Eq. (32), we arrive

$$c(x,t)=\frac{2}{L}a\exp\left(\frac{\upsilon x}{2a}\right)\sum_{k=1}^{\infty}\xi_k\exp\left(-\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)t\right)\sin(\xi_k x). \quad (33)$$

In the Figures (2a) and (2b), we illustrate the concentration profiles belonging to the 1. case for Dirichlet problem under the variations of drift parameter. As seen in Figure (2a), the sharps disappear at the maximum values of concentrations. We can clearly see the drift parameter affects the skewness of the solutions.



(a) (b)

Figure 2. Dependence of concentration function resulting from 1. case of Dirichlet problem: (a) $\alpha = 0.5$ and (b) $\alpha = 1$.

2. Case:

Moreover, we investigate fundamental solution of Eq. (26) under the influence of zero initial condition and the non-dimensional constant concentration at the boundary:

$$\begin{aligned} x=0: &\quad c(x,t)=c_0, \\ x=L: &\quad c(x,t)=0. \end{aligned} \tag{34}$$

Similarly, applying the integral transforms yields

$$\bar{u}^*(\xi_k,s)=\frac{\xi_k c_0}{\xi_k^2+\dfrac{\upsilon^2}{4a^2}}\left(\frac{1}{s}-\frac{\gamma}{\left(\gamma+a\xi_k^2+\dfrac{\upsilon^2}{4a}\right)}\frac{s^{\alpha-1}}{s^{\alpha}+\dfrac{\alpha\gamma\left(a\xi_k+\dfrac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\dfrac{\upsilon^2}{4a}}}\right). \tag{35}$$

Inverting the transforms leads to,

$$u(x,t)=c_0\left[\frac{\sinh\left(\dfrac{v}{2a}(L-x)\right)}{\sinh\left(\dfrac{vL}{2a}\right)} -\frac{2}{L}\sum_{k=1}^{\infty}\frac{\xi_k\gamma}{\left(\xi_k^2+\dfrac{\upsilon^2}{4a^2}\right)\left(\gamma+a\xi_k^2+\dfrac{\upsilon^2}{4a}\right)}E_\alpha\left(-\frac{\alpha\gamma\left(a\xi_k^2+\dfrac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\dfrac{\upsilon^2}{4a}}t^{\alpha}\right)\sin(\xi_k x)\right] \tag{36}$$

and then we have the concentration function $c(x,t)$:

$$c(x,t)=c_0\exp\left(\frac{vx}{2a}\right)\left[\frac{\sinh\left(\frac{\upsilon}{2a}(L-x)\right)}{\sinh\left(\frac{\upsilon L}{2a}\right)}\right.$$
$$\left.-\frac{2}{L}\sum_{k=1}^{\infty}\frac{\xi_k\gamma}{\left(\xi_k^2+\frac{\upsilon^2}{4a^2}\right)\left(\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}\right)}E_\alpha\left(-\frac{\alpha\gamma\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)}{\gamma+a\xi_k^2+\frac{\upsilon^2}{4a}}t^\alpha\right)\sin(\xi_k x)\right]. \tag{37}$$

In the usual sense for $\alpha=1$ for $t>0$, we obtain

$$c(x,t)=c_0\exp\left(\frac{vx}{2a}\right)\left[\frac{\sinh\left(\frac{\upsilon}{2a}(L-x)\right)}{\sinh\left(\frac{\upsilon L}{2a}\right)}-\frac{2}{L}\sum_{k=1}^{\infty}\frac{\xi_k}{\xi_k^2+\frac{\upsilon^2}{4a^2}}\exp\left(-\left(a\xi_k^2+\frac{\upsilon^2}{4a}\right)t\right)\sin(\xi_k x)\right]. \tag{38}$$

Consequently, we plot the profiles of concentration for changing values of α in the Figure (3). The characters of the solutions naturally change because of constant boundary condition. There has not been a meaningful effect of fractional order on the character of profiles. The similar results have been also obtained in Caputo sense [21] from the physical point of view.

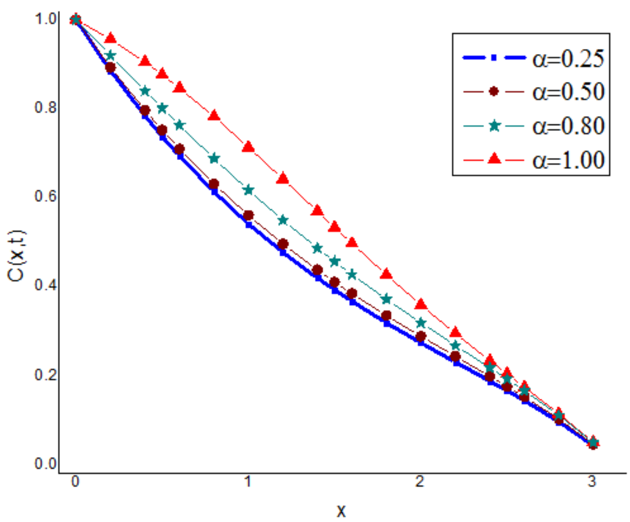


Figure 3. Dependence of concentration function resulting from 2. Case of Dirichlet problem: drift velocity $\upsilon=1$.

4. Conclusions

In the present study, we have aimed to investigate the diffusion profiles of advection-diffusion process modelled by AB derivative. For this purpose, Cauchy and Dirichlet problems have been considered in a line segment. By using an exponential auxiliary function, we have focused on the fundamental solutions of the reduced problems. Laplace and finite-Fourier transformation methods have been applied. Results have been obtained in terms of one or bi- parameter Mittag-Leffler functions. As seen in the literature and also in the current study, AB derivative has been more advantageous than the Caputo fractional derivative for different types of diffusive transport models. It is due to the non-singular Mittag-Leffler kernel [35, 36]. It can be obviously observed that the current results for concentration function differ from the results of ADE model with Caputo fractional derivative with only some parameter coefficients if examined the results in [21]. This is a noteworthy proof to show the AB derivative is an effective alternative to the Caputo fractional derivative for advection-diffusion models.

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