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Riemannian Curvature of a Sliced Contact Metric Manifold

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Abstract

Contact geometry become a more important issue in the mathematical world with the works which had done in the 19th century. Many mathematicians have made studies on contact manifolds, almost contact manifolds, almost contact metric manifolds and contact metric manifolds. Many different studies have been done and papers have been published on Sasaki manifolds, Kähler manifolds, the other manifold types and submanifolds of them. In our previous studies we get the characterization of indefinite Sasakian manifolds. In order to get the characterization of indefinite Sasakian manifolds, firstly we defined sliced contact metric manifolds and then we examined the features of them. As a result we obtain a sliced almost contact metric manifold which is a wider class of almost contact metric manifolds. Thus, we constructed a sliced which is a contact metric manifold on an almost contact metric manifold where the manifold is not a contact metric manifold. Sliced almost contact metric manifolds generalized the almost contact metric manifolds. Then, we study on the sliced Sasakian manifolds and the submanifolds of them. Moreover we analyzed some important properties of the manifold theory on sliced almost contact metric manifolds.
In this paper we calculated the ϕ_π-sectional curvature and the Riemannian curvature tensor of the sliced almost contact metric manifolds. Hence we think that all these studies will accelerate the studies on the contact manifolds and their submanifolds.

Keywords: contact geometry, sectional curvature, riemannian curvature, Sliced almost contact metric manifolds, Sliced contact metric manifolds.

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Bir Dilimlenmiş Kontak Metrik Manifoldun Riemann Eğriliği

Özet

Kontak geometri 19. yüzyılda yapılan çalışmalar sonucunda gittikçe artan bir öneme sahip olmuştur. Birçok matematikçi kontak manifoldlar, hemen hemen kontak manifoldlar, hemen hemen kontak metrik manifoldlar ve kontak metrik manifoldlar üzerine çalışmalar yapmışlardır. Ayrıca, Sasaki manifoldların, Kähler manifoldların ve diğer manifold türlerinin lightlike altmanifoldları üzerine çok sayıda çalışma yapılmış ve farklı makaleler yayımlanmıştır. Yapılan önceki çalışmamızda belirsiz Sasaki manifoldların karakterizasyonunu elde ettik. Bu karakterizasyonu yapmak için önce dilimlenmiş kontak metrik manifoldlar tanımladık ve özelliklerini inceledik. Sonuç olarak, kontak metrik manifoldların ve diğerlerinin daha geniş bir sınıfı olan dilimlenmiş hemen hemen kontak metrik manifoldları elde ettik. Böylece, kontak metrik manifold olmayıp hemen hemen kontak metrik manifold olan bir manifold üzerinde kontak metrik manifold olacak şekilde bir dilim oluşturduk. Dilimlenmiş hemen hemen kontak metrik manifoldlar, hemen hemen kontak metrik manifoldları genelleştirmiştir. Daha sonra dilimlenmiş Sasaki manifoldları ve bu manifoldların altmanifoldlarını çalıştık. Ayrıca, dilimlenmiş hemen hemen kontak metrik manifoldlarda manifoldlar teorisinin bazı önemli özelliklerini inceledik.
Bu makalede ise dilimlenmiş kontak metrik manifoldların ϕ_π-kesitsel eğriliği ile Riemann eğrilik tensörünü hesapladık. Böylece bu çalışmaların kontak manifoldlar ve onların altmanifoldları üzerine çalışmalara yeni bir ivme kazandıracağını düşünüyoruz.

Anahtar Kelimeler: dilimlenmiş hemen hemen kontak metrik manifoldlar, dilimlenmiş kontak metrik manifoldlar, desitsel eğrilik, kontak geometri, riemann eğriliği

1. Introduction

Contact geometry and its applications are important for 3-dimensional physical world, optics, solutions of differential equations and our world. The first works on contact geometry were started in the 19th century. In 1900s many mathematicians worked on contact geometry. In the 20th century, the works of researchers (Sasaki, 1962; Gray, 1959; Ogiue,1964 and Bootby, 1986) were took an important role in contact geometry. After 1960s mathematicians have started to study on the main properties of the manifolds and their submanifolds. We can see some of these in the works of Blair, 1976, Yano Kon, 1984 and Chen, 1973. The curvatures are important for manifolds because we can understand the characteristic properties of the geometric objects with them. In 1960s, Ogiue, 1964 calculated the Riemannian curvature tensor for Sasakian manifolds. Riemannian curvature, sectional curvature and the other curvature tensors are important characteristic properties of manifolds. In the present paper, we calculated the Riemannian curvature tensor for sliced almost contact metric manifolds.

