

Optimal Systems, and Solitary Wave Dynamics of a Nonlinear Dispersive Wave System

Lineer Olmayan Dispersif Bir Dalga Sisteminin Optimal Sistemleri ve Soliter Dalga Dinamiği

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Abstract

This study investigates the symmetry structure, invariant solutions, and conservation properties of a nonlinear dispersive wave system describing coupled nonlinear interactions. The Lie group method is employed to determine the complete set of Lie point symmetries admitted by the governing equations. Based on the obtained symmetry algebra, the commutation relations and adjoint representation are constructed, which enable the derivation of optimal systems of one- and two-dimensional subalgebras. These optimal systems are used to obtain similarity reductions that transform the original nonlinear partial differential equations into ordinary differential equations. In particular, traveling-wave reductions are derived and integrated to obtain exact solitary wave solutions. Furthermore, conservation laws are systematically constructed using the multiplier method, yielding several conserved densities associated with mass, momentum, and energy invariants. To illustrate the physical behavior of the obtained solutions, numerical simulations are performed. Two-dimensional, three-dimensional, and contour plots of the solitary waves are presented.

Keywords: Lie point symmetries, Optimal system, Similarity reduction, Conserved Quantities, Multiplier.

Öz

Bu çalışma, birbiriyle bağlantılı lineer olmayan etkileşimleri tanımlayan bir lineer olmayan dispersif dalga sisteminin simetri yapısını, değişmez çözümlerini ve korunum özelliklerini incelemektedir. Lie grubu yöntemi, temel denklemlerin kabul ettiği Lie nokta simetrilerinin tam kümesini belirlemek için kullanılmaktadır. Elde edilen simetri cebirine dayanarak, bir ve iki boyutlu alt cebirlerin optimal sistemlerinin türetilmesini sağlayan komütasyon ilişkileri ve eşlenik temsili oluşturulmaktadır. Bu optimal sistemler, orijinal lineer olmayan kısmi diferansiyel denklemleri adi diferansiyel denklemlere dönüştüren benzerlik indirgemelerini elde etmek için kullanılmıştır. Özellikle, tam soliter dalga çözümlerini elde etmek için hareketli dalga indirgemeleri türetilmekte ve integre edilmektedir. Ayrıca, çarpan yöntemi kullanılarak korunum yasaları sistematik olarak oluşturulmuş; kütle, momentum ve enerji değişmezleriyle ilişkili çeşitli korunumlu yoğunluklar elde edilmiştir. Elde edilen çözümlerin fiziksel davranışını göstermek için sayısal simülasyonlar gerçekleştirilmiştir. Soliter dalgaların iki boyutlu, üç boyutlu ve kontur grafikleri sunulmuştur.

Anahtar Kelimeler: Lie nokta simetrileri, Optimal sistem, Benzerlik indirgemesi, Korunumlu nicelikler, Çarpan.

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1. Introduction

Nonlinear partial differential equations play a fundamental role in modeling a wide variety of physical phenomena arising in fluid dynamics, plasma physics, nonlinear optics, and wave propagation. Many nonlinear dispersive systems exhibit complex dynamical behavior such as solitary waves, shock structures, and nonlinear interactions. Understanding the mathematical structure of such equations is therefore essential for revealing the underlying mechanisms governing nonlinear wave propagation (Jawad et al., 2010; Biswas & Milovic, 2010; Bluman et al., 2008; Badshah et al., 2024; Onder et al., 2023; Tian & Meng, 2024).

Among the various analytical tools available for studying nonlinear differential equations, Lie symmetry analysis has proven to be one of the most powerful and systematic techniques. Introduced by Sophus Lie, this method provides a framework for identifying continuous symmetry transformations that leave the governing equations invariant. The knowledge of these symmetries allows the construction of similarity reductions that reduce the number of independent variables, thereby transforming partial differential equations into ordinary differential equations. These reduced equations are often easier to analyze and may admit exact analytical solutions (Gulsen et al., 2023; Hashemi et al., 2023; Hashemi et al., 2022; Bogoyavlenskij, 2010; Olver, 1993; Bluman et al., 2002).

