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On (m, n)-bi-ideals in LA-semigroups

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Abstaract − In this paper we study define of an (m, n)-bi-ideals in LA-semigroup and study basic properties of it.

Keywords − *LA-semigroup, ideal, quasi-ideal,* (m, n)*-bi-ideals.*

1 Introduction

The concepts of on (m, n)-bi-Γ-ideals in Γ-semigroup of a semigroup was introduced by Ansari, M.A. and Khan, M.R. [2], in 1993. The left almost semigroup (LA-semigroup) was first introduced by Kazin and Naseerudin [3], in 1972. An LA-semigroup is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An LA-semigroup is non-associative and non-commutative in general, however, there is a close relationship with semigroup as well as with commutative structures.

Later the concept of an (m, n)-ideal in LA-semigroup was first introduced and by M. Akram and N. Yaqood [3] in 2013 and study properties of (m, n)-ideal in LA-semigroup. In 2015 T. Gaketem [9] introduced concept of an (m, n)-quasi ideal in LA-semigroup and study properties of it.

In this paper, we discussed some properties of (m, n)-bi-ideal in LA-semigroup.

2 Preliminaries and basic definitions

Definition 2.1. [3, p.2188] A groupoid $(S, \cdot)$ is called an *LA-semigroup* or an *AG-groupoid*, if its satisfies left invertive law

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \quad \text{for all} \quad a, b, c \in S.$$

Definition 2.2. [3, p.2188] An LA-semigroup S is called a *locally associative* LA-semigroup if its satisfies

$$(aa)a = a(aa), \quad \text{for all} \quad a \in S.$$

Lemma 2.3. [5, p.1] In an LA-semigroup S its satisfies the *medial law* if

$$(ab)(cd) = (ac)(bd), \quad \text{for all} \quad a, b, c, d \in S.$$

Definition 2.4. [7, p.1759] An element $e \in S$ is called *left identity* if $ea = a$ for all $a \in S$.

Lemma 2.5. [3, p.2188] If S is an LA-semigroup with left identity, then

$$a(bc) = b(ac), \quad \text{for all} \quad a, b, c \in S.$$

Lemma 2.6. [5, p.1] An LA-semigroup S with left identity its satisfies the *paramedial* if

$$(ab)(cd) = (dc)(ba), \quad \text{for all} \quad a, b, c, d \in S.$$

Definition 2.7. [4, p.2] Let S be an LA-semigroup. A non-empty subset A of S is called an LA-subsemigroup of S if $AA \subseteq A$.

Definition 2.8. [4, p.2] A non-empty subset A of an LA-semigroup S is called a *left (right) ideal* of S if $SA \subseteq A(AS \subseteq A)$. As usual A is called an *ideal* if it is both left and right ideal.

Definition 2.9. [4, p.2] Let S be an LA-semigroup. An LA-subsemigroup B of S is said to be *bi-ideal* of S if $(BS)B \subseteq B$.

Definition 2.10. [4, p.2] A non-empty subset A of an LA-semigroup S is called a *quasi-ideal* of S if $SA \cap AS \subseteq A$.

Definition 2.11. [3, p.107] A non-empty subset A of an LA-semigroup S is called an *(m, n)-ideal* of S if $(A^m S)A^n \subseteq A$ where m and n are positive integers.

Definition 2.12. [9, p.58] A non-empty subset Q of an LA-semigroup S is called an *(m, n)-quasi ideal* of S if $S^m Q \cap QS^n \subseteq Q$ where m and n are positive integers.

3 (m, n)-bi-ideal in LA-semigroups

In section we definition and study of (m, n)-bi-ideal in LA-semigroup is define the same as an (m, n)-bi-ideal in semigroup.

Definition 3.1. Let S be an LA-semigroup. An LA-subsemigroup B of S is called a *(m, n)-bi-ideal* of S if $(B^m S)B^n \subseteq B$, where m and n are arbitrary positive integers.

Note: The power B^m is canceled when $m = 0$ i.e. $(B^0 S) = S = (SB^0)$. Now we have the following definition:

Definition 3.2. Let S be an LA-semigroup. An LA-subsemigroup B of S is said to be $(m, 0)$-*bi-ideal* of S if $(B^m S)B^0 \subseteq (B^m S) \subseteq B$ and $(0, n)$-*bi-ideal* of S if $(B^0 S)B^n \subseteq (SB^n) \subseteq B$

In another words we can say that $(m, 0)$-bi-ideal of S is exactly the m-left-ideal and $(0, n)$-bi-ideal of S is exactly the n-right-ideal.