2. Preliminaries

In differential geometry if M is a $(2\mathrm{n}+1)$ $-$dimensional differantiable manifold and η is a 1-form on M which satisfies

$$\eta \Lambda (d\eta)^n \neq 0 \quad (2.1)$$

everywhere on M, then M is called a *contact manifold* and η is named as a *contact form*.

On a contact manifold M, contact distrubition denoted by $\mathrm{D_p}$ and it is defined by the set

$$\mathrm{D_p} = \{\mathrm{X} \in T_pM \mid \eta(\mathrm{X}) = 0\}. \tag{2.2}$$

It satisfy

$$\eta(\xi) = 1 \text{ and } d\eta(X,\xi) = 0 \tag{2.3}$$

for all X on M. If ϕ, ξ, η satisfy

$$\phi^2\mathrm{X} = -\mathrm{X} + \eta(\mathrm{X})\xi\ , \quad \phi(\xi) = 0 \text{ and } \eta \circ \phi = 0 \tag{2.4}$$

then M is called an *almost contact manifold* with an almost contact structure (ϕ, ξ, η).

M becomes an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) if

$$g\big(\phi(X), \phi(Y)\big) = g(X,Y) - \eta(\mathrm{X})\eta(\mathrm{Y}), \tag{2.5}$$

$$g\big(X, \phi(Y)\big) = -g(\phi(X), Y), \tag{2.6}$$

$$\eta(\mathrm{X}) = \mathrm{g}(\mathrm{X}, \xi) \tag{2.7}$$

where $\mathrm{X}, \mathrm{Y} \in \chi(M)$ and g is a Riemannian tensor of M (Blair, 2002). Also, in 3-dimensional almost contact metric manifold, (Olszak, 1986) showed that

$$(\nabla_X\phi)Y = g(\phi\nabla_X\xi, Y)\xi - \eta(Y)\phi\nabla_X\xi \tag{2.8}$$

for all $\mathrm{X}, \mathrm{Y} \in \chi(M)$.

Definition 2.1 Let (M, g) be a semi-Riemannian manifold. The tensor R defined by following equation

$$R(X,Y)Z = \nabla_X\nabla_Y\mathrm{Z} - \nabla_Y\nabla_X\mathrm{Z} - \nabla_{[X,Y]}\mathrm{Z} \qquad \forall X, Y, Z \in \chi(M) \tag{2.9}$$

is called the *curvature tensor* of the connection ∇ (Yano Kon, 1984).

Definition 2.2 Let (M, g) be a semi-Riemannian manifold. The tensor R of type (0,4) defined by

$$R: \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) \to \mathrm{C}^\infty(M, \mathbb{R})$$

$$(W, Z, X, Y) \to R(W, Z, X, Y) = g(R(X,Y)Z, W) \tag{2.10}$$

is called the *Riemann Christoffel curvature tensor* (Yano Kon, 1984).

Theorem 2.1 Let (M, ϕ, η, ξ) be an almost contact metric manifold. (M, ϕ, η, ξ) is Sasakian if and only if the following equation is satisfied (Sasaki, 1962).

$$(\boldsymbol{\nabla_X}\phi)Y = g(X,Y)\xi - \eta(Y)X$$

3. Sliced Almost Contact Metric Manifolds

Gümüş, 2018 defined the sliced almost contact manifolds as a wider class of almost contact manifolds by the following definition in doctoral thesis.