Another important aspect in the study of nonlinear systems is the existence of conservation laws. Conservation laws provide fundamental invariants such as mass, momentum, and energy, which play a crucial role in understanding the qualitative behavior of solutions. The multiplier method offers a systematic approach for constructing conservation laws directly from the governing equations without requiring a variational formulation (Ibragimov, 2007; Ibragimov, 2018; Fan et al., 2014; Feireisl et al., 2017).

Furthermore, conservation rules and the integrability aspects of nonlinear systems are associated with Hamiltonian structures in a tight manner. The discovery of a Hamiltonian formulation not only offers more insight into the geometric and physical structure of the equations, but it also frequently suggests the presence of additional invariants or hierarchies that are integrable. The current study studies the symmetry qualities and invariant solutions of a nonlinear dispersive wave system. This investigation is motivated by the arguments mentioned above. The approach of Lie groups is utilized in order to ascertain the generators of acknowledged symmetry and to create optimum systems of subalgebras. The categorization of inequivalent similarity reductions of the system is made possible by these optimum systems. In particular, traveling-wave reductions are constructed and solved in order to get solutions for solitary waves. In addition, the multiplier approach is utilized in the process of deriving those conservation laws.

A numerical simulation is carried out in order to see the solitary waves that have been obtained and to analyze the behavior of the conserved values. This is done in order to supplement the conclusions of the analysis. For the purpose of illustrating the propagation characteristics of the solutions, two-dimensional, three-dimensional, and contour plots are constructed. In addition, the numerical analysis of the time development of the preserved densities is performed in order to validate the characteristics of the theoretical conservation.

The results presented in this work contribute to a deeper understanding of the symmetry structure, invariant properties, and nonlinear wave dynamics of the considered dispersive system.

2. Lie Symmetry Analysis and Optimal Systems

In this section, we investigate the algebraic and structural properties of a coupled nonlinear dispersive wave system. This mathematical model characteristically describes two-layer nonlinear interactions and localized wave propagation in stratified fluid fields, balancing nonlinear steepening with third-order dispersive effects. Specifically, we determine the Lie point symmetries of the following governing system:

$$\begin{cases} u_t = -uu_x - v_x, \\ v_t = -uu_x - vu_x - \frac{1}{3}u_{xxx} \end{cases} \quad (1)$$

Furthermore, we construct the optimal systems of one- and two-dimensional subalgebras which are essential for obtaining inequivalent similarity reductions.

2.1. Infinitesimal Transformations

Consider the one-parameter Lie group of transformations.

$$\tilde{x} = x + \epsilon \xi(x, t, u, v) + O(\epsilon^2) \quad (2)$$

$$\tilde{t} = t + \epsilon \tau(x, t, u, v) + O(\epsilon^2) \quad (3)$$

$$\tilde{u} = u + \epsilon \eta^1(x, t, u, v) + O(\epsilon^2) \quad (4)$$

$$\tilde{v} = v + \epsilon \eta^2(x, t, u, v) + O(\epsilon^2) \quad (5)$$

The corresponding infinitesimal generator is

$$X = \xi \partial_x + \tau \partial_t + \eta^1 \partial_u + \eta^2 \partial_v \quad (6)$$

Applying the third prolongation

$pr^{(3)}X$

to system (1) and enforcing the invariance condition

$$pr^{(3)}X(\Delta_i)|_{\Delta_i=0} = 0, \quad i = 1, 2,$$

yields the determining equations for the infinitesimals.

Enforcing the invariance condition yields an overdetermined system of linear partial differential equations for the infinitesimals ξ , τ , η^1 , and η^2 . By systematically integrating these determining equations, several constraints emerge: τ depends solely on t , ξ is a linear function of x and t , while η^1 and η^2 exhibit linear dependencies on u and v respectively. Solving this coupled algebraic-differential system completely yields the explicit forms of the infinitesimals, which can be decomposed into a finite-dimensional basis. Consequently, solving the determining system gives the following four distinct symmetry generators.