Next following we will study basic properties of (m, n)-bi-ideal.

Theorem 3.3. Let S be an LA-semigroup and and B, C be an (m, n)-bi-ideal of S. then the intersection $B \cap C$ is an (m, n)-bi-ideal of S.

Proof. Since $B \cap C \subseteq B$ and $B \cap C \subseteq C$ we have $B \cap C \subseteq B \cap C$. Thus $B \cap C$ is an LA-subsemigroup. Next to show that $B \cap C$ is an (m, n)-bi-ideal of S. Consider

$$((B \cap C)^m S](B \cap C)^n \subseteq (B^m S]B^n \subseteq B,$$

since B is an (m, n)-bi-ideal of S. Secondly

$$((B \cap C)^m S](B \cap C)^n \subseteq (C^m S]C^n \subseteq (CC^n] \subseteq C.$$

Therefore from the above we get $((B\cap C)^m S](B\cap C)^n \subseteq B\cap C$. Thus the intersection $B \cap C$ is an (m, n)-bi-ideal of S. □

Theorem 3.4. Let S be an LA-semigroup and C is an LA-subsemigroup of S. Further let B be an (m, n)-bi-ideal of S. If $B \cap C \neq \emptyset$ then the intersection $B \cap C$ is an (m, n)-bi-ideal of C.

Proof. Assume that $B \cap C \neq \emptyset$ and $x, y \in B \cap C$. Then $x, y \in B$ and $x, y \in C$. Since B, C is an LA-subsemigroup of S we have $xy \in B \cap C$. Then $B \cap C$ is an LA-subsemigroup. Next to show that $B \cap C$ is an (m, n)-bi-ideal of C. Consider

$$((B \cap C)^m C)(B \cap C)^n \subseteq (B^m C)B^n \subseteq (B^m S)B^n \subseteq B,$$

since B is an (m, n)-bi-ideal of S. Secondly

$$((B \cap C)^m C)(B \cap C)^n \subseteq (C^m C)C^n \subseteq C.$$

Therefore from the above we get $((B\cap C)^m C)(B\cap C)^n \subseteq B\cap C$. Thus the intersection $B \cap C$ is an (m, n)-bi-ideal of C. □

In the Theorem 3.5 we can show that arbitrary intersection (m, n)-bi-ideal is an (m, n)-bi-ideal with can prove analogous [3, p.2190]

Theorem 3.5. Let $\{A_i : i \in I\}$ be a family of (m, n)-bi-ideal of an LA-semigroup S. Then $B = \bigcap_{i=1}^{k} A_i \neq \emptyset$ is an (m, n)-bi-ideal of S.

Proof. Since $\{A_i : i \in I\}$ be a family of (m, n)-bi-ideal of an LA-semigroup S we have the intersection of an LA-subsemigroup is an LA-subsemigroup. Next show that $B = \bigcap_{i=1}^{k} A_i$ is an (m, n)-bi-ideal of S. It suffice to prove that $(B^m S)B^n \subseteq B$. Let $x \in (B^m S)B^n$ then $x = (b_1^m s)b_2^n$ for some $b_1^m, b_2^n \in B$ and $s \in S$. Thus for any arbitrary $i \in I$ as $b_1^m, b_2^n \in B_i$ so $x \in (B_i^m S)B_i^n$. Since B_i is an (m, n)-bi-ideal of S we have $(B_i^m S)B_i^n \subseteq B_i$. Then $x \in B_i$. Since i was chosen arbitrarily so $x \in B_i$ for all $i \in I$ and hence $x \in B$. So $(B^m S)B^n \subseteq B$. Hence $B = \bigcap_{i=1}^{k} A_i$ is an (m, n)-bi-ideal of S. □

Theorem 3.6. Let A and B be LA-subsemigroups of a locally associative LA-semigroup S. If A is an $(m, 0)$-ideal and B is a $(0, n)$-ideal of S, then the product AB is an (m, n)-bi-ideal of S if $AB \subseteq A$.

Proof. By medial law we get

$$(AB)(AB) = (AA)(BB) \subseteq AB.$$

This shows that AB is an LA-subsemigroup. Now

$$((AB)^m S)(AB)^n \subseteq (A^m S)(A^n B^n) \subseteq A(SB^n) \subseteq AB$$

Hence the product AB is an (m, n)-ideal of S. □

Theorem 3.7. Let A and B be LA-subsemigroups of a locally associative LA-semigroup S with left identity . If A is an $(0, n)$-bi-ideal and B is a (m, n)-ideal of S, then the product BA is an (m, n)-bi-ideal of S.