Definition 3.1 Let M be a manifold and TM be the tangent bundle of the manifold M. Assume that H is a distrubition on TM and $\xi \in$ H. We define the projection π, ω tensor field of type (0,1) and ϕ_π tensor field of type (1,1) by the following, $\pi, \phi_\pi : TM \to \mathrm{H}$ and $\omega : TM \to \mathrm{C}^\infty(\mathrm{M}, \mathbb{R})$. If these tensor fields satisfy the following conditions,

$$\phi_\pi^2 \mathrm{X} = -\pi(\mathrm{X}) + \omega(\mathrm{X})\xi \tag{3.1}$$

$$\omega(\xi) = 1 \tag{3.2}$$

then $(M, \phi_\pi, \omega, \pi, \xi)$ is called a *sliced almost contact manifold* (Gümüş, 2018).

Definition 3.2 Let (M, ϕ, η, ξ) be an almost contact manifold and H is a distrubition on M. If $(M, \phi_\pi, \omega_\pi, \pi, \xi)$ is a sliced almost contact manifold and the equalities

$$i)\ \phi \circ \pi = \phi_\pi \tag{3.3}$$

$$ii)\ \eta \circ \pi = \omega_\pi \tag{3.4}$$

are satisfied then the manifold $(M, \phi_\pi, \omega_\pi, \pi, \xi)$ is called *compatible sliced almost contact manifold* with (M, ϕ, η, ξ) (Gümüş, 2018).

Definition 3.3 Let $(M, \phi_\pi, \omega_\pi, \pi, \xi)$ be a sliced almost contact manifold. If there exists a Riemannian metric $g : TM \times TM \to \mathrm{C}^\infty(M, \mathbb{R})$ defined on M which satisfies

$$g(\phi_\pi \mathrm{X}, \phi_\pi \mathrm{Y}) = g(\pi\mathrm{X}, \pi\mathrm{Y}) - \omega_\pi(\mathrm{X})\omega_\pi(\mathrm{Y}) \tag{3.5}$$

then $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ is called a *sliced almost contact metric manifold* (Gümüş, 2018).

Example 3.1 Let the coordinate functions be (x_1, x_2, y_1, y_2, z) in $\mathbb{R}^5$. If we define the tensor field $\omega : T\mathbb{R}^5 \to \mathrm{C}^\infty(\mathbb{R}^5, \mathbb{R})$ and the characteristic vector field ξ on $\mathbb{R}^5$ as

$$\omega = \frac{1}{2}(dz - y_1 dx_1)$$

$$\xi = 2\frac{\partial}{\partial z}$$

then it is easily seen that $\omega(\xi) = 1$. If we choose the subspace H in $\mathbb{R}^5$ as $H = Sp\{\partial x_1, \partial y_1, \partial z\}$ then the projection π becomes:

$$\pi : \mathbb{R}^5 \to H$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \rightarrow \pi X = \begin{pmatrix} X_1 \\ 0 \\ X_3 \\ 0 \\ X_5 \end{pmatrix}.$$

On the other hand let the tensor field $\phi_\pi: T\mathbb{R}^5 \rightarrow \mathrm{H}$ be the following.

$$\phi_\pi = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_1 & 0 & 0 \end{pmatrix}$$

If we make the necessary computations, we get $\phi_\pi^2 X = -\pi(X) + \omega(X)\xi$. As a result, the structure $(\mathbb{R}^5, \phi_\pi, \omega, \pi, \xi)$ is a sliced almost contact manifold. If we define the semi-Riemannian metric g on $\mathbb{R}^5$ by the following,

$$g := -\frac{1}{4}(dx_1^2 + dy_1^2) + \frac{1}{4}(dx_2^2 + dy_2^2) + \omega \otimes \omega \tag{3.6}$$

then, we get the following equations

$$g(\phi_\pi X, \phi_\pi Y) = -\frac{1}{4}(X_3 Y_3 + X_1 Y_1) \tag{3.7}$$

$$g(\pi X, \pi Y) = -\frac{1}{4}(X_1 Y_1 + X_3 Y_3) + \omega(X)\omega(Y). \tag{3.8}$$

From the equations (3.7) and (3.8) we get $g(\phi_\pi X, \phi_\pi Y) = g(\pi X, \pi Y) - \omega(X)\omega(Y)$. Thus, we say that the structure $(\mathbb{R}_2^5, \phi_\pi, \xi, \omega, \pi, g)$ is a sliced almost contact metric manifold.

