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On some new sequence spaces

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Abstract

In this paper, we investigate some new sequence spaces which arise from the notation of generalized de la Vallée-Poussin means and introduce the spaces of strongly λ- invariant summable sequences which happen to be complete paranormed spaces under certain conditions.

Keywords: *σ- convergence, absolutely lambda- invariant, strongly lambda invariant summability.*

Bazı yeni dizi uzayları üzerine

Özet

Bu makalede, genelleştirilmiş de la Vallée-Poussin ortalamalarından ortaya çıkan bazı yeni dizi uzayları incelenmiş ve belirli koşullar altında tam paranormlu uzay olan kuvvetli λ-değişmez toplanabilir dizi uzayları tanıtılmıştır.

Anahtar kelimeler: *σ- yakınsama, mutlak lambda- değişmez, güçlü lambda değişmez toplanabilirlik.*

1. Introduction

Let w be the set of all sequences real or complex and ℓ_∞ denote the Banach space of bounded sequences $x = \{x_k\}_{k=0}^{\infty}$ normed by $\|x\| = \sup_{k\geq 0}|x_k|$. Let D be the shift operator on

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w, that is, $Dx = \{x_k\}_{k=1}^{\infty}$, $D^2x = \{x_k\}_{k=2}^{\infty}$ and so on. It may be recalled that [see Banach [1]] Banach limit L is a nonnegative linear functional on ℓ_∞ such that L is invariant under the shift operator (that is, $L(Dx) = L(x)\ \forall x \in \ell_\infty$) and that $L(e) = 1$ where $e = \{1,1,...\}$. A sequence $x \in \ell_\infty$ is called almost convergent (see, [5]), if all Banach limits of x coincide. Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [5] proved that

$$\hat{c} = \left\{ x : \lim_{m\to\infty} \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \text{ exists uniformly in } n \right\}.$$

Several authors including Duran [2], Lorentz [5], King [6], Nanda[12], [9] and Savas [17] have studied almost convergent sequences.

Let σ be a one-to-one mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ- mean if and only if

1. $\varphi \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n.
2. $\varphi(e) = 1$, where $e = (1,1,\ldots)$ and
3. $\varphi\left(x_{\sigma(n)}\right) = \varphi(x)$ for all $x \in l_\infty$.

For a certain kinds of mapping σ every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the bounded sequences all of whose σ-means are equal, (see, [19]).

If $x = (x_k)$, set $Tx = (Tx_k) = \left(x_{\sigma(k)}\right)$ it can be shown that (see, Schaefer [19]) that

$$V_\sigma = \left\{ x \in l_\infty : \lim_k t_{km}(x) = Le \text{ uniformly in } m \text{ for some } L = \sigma - \lim x \right\} \tag{1.1}$$

where

$$t_{km}(x) = \frac{x_m + Tx_m + \ldots + T^k x_m}{k+1} \text{ and } t_{-1,m} = 0.$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$ such that $\sigma^k(n) \neq n$ for all $n \geq 0$, $k \geq 1$.

Just as the concept of almost convergence lead naturally to the concept of strong almost convergence, σ- convergence leads naturally to the concept of strong σ-convergence. A sequence $x = (x_k)$ is said to be strongly σ-convergent (see Mursaleen [10]) if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k}\left|x_{\sigma^{i}(m)}-L\right|\to 0 \tag{1.2}$$

as $k\to\infty$ uniformly in m. We write $[V_\sigma]$ as the set of all strong σ - convergent sequences. When (1.2) holds we write $[V_\sigma]-\lim x=\ell$. Taking $\sigma(m)=m+1$, we obtain $[V_\sigma]=[\hat{c}]$ so strong σ - convergence generalizes the concept of strong almost convergence.

Note that

$[V_\sigma]\subset V_\sigma\subset l_\infty.$

σ -convergent sequences are studied by Savas ([13]-[16]) and others.