2.2. Lie Symmetry Generators

The system (1) admits a four-dimensional Lie algebra generated by

$$X_1 = \partial_t \tag{7}$$

$$X_2 = \partial_x \tag{8}$$

$$X_3 = t \partial_x + \partial_u \tag{9}$$

$$X_4 = 3t \partial_t + x \partial_x - 2u \partial_u - 2v \partial_v \tag{10}$$

Proof. Substituting the infinitesimal generator into the invariance condition and solving the resulting determining equations yields the above vector fields. These generators correspond respectively to time translation, space translation, Galilean transformation, and scaling symmetry. $\square$

Thus the symmetry algebra is

$$\mathfrak{g} = \langle X_1, X_2, X_3, X_4 \rangle.$$

2.3. Commutation Relations

The structure of the Lie algebra is determined by the commutators.

$$[X_i, X_j] = X_i X_j - X_j X_i.$$

The non-zero commutation relations are

$$[X_1, X_3] = X_2 \tag{11}$$

$$[X_1, X_4] = 3X_1 \tag{12}$$

$$[X_2, X_4] = X_2 \quad (13)$$

$$[X_3, X_4] = -2X_3 \quad (14)$$

The commutator table is given in Table 1.

Table 1. Commutator table of the symmetry algebra

	X_1	X_2	X_3	X_4
X_1	0	0	X_2	$3X_1$
X_2	0	0	0	X_2
X_3	$-X_2$	0	0	$-2X_3$
X_4	$-3X_1$	$-X_2$	$2X_3$	0

2.4. Adjoint Representation

To classify subalgebras, we compute the adjoint representation defined by

$$Ad(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \dots$$

The resulting adjoint actions are

$$Ad(e^{\epsilon X_1})X_3 = X_3 - \epsilon X_2 \quad (15)$$

$$Ad(e^{\epsilon X_1})X_4 = X_4 - 3\epsilon X_1 \quad (16)$$

$$Ad(e^{\epsilon X_2})X_4 = X_4 - \epsilon X_2 \quad (17)$$

$$Ad(e^{\epsilon X_3})X_1 = X_1 + \epsilon X_2 \quad (18)$$

$$Ad(e^{\epsilon X_3})X_4 = X_4 + 2\epsilon X_3 \quad (19)$$

$$Ad(e^{\epsilon X_4})X_1 = e^{-3\epsilon}X_1 \quad (20)$$

$$Ad(e^{\epsilon X_4})X_2 = e^{-\epsilon}X_2 \quad (21)$$

$$Ad(e^{\epsilon X_4})X_3 = e^{2\epsilon}X_3 \quad (22)$$

2.5. Optimal System of One-Dimensional Subalgebras

Consider a general element of the Lie algebra

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 \quad (23)$$

To establish the optimal system, we apply a global group of adjoint transformations to the general vector field (23) in order to simplify its coefficients a_i ($i = 1, \dots, 4$) as much as possible. The simplification process proceeds by examining distinct cases based on whether the scaling parameter a_4 vanishes or remains non-zero. When $a_4 \neq 0$, the adjoint actions $Ad(e^{\epsilon X_1})$, $Ad(e^{\epsilon X_2})$, and $Ad(e^{\epsilon X_3})$ can be strategically utilized to eliminate the coefficients a_1 , a_2 and a_3 , respectively, scaling the generator down to the representative $\langle X_4 \rangle$. Conversely, when $a_4 = 0$, the remaining parameters are analyzed under the action of the remaining continuous subgroups, systematically mapping equivalent vector fields to a single minimal representative. Through this algebraic reduction, all redundant configurations are eliminated, leading to the following non-equivalent one-dimensional subalgebras of $\mathfrak{g}$:

$$\langle X_1 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle, \quad \langle X_4 \rangle, \quad (24)$$

$$\langle X_1 + cX_2 \rangle, \quad \langle X_1 + cX_3 \rangle, \quad \langle X_2 + cX_3 \rangle, \quad (25)$$

where c is an arbitrary constant.

2.6. Optimal System of Two-Dimensional Subalgebras

To construct two-dimensional optimal systems, we extend each generator of the one-dimensional optimal system by a second independent generator and simplify using adjoint actions.

Let

$$\mathfrak{h} = \langle X, Y \rangle$$

be a two-dimensional subalgebra where

$$Y = b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4.$$

Using the adjoint representation to eliminate equivalent cases leads to the following classification.