Proof. By medial law we get

$$(BA)(BA) = (BB)(AA) \subseteq BA.$$

This shows that BA is an LA-subsemigroup. Now

$$\begin{aligned} ((BA)^m S)(BA)^n &= ((B^m A^m)S)(B^n A^n) \\ &= ((SA^m)B^m)(B^n A^n) \\ &= ((B^n A^n)B^m)(SA^n) \\ &= ((B^m A^n)B^n)(SA^n) \\ &= ((B^m S)B^n)(SA^n) \\ &\subseteq BA \end{aligned}$$

Hence BA is an (m, n)-ideal of S. □

Definition 3.8. [3, p.2190] An element a of an LA-semigroup S is called idempotent if $aa = a$. A subset I of an LA-semigroup S is called *idempotent* if all of its elements are idempotent.

Theorem 3.9. Suppose that S be a locally associative LA-semigroup, C be an (m, n)-bi-ideal of S and B be an (m, n)-bi-ideal of the LA-semigroup C such that B is an idempotent. Then B is an (m, n)-bi-ideal of S.

Proof. Since C be an (m, n)-bi-ideal of S and B be an (m, n)-bi-ideal of the LA-semigroup C we have B is an LA-subsemigroup of S.

Next show that $(B^m S)B^n$ is an (m, n)-bi-ideal of S. It is by media law. Thus

$$\begin{aligned} (B^m S)B^n &= ((B^2)^m (SS))(B^2)^n \\ &= ((B^m)^2 (SS))(B^n)^2 \\ &= ((B^m B^m)(SS))((B^n B^n) \\ &= ((B^m S)(B^m S))((B^n B^n) \\ &= ((B^m S)B^n)((B^m S)B^n) \\ &\subseteq BB \\ &\subseteq B. \end{aligned}$$

Then B is an (m, n)-bi-ideal of S. □

Theorem 3.10. Let S be a locally associative LA-semigroup. Let A and B are (m,n)-bi-ideal of S. Then the following assertions are true:

(1) AB is an (m,n)-bi-ideal of S.

(2) BA is an (m,n)-bi-ideal of S.

Proof. (1) Consider by media law

$$(AB)(AB) \subseteq (AB)(AB) \subseteq (AA)(BB) \subseteq AB.$$

This show that AB is a LA-subsemigroup S. Next we shows that AB is an (m,n)-bi-ideal of S. By medial law we have

$$\begin{aligned} ((AB)^m S)(AB)^n &= ((A^m B^m)S)(A^n B^n) \\ &= ((A^m B^m)(SS))(A^n B^n) \\ &= ((A^m S)(B^m S))(A^n B^n) \\ &= ((A^m S)A^n)((B^m S)B^n) \\ &\subseteq AB. \end{aligned}$$

Therefore AB is an (m,n) bi-ideal of S.

(2) Consider by media law

$$(BA)(BA) \subseteq (BB)(AA) \subseteq BA.$$

This show that BA is a sub LA-semigroup S. Next we shows that BA is an (m,n)- bi-ideal of S. By medial law we have

$$\begin{aligned} ((BA)^m S)(BA)^n &= ((B^m A^m)S)(B^n A^n) \\ &= ((B^m A^m)(SS))(B^n A^n) \\ &= ((B^m S)(A^m S))(B^n A^n) \\ &= ((B^m S)B^n)((A^m S)A^n) \\ &\subseteq BA. \end{aligned}$$

Therefore BA is an (m,n) bi-ideal of S.

□

Corollary 3.11. Suppose that S be an LA-semigroup and B is an (m,n)-bi-ideal and b be an element of S. Then the product Bb and bB are (m,n)-bi-ideal of S.

Proof. It followed by Theorem 3.10. □

Theorem 3.12. Let A be an ideal of an LA-subsemigroup S and Q an (m,n)-quasi-ideal of A, then Q is an (m,n)-bi-ideal of S.

Proof. Since $Q \subseteq A$ we have $Q^m S Q^n \subseteq Q^m SA \cap ASQ^n \subseteq Q^m A \cap AQ^n \subseteq Q$. Thus Q is an (m,n)-bi-ideal of S. □

Theorem 3.13. If B is an (m,n)-bi-ideal of an LA-semigroup S and A is an LA-subsemigroup of S such that $(B^m S)B^n \subseteq A \subseteq B$, then A is an (m,n)-bi-ideal of an LA-semigroup S.