The summability methods of real or complex sequences by infinite matrices are of three types [see, Maddox [7], p.185] ordinary, absolute and strong. In the same vein, it is expected that the concept of invariant convergence must give rise to three types of summability methods-invariant, absolutely invariant and strongly invariant. The invariant summable sequences have been discussed by Schafer [19] and some others. More recently Mursaleen [11] have considered absolute invariant convergent and absolute invariant summable sequences. Also the strongly invariant summable sequences was studied by Saraswat and Gupta[18]. The strongly summable sequences have been systematically investigated by Hamilton and Hill [3], Kuttner [4] and some others. The spaces of strongly summable sequences were introduced and studied by Maddox [7, 8]. It is naturel to ask that how we can define a new sequence spaces by using $(\lambda,\sigma)-$ summable sequences. In this paper, we will give answer of this question and study the spaces of strongly $(\lambda,\sigma)-$ summable sequences, which naturally come up for investigation and which will fill up a gap in the existing literature.

Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$\lambda_{n+1}\le\lambda_n+1,\lambda_1=1.$

The generalized de la Valèe-Poussin mean of a sequence x is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k$$

where $I_n=[n-\lambda_n+1,n]$, for $n=1,2,...$. A sequence $x=(x_k)$ is said to be (V,λ) -summable to a number L, if $t_n(x)\to L\ as\ n\to\infty.$

Let $A=(a_{nk})$ be an infinite matrix of nonnegative real numbers and $p=(p_k)$ be a sequence such that $p_k>0$. (These assumptions are made throughout.) We write $Ax=$

$\{A_n(x)\}$ if $A_n(x)=\sum_k a_{nk}|x_k|^{p_k}$ converges for each n. We write

$$d_{mn}(x)=\frac{1}{\lambda_m}\sum_{i\in I_m}A_{\sigma^n(i)}(x)=\sum_k a(n,k,m)|x_k|^{p_k}$$

where

$$a(n,k,m)=\frac{1}{\lambda_m}\sum_{i\in I_m}a_{\sigma_n(i),k}.$$

If $\lambda_m=m, m=1,2,3,....$

$$d_{mn}(x)=\frac{1}{\lambda_m}\sum_{i\in I_m}A_{\sigma_n(i)}(x)=\sum_k a(n,k,m)|x_k|^{p_k}$$

and

$$a(n,k,m)=\frac{1}{\lambda_m}\sum_{i\in I_m}a_{\sigma^n(i)}$$

reduces to

$$t_{mn}(x)=\frac{1}{m+1}\sum_{i=0}^{m}A_{\sigma^n(i)}(x)=\sum_k a(n,k,m)|x_k|^{p_k}$$

where

$$a(n,k,m)=\frac{1}{m+1}\sum_{i=0}^{m}a_{\sigma^n(i),k}.$$

We now define

$$\left[A_{(\lambda,\sigma)},p\right]_0=\{x:d_{mn}(x)\to 0 \text{ uniformly in } n\};$$
$$\left[A_{(\lambda,\sigma)},p\right]=\{x:d_{mn}(x-le)\to 0 \text{ for some } l \text{ uniformly in } n\}$$

and

$$\left[A_{(\lambda,\sigma)},p\right]_\infty=\left\{x:\sup_n t_{mn}(x)<\infty\right\}.$$

The sets $[A_{(\lambda,\sigma)},p]_0$, $[A_{(\lambda,\sigma)},p]$ and $[A_{(\lambda,\sigma)},p]_\infty$ will be respectively called the spaces of strongly (λ,σ) -summable to zero, strongly (λ,σ) -summable and strongly (λ,σ) -bounded sequences. If $\lambda_m = m, m = 1,2,3,....$, the above spaces reduces to the following sequence spaces.