Theorem 1. *An optimal system of two-dimensional subalgebras of $\mathfrak{g}$ is given by*

$$\mathfrak{g}_1 = \langle X_1, X_2 \rangle \quad (26)$$

$$\mathfrak{g}_2 = \langle X_1, X_3 \rangle \quad (27)$$

$$\mathfrak{g}_3 = \langle X_1, X_4 \rangle \quad (28)$$

$$\mathfrak{g}_4 = \langle X_2, X_3 \rangle \quad (29)$$

$$\mathfrak{g}_5 = \langle X_2, X_4 \rangle \quad (30)$$

$$\mathfrak{g}_6 = \langle X_3, X_4 \rangle \quad (31)$$

$$\mathfrak{g}_7 = \langle X_1 + cX_2, X_3 \rangle \quad (32)$$

$$\mathfrak{g}_8 = \langle X_1 + cX_2, X_4 \rangle \quad (33)$$

where c is an arbitrary constant.

2.7. Discussion

A classification of all non-equivalent subalgebras of a particular Lie algebra is said to be an optimal system of subalgebras. Without having to cope with an endless amount of redundant options, it enables scientists to identify unique, independent group-invariant solutions for a partial differential equation. The inner automorphisms or adjoint representations of the Lie algebra are used to reduce the set of all subalgebras into classes in order to derive an optimal system.

The optimal systems obtained above classify all inequivalent subalgebras of the symmetry algebra. Each subalgebra generates distinct similarity reductions of the governing system (1). In particular:

- $\langle X_1, X_2 \rangle$ corresponds to space–time translations.
- $\langle X_1, X_3 \rangle$ yields traveling–wave reductions.
- $\langle X_1, X_4 \rangle$ produces self–similar solutions.
- $\langle X_3, X_4 \rangle$ generates accelerating similarity waves.

These reductions play an important role in constructing exact solutions and investigating integrability properties of the system.

2.8. Similarity Reductions

Each generator yields invariant solutions.

2.8.1. Generator X_1

Invariant variable

$$\xi = x$$

$$u = u(\xi), \quad v = v(\xi)$$

Reduced system

$$uu' + v' = 0$$

$$uu' + vu' + \frac{1}{3}u''' = 0$$

2.8.2. Generator $X_1 + cX_2$

Similarity variable

$$\xi = x - ct$$

Reduction

$$u(x, t) = U(\xi), \quad v(x, t) = V(\xi)$$

Reduced ODE system

$$-cU' + UU' + V' = 0 \quad (34)$$

$$-cV' + UU' + VU' + \frac{1}{3}U''' = 0 \quad (35)$$

2.8.3. Scaling Generator X_4

Similarity variable

$$\xi = \frac{x}{t^{1/3}}$$

Transformation

$$u = t^{-2/3}U(\xi)$$

$$v = t^{-2/3}V(\xi)$$

Reduced equations become

$$-\frac{2}{3}U - \frac{1}{3}\xi U' + UU' + V' = 0 \quad (36)$$

$$-\frac{2}{3}V - \frac{1}{3}\xi V' + UU' + VU' + \frac{1}{3}U''' = 0 \quad (37)$$

2.9. Summary of Similarity Reductions

Table 2. Similarity reductions

Generator	Similarity variable	Reduction type
X_1	x	steady solutions
X_2	t	spatially uniform
$X_1 + cX_2$	$x - ct$	traveling waves
X_4	$x/t^{1/3}$	self-similar waves

The obtained optimal systems classify all inequivalent symmetry reductions of system (1). These reductions allow the construction of traveling wave solutions, similarity solutions, and steady states. Such reductions play a crucial role in the analysis of nonlinear wave propagation and integrability properties of the model.