Proof. Suppose that A is an LA-subsemigroup of S. We must show show that $(A^mS)A^n \subseteq A$. By assumption $(A^mS)A^n \subseteq (B^mS)B^n \subseteq A$. By the definition of bi-ideal of LA-semigroup S. Hence A is an (m,n) bi-ideal of LA-semigroup S. □

Theorem 3.14. Suppose that S be a locally associative LA-semigroup and B_1 be an m-left ideal and B_2 be an n-right ideal of S. Then the product B_1B_2 is an (m,n)-bi-ideal of S where m,n are arbitrary positive integers.

Proof. Consider by media law

$$(B_1B_2)(B_1B_2) = (B_1B_1)(B_2B_2) \subseteq B_1B_2.$$

This show that B_1B_2 is an LA-subsemigroup of S. Next to show that product B_1B_2 is an (m,n)-bi-ideal of S. By media law

$$\begin{aligned}((B_1B_2)^mS)(B_1B_2)^n &= ((B_1B_2)^m(SS))(B_1B_2)^n \\ &= ((B_1^mB_2^m)(SS))(B_1B_2)^n \\ &= ((B_1^mS)(B_2^mS))(B_1B_2)^n \\ &\subseteq (B_1^mB_2^m)(B_1B_2)^n \\ &= (B_1B_2)^m(B_1B_2)^n.\end{aligned}$$

Similar

$$\begin{aligned}((B_1B_2)^mS)(B_1B_2)^n &= ((B_1B_2)^m(SS))(B_1B_2)^n \\ &= ((B_1B_2)^m(SS))(B_1^nB_2^n) \\ &= (B_1B_2)^m(SB_1^n)(SB_2^n) \\ &\subseteq (B_1B_2)^m(B_1^nB_2^n) \\ &= (B_1B_2)^m(B_1B_2)^n.\end{aligned}$$

Then B_1B_2 is an (m,n)-bi-ideal of S. □

Theorem 3.15. Let I an (m,n)-ideal of an LA-semigroup S and Q be an (m,n)-quasi-ideal of I then Q is an (m,n)-bi-ideal of S.

Proof. Let Q be an (m,n)-quasi-ideal of I where I is (m,0) ideal and (0,n) ideal of LA-semigroup S. Then $Q \subseteq I$. Thus

$$(Q^mS)Q^n \subseteq (Q^mS)I^n \cap (I^mS)Q^n \subseteq Q.$$

By assumption we have $Q^mI \cap IQ^n \subseteq Q$, $SI^n \subseteq I$ and $I^mS \subseteq I$. Then

$$\begin{aligned}(Q^mS)Q^n &\subseteq (Q^mS)I^n \cap (I^mS)Q^n \\ &\subseteq Q^mI^n \cap I^mQ^n \\ &\subseteq Q^mI \cap IQ^m \subseteq Q\end{aligned}$$

This show that Q is an (m,n)-bi-ideal of S. □

Corollary 3.16. Let Q be an (m,n)-quasi-ideal of an LA-semigroup S. Then Q is an (m,n)-bi-ideal of S.

Proof. Let Q be an (m,n)-quasi-ideal of an LA-semigroup S. Then $Q^mS \cap SQ^n \subseteq Q$. Thus $Q^mS \subseteq Q$ and $SQ^n \subseteq Q$. Hence $(Q^mS)SQ^n \subseteq QQ$ implies that $(Q^mS)Q^n \subseteq Q$. Therefore Q is an (m,n)-bi-ideal of S. □

Definition 3.17. An LA-semigroup S is called (m,n)-*simple* if $SS \neq 0$ and S has no (m,n)-bi-ideal other than 0 and S. In other words S is said to be (m,n)-simple LA-semigroup if S is the unique (m,n)-bi-ideal of S.

Next we define (m,n)-simple and study relation of (m,n)-simple and (m,n)-bi-ideal.

Theorem 3.18. An LA-semigroup S is (m,n)-simple if $S = (A^m S)A^n$ for $A \subseteq S$.

Proof. Let S is an (m,n)-simple LA-semigroup. Further suppose that $B \subseteq S$. Then $(B^m S)B^n$ is an (m,n)-bi-ideal of S. Hence $S = (B^m S)B^n$. Further let $B \subseteq A$ be another (m,n)-bi-ideal of S. Then $S = (B^m S)B^n \subseteq B \subseteq A$. Hence $S = A$. Whence S is an (m,n)-simple LA-semigroup. □

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