$$\left[A_\sigma, p\right]_0 = \left\{x : t_{mn}(x) \to 0 \text{ uniformly in } n\right\};$$
$$\left[A_\sigma, p\right] = \left\{x : t_{mn}(x-le) \to 0 \text{ for some } l \text{ uniformly in } n\right\}$$

and

$$\left[A_\sigma, p\right]_\infty = \left\{x : \sup_n t_{mn}(x) < \infty\right\}.$$

If x is strongly (λ,σ) - summable to l we write $x_k \to l[A_{(\lambda,\sigma)},p]$. A pair (A,p) will be called strongly λ - invariant regular if

$$x_k \to l \Rightarrow x_k \to l[A_{(\lambda,\sigma)},p].$$

In the next Theorem, we have suitable conditions for the above sets to be complete linear topological spaces.

2. The main results

We first establish a number of useful propositions.

Proposition 2.1 *If* $p \in \ell_\infty$, *then* $[A_{(\lambda,\sigma)},p]_0$, $[A_{(\lambda,\sigma)},p]$ *and* $[A_{(\lambda,\sigma)},p]_\infty$ *are linear spaces over* $\mathbb{C}$.

Proof. We consider only $[A_{(\lambda,\sigma)},p]$. If $H = \sup p_k$ and $K = \max(1, 2^{H-1})$, we have [see, Maddox [6, p. 346].

$$\left|a_k + b_k\right|^{p_k} \leqq K(\left|a_k\right|^{pk} + \left|b_k\right|^{pk}) \tag{2.1}$$

and for $\lambda \in \mathbb{C}$,

$$\left|\lambda\right|^{p_k} \leqq \max(1, \left|\lambda\right|^H). \tag{2.2}$$

Suppose that $x_k \to l[A_{(\lambda,\sigma)},p]$, $y_k \to l[A_{(\lambda,\sigma)},p]$ and $\lambda,\mu \in \mathbb{C}$. Then we have

$$d_{mn}(\lambda x + \mu y - (\lambda l + \mu l\ )'e) \leqq KK_1 d_{mn}(x - le) + KK_2 d_{mn}(y - l\ e')$$

where $K_1 = \sup|\lambda|^{p_k}$ and $K_2 = \sup|\mu|^{p_k}$, and this implies that $\lambda x + \mu y \to (\lambda l + \mu l\ )[A_{(\lambda,\sigma)}, p]$. This completes the proof.

We have

Proposition 2.2 $[A_{(\lambda,\sigma)}, p] \subset [A_{(\lambda,\sigma)}, p]_\infty$ *, if*

$$\|A\| = \sup_m \sum_k a(n,k,m) < \infty. \tag{2.3}$$

Proof. Assume that $x_k \ \to l[A_{(\lambda,\sigma)}, p]$ and (2.3) holds. Now by the inequality (2.1),

$$\begin{aligned} d_{mn}(x) &= t_{mn}(x - le + le)(4) \qquad (2.4)\\ &\leqq Kd_{mn}(x - le) + K\sum_k a(n,k,m)|l|^{p_k} \\ &\leqq Kd_{mn}(x - le) + K(\sup|l|^{p_k})\sum_k a(n,k,m). \end{aligned}$$

Therefore $x \in [A_{(\lambda,\sigma)}, p]_\infty$ and this completes the proof.

Remark 2.3 *Some known sequence spaces are obtained by specializing A and therefore some of the results proved here extend the corresponding results obtained for the special cases.*

Proposition 2.4 *Let $p \in \ell_\infty$ then $[A_{(\lambda,\sigma)}, p]_0$ and $[A_{(\lambda,\sigma)}, p]_\infty$ $(\inf p_k > 0)$ are linear topological spaces paranormed by g defined by*

$$g(x) = \sup_{m,n}\left[d_{m,n}(x)\right]^{1/M}$$

where $M = \max(1, H = \sup p_k)$. If (2.3) holds, then $[A_\lambda, p]$ has the same paranorm.