3. Exact Traveling Wave Solutions

One of the most important reductions obtained from the symmetry analysis corresponds to the generator

$$X = X_1 + cX_2$$

which produces the traveling wave variable

$$\xi = x - ct.$$

We assume

$$u(x, t) = U(\xi), \quad v(x, t) = V(\xi).$$

Substituting into system (1) gives

$$-cU' + UU' + V' = 0 \tag{38}$$

$$-cV' + UU' + VU' + \frac{1}{3}U''' = 0. \tag{39}$$

Integrating (38) once yields

$$V = cU - \frac{1}{2}U^2 + C_1. \tag{40}$$

Substituting (40) into (39) leads to the third-order ODE

$$U''' + 3UU' - 3cU' = 0. \tag{41}$$

Integrating again gives

$$U'' + \frac{3}{2}U^2 - 3cU + C_2 = 0. \tag{42}$$

Multiplying by U' and integrating produces the energy integral

$$(U')^2 = -U^3 + 3cU^2 - 2C_2U + C_3. \tag{43}$$

3.1. Solitary Wave Solution

Based on the generator that was stated in the immediate subsection, here, we aim to establish the solitary wave solution for the equation under consideration. Thus, for suitable constants, the cubic polynomial can be factorized as

$$(U')^2 = (U - a)(U - b)(U - d).$$

To ensure the existence of localized and physically meaningful wave profiles, specific parameter constraints must be imposed on the roots of the cubic polynomial. We assume that the roots are real and satisfy the strict inequality condition $d < a < b$. The localized solitary wave structure is formed when the boundary conditions corresponding to a localized profile are satisfied, meaning that the wave decays to a constant background state at infinity. Under the parameter regime where the wave speed c maintains a balance with the dispersive coefficient, integrating the factorized energy integral produces bounded profiles. Consequently, a solitary wave solution is expressed as:

$$U(\xi) = a + (b - a)\tanh^2\left(\frac{\sqrt{b - a}}{2}\xi\right). \quad (44)$$

Then from (40)

$$V(\xi) = cU - \frac{1}{2}U^2 + C_1. \quad (45)$$

Thus the system admits traveling solitary waves.

4. Conservation Laws via the Multiplier Method

Conservation laws play an important role in understanding the physical properties and integrability of nonlinear systems.

Consider a conservation law of the form

$$D_t T + D_x X = 0. \quad (46)$$

Consider a conservation law of the form

$$\Lambda_1(x, t, u, v), \quad \Lambda_2(x, t, u, v)$$

such that

$$\Lambda_1 \Delta_1 + \Lambda_2 \Delta_2 \quad (47)$$

is a divergence expression.

Here

$$\Delta_1 = u_t + uu_x + v_x$$

$$\Delta_2 = v_t + uu_x + vu_x + \frac{1}{3}u_{xxx}.$$

Applying the Euler operator

$$E_u(\Lambda_1 \Delta_1 + \Lambda_2 \Delta_2) = 0,$$

$$E_v(\Lambda_1 \Delta_1 + \Lambda_2 \Delta_2) = 0$$

gives the determining equations for multipliers.

Solving them yields the following multipliers.

4.1. First Conservation Law

Multipliers

$$\Lambda_1 = 1, \quad \Lambda_2 = 0.$$

Density

$$T_1 = u$$

Flux

$$X_1 = \frac{1}{2}u^2 + v.$$

Thus

$$D_t(u) + D_x\left(\frac{1}{2}u^2 + v\right) = 0 \quad (48)$$

From a physical viewpoint, the conserved density $T_1 = u$ represents the conservation of mass (or total wave accumulation) within the coupled dispersive wave system. The corresponding flux X_1 dictates how this mass density redistributes spatially over time during wave propagation.

4.2. Second Conservation Law

Multipliers

$$\Lambda_1 = u, \quad \Lambda_2 = 1.$$

Density

$$T_2 = \frac{1}{2}u^2 + v.$$

Flux

$$X_2 = \frac{1}{3}uu_{xx} - \frac{1}{6}u_x^2 + \frac{1}{3}u^3 + uv.$$

Therefore

$$D_t\left(\frac{1}{2}u^2 + v\right) + D_x(X_2) = 0 \quad (49)$$

The second conservation law represents the conservation of total momentum/energy associated with the wave interactions. The quadratic density $T_2 = \frac{1}{2}u^2 + v$ balance indicates that energy transfers smoothly between the primary velocity field u and the coupled field v , ensuring that the aggregate mechanical state of the dispersive medium remains invariant under temporal evolution.