Definition 3.4 Let $(M, \phi_\pi, \omega_\pi, \pi, \xi)$ be a compatible sliced almost contact manifold with (M, ϕ, η, ξ). If g is a Riemannian metric and (M, ϕ, η, g, ξ) is an almost contact metric manifold which satisfy

$$g(\phi \mathrm{X}, \phi \mathrm{Y}) = g(\mathrm{X}, \mathrm{Y}) - \eta(\mathrm{X})\eta(\mathrm{Y}) \tag{3.9}$$

then $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ is called *compatible sliced almost contact metric manifold* with (M, ϕ, η, g, ξ). Here if we use $g|_{\mathrm{H}} = \overline{g}$ then we get

$$\overline{g}(\phi_\pi \mathrm{X}, \phi_\pi \mathrm{Y}) = \overline{g}(\pi \mathrm{X}, \pi \mathrm{Y}) - \omega_\pi(\mathrm{X})\omega_\pi(\mathrm{Y}) \tag{3.10}$$

where $\omega_\pi(\mathrm{X}) = g(\pi \mathrm{X}, \xi)$ (Gümüş, 2018).

Definition 3.5 Let $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ be a sliced almost contact metric manifold. Then Φ_π is called *second fundamental form* and it is defined in the following sense (Gümüş, 2018).

$$\Phi_\pi(X, Y) = g(\pi X, \phi_\pi Y) \tag{3.11}$$

Definition 3.6 Let $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ be a sliced almost contact metric manifold. If $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ satisfies the equation $\varepsilon d\omega_\pi = \Phi_\pi$ then $(M, \phi_\pi, \omega_\pi, \pi, g, \xi, \varepsilon)$ is called an *ε-sliced contact metric manifold* (Gümüş, 2018).

Let F be a tensor field of type $(1,1)$ on manifold M. If we define the tensor field N_F by

$$N_F: \chi(M) \times \chi(M) \to \chi(M)$$

$$(X, Y) \to N_F(X, Y) \text{ as}$$

$$N_F(X, Y) = F^2[X, Y] + [F(X), F(Y)] - F[F(X), Y] - F[X, F(Y)] \tag{3.12}$$

then N_F is a tensor field of type (1,2) (Yano Kon, 1984).

Definition 3.7 If J_π is a sliced almost complex structure on manifold M and $N_{J_\pi} \equiv 0$ then J_π is *integrable* on M (Gümüş, 2018).

Definition 3.8 Let J_π be a sliced almost complex structure on $M \times \mathbb{R}$. If J_π is integrable then $(\phi_\pi, \omega_\pi, \xi)$ is called a *sliced normal structure* (Gümüş, 2018).

Definition 3.9 Let $(M, \phi_\pi, \pi, \omega_\pi, g, \xi)$ be a compatible sliced almost contact metric manifold with (M, ϕ, η, g, ξ). If (M, ϕ, η, g, ξ) is a Sasakian manifold then $(M, \phi_\pi, \pi, \omega_\pi, g, \xi)$ is a *sliced Sasakian manifold* (Gümüş, 2018).

Theorem 3.1 Let $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ be a sliced almost contact metric manifold. If the structure $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ is sliced Sasakian manifold if and only if the following equation is satisfied (Gümüş, 2018).

$$(\nabla_X \phi_\pi)Y = g(\pi X, \pi Y)\xi - \omega_\pi(Y)\pi X$$

4. Riemannian Curvature of a Sliced Contact Metric Manifold

Let $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ is a sliced contact metric manifold. In this section we introduce the curvature properties of M. For this aim we started with some usual definitions.