Proof. Clearly $g(0) = 0$ and $g(x) = g(-x)$. Since $M \geqq 1$, by Minkowski's inequality it follows that g is subadditive. We now show that the scalar multiplication is continuous. It follows from the inequality (2.2) that

$$g(\lambda x) \leqq \sup|\lambda|^{p_k/M}\ g(x).$$

Therefore $x \to 0 \ \Rightarrow \lambda x \to 0$ (for fixed λ). Now let $\lambda \to 0$ and x be fixed. Given $\varepsilon > 0$ $\exists N$ such that

$$d_{m,n}(\lambda x) < \varepsilon / 2\left(\forall n, \forall m > N\right). \tag{2.5}$$

Since $d_{m,n}(x)$ exists for all m, we write

$$d_{m,n}(x) = K(m), \left(1 \leqq m \leqq N\right)$$

and

$$\delta = \left(\frac{\varepsilon}{2K(m)}\right)^{1/p_k}.$$

Then $|\lambda| < \delta$,

$$d_{m,n}(\lambda x) < \frac{\varepsilon}{2}\left(\forall n, 1 \leqq m \leqq N\right). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\lambda \to 0 \Rightarrow \lambda x \to 0\left(x \text{fixed}\right)$$

This proves the assertion about $[A_{(\lambda,\sigma)}, p]_0$. If $\inf p_k = \theta > 0$ and $0 < |\lambda| < 1$, then $\forall x \in [A_{(\lambda,\sigma)}, p]_\infty$,

$$g^M(\lambda x) \leqq |\lambda|^\theta g^M(x).$$

Therefore $[A_\lambda, p]_\infty$ has the paranorm g. If (2.3) holds it is clear from Proposition 2.2 that $g(x)$ exists for each $x \in [A_{(\lambda,\sigma)}, p]$. This completes the proof.

Remark 2.5 *It is evident that g is not a norm in general. But if $p_k = p$ $\forall k$, then clearly g is a norm for $1 \leqq p \leqq \infty$ and a $p-$ norm for $0 < p < 1$.*

Proposition 2.6 $[A_\lambda, p]_0$ *and* $[A_{(\lambda,\sigma)}, p]_\infty$ *are complete with respect to their paranorm topologies* $[A_{(\lambda,\sigma)}, p]$ *is complete if* (2.3) *holds and*

$$\sum_k a(n,k,m) \to 0 \text{ uniformly in } n. \tag{2.7}$$

Proof. Let $\{x^i\}$ be a Cauchy sequence in $[A_{(\lambda,\sigma)}, p]_0$. Then there exists a sequence x such that $g(x^i - x) \to 0$ $\left(i \to \infty\right)$. Since g is subadditive it follows that $x \in [A_\lambda, p]_0$. The completness of $[A_{(\lambda,\sigma)}, p]_\infty$ can be similarly obtained. We now consider $[A_{(\lambda,\sigma)}, p]$. If (2.3) holds and $\{x^i\}$ is a Cauchy sequence in $[A_\lambda, p]$, Then there exists x such that $g(x^i - x) \to 0$. If (2.7) holds then from inequality (2.4) it is clear that $[A_{(\lambda,\sigma)}, p] = [A_{(\lambda,\sigma)}, p]_0$. This completes the proof.

Combining the above facts we obtain the main result.

Theorem 2.7 *Let* $p \in \ell_\infty$. *Then* $[A_{(\lambda,\sigma)}, p]_0$ *and* $[A_{(\lambda,\sigma)}, p]_\infty$ $(\inf p_k > 0)$ *are complete linear topological spaces paranormed by* g. *If* (2.3) *and* (2.7) *hold then* $[A_{(\lambda,\sigma)}, p]$ *has the same property. If further* $p_k = p$ *for all* k, *they are Banach spaces for* $1 \leqq p < \infty$ *and* $p-$*normed spaces for* $0 < p < 1$.

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