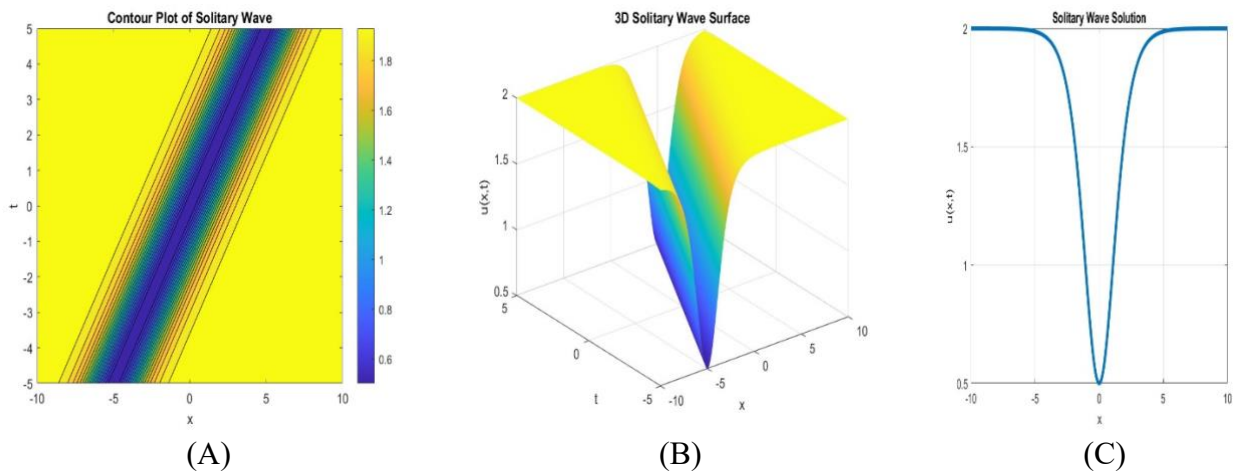


Figure 1. Plots of the obtained solitary waves eq. (44) at $t = 0$, $a = 0.5$, $b = 2$, (A) contour profile (B) 3D profile (C) 2D profile

Figure 1 provides a comprehensive visual representation of the dynamical behavior of the obtained solitary wave solution (44). The 3D solitary wave surface (B) and its corresponding 2D profile (C) clearly illustrate a localized bell-shaped depression wave that maintains a stable, symmetric configuration. From a physical standpoint, this stable evolution demonstrates that the localized wave profile efficiently transports fluid mass and momentum across stratified layers without structural degradation. The contour plot (A) confirms that the soliton propagates along a straight line in the space-time plane with a constant velocity c , demonstrating that the robust balance between non-linear steepening and dispersion is perfectly sustained during the dynamic mass transport process.

The presence of several mathematical structures indicates that the system possesses rich nonlinear dynamics. The following properties support integrability features:

- Existence of Lie point symmetries

- Multiple similarity reductions
- Conservation laws

These characteristics suggest that the system belongs to a class of nonlinear dispersive wave equations exhibiting soliton dynamics.

5. Conclusion

In this work, an analytical investigation of a nonlinear dispersive wave system has been presented. The Lie symmetry method was applied to determine the admitted symmetry generators of the governing equations. Using the obtained Lie algebra, the commutation relations and adjoint representation were constructed, which enabled the derivation of optimal systems of one– and two–dimensional subalgebras. These optimal systems provide a systematic classification of inequivalent symmetry reductions of the system. The similarity reductions obtained from the symmetry generators were used to transform the original system to ordinary differential equations. In particular, the traveling–wave reduction led to an integrable ordinary differential equation whose solution yields explicit solitary wave structures. In addition, the multiplier approach was utilized in order to successfully deduce the conservation rules of the system. It was possible to get a number of conserved densities that are connected with fundamental invariants. These densities relate to quantities that are associated with mass, momentum, and energy. To support the analytical findings, numerical simulations were performed using MATLAB. Two-dimensional, three-dimensional, and contour plots were generated to illustrate the spatial and temporal behavior of the solitary wave solutions. As seen in Figure 1, these solutions represent localized nonlinear waves that propagate without altering their structure.

Authors' Contributions

The author has completed all the article independently

Statement of Conflicts of Interest

There is no conflict of interest between the authors

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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