Definition 4.1 Let (M, ϕ, η, g, ξ) be a $(2n + 1)$ −dimensional contact metric manifold and the unit vector field $X \in \chi(M)$ is perpendicular to the characteristic vector field ξ. If the set $\{X, \phi X\}$ is the base of a plane section then κ given with the equation under is called the *ϕ −sectional curvature* (Yano Kon, 1984).

$$\kappa(X, \phi X) = g(R(X, \phi X)\phi X, X)$$

Definition 4.2. Let the structure $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ be a sliced contact metric manifold and the unit vector field $X \in \chi(M)$ is perpendicular to the characteristic vector field ξ. If the set $\{\pi X, \phi_\pi X\}$ is a base for a plane section then

$$\kappa_\pi(\pi X, \phi_\pi X) = g(R(\pi X, \phi_\pi X)\phi_\pi X, \pi X) \tag{4.1}$$

the value κ_π is called as ϕ_π-*sectional curvature*.

In this work we used the methods in the doctoral thesis of (Camcı, 2007) to define new tensors to calculate the Riemannian curvature tensor for sliced contact metric manifolds. In order to calculate the Riemannian curvature tensor, we define the tensor B similar to the Definition 3.4.2 (Camcı, 2007).

Definition 4.3 Let $(M, \phi_\pi, \omega_\pi, \pi, g, \xi)$ be a $(2\mathrm{n}+1)-$dimensional sliced contact metric manifold and B is a tensor of type $(0,4)$ and defined as

$$B: \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) \to \mathrm{C}^\infty(M, \mathbb{R})$$

Assume that B satisfies the conditions at the below for all $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{W} \in \chi(M)$ and $\overline{\mathrm{X}}, \overline{\mathrm{Y}}, \overline{\mathrm{Z}}, \overline{\mathrm{W}} \in \mathrm{D}$

1) $B(\pi W, \pi Z, \pi X, \pi Y) = -B(\pi Z, \pi W, \pi X, \pi Y) = -B(\pi W, \pi Z, \pi Y, \pi X)$
2) $B(\pi W, \pi Z, \pi X, \pi Y) = B(\pi X, \pi Y, \pi W, \pi Z)$
3) $B(\pi W, \pi Z, \pi X, \pi Y) + B(\pi W, \pi X, \pi Y, \pi Z) + B(\pi W, \pi Y, \pi Z, \pi X) = 0$
4) $B(\pi \overline{W}, \pi \bar{Z}, \pi \bar{X}, \pi \bar{Y}) = B(\phi_\pi \overline{W}, \phi_\pi \bar{Z}, \pi \bar{X}, \pi \bar{Y}) = B(\pi \overline{W}, \pi \bar{Z}, \phi_\pi \bar{X}, \phi_\pi \bar{Y})$
5) $B(\xi, \pi \bar{Z}, \pi \bar{X}, \pi \bar{Y}) = B(\pi \overline{W}, \xi, \pi \bar{X}, \pi \bar{Y}) = B(\pi \overline{W}, \pi \bar{Z}, \xi, \pi \bar{Y}) =$
$B(\pi \overline{W}, \pi \bar{Z}, \pi \bar{X}, \xi) = B(\pi \bar{X}, \xi, \pi \bar{Y}, \xi) = 0.$

Theorem 4.1 Let B and T be two tensors of type $(0,4)$ and satisfy the all conditions from 1 to 5 in Definition 4.3. In this case, if the following equation is satisfied $\forall \mathrm{X}, \mathrm{Y} \in \chi(M)$

$$B(\pi \mathrm{X}, \pi \mathrm{Y}, \pi \mathrm{X}, \pi \mathrm{Y}) = T(\pi \mathrm{X}, \pi \mathrm{Y}, \pi \mathrm{X}, \pi \mathrm{Y}) \tag{4.2}$$

then the following equation

$$B(\pi \mathrm{W}, \pi \mathrm{Z}, \pi \mathrm{X}, \pi \mathrm{Y}) = T(\pi \mathrm{W}, \pi \mathrm{Z}, \pi \mathrm{X}, \pi \mathrm{Y}) \tag{4.3}$$

is true $\forall \mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{W} \in \chi(M)$.

Proof. If B and T satisfy the all conditions from 1 to 5 in Definition 4.3 then it is clear that the tensor $B - T$ satisfies the all conditions too. From the assumption we can conclude that

$$(B - T)(\pi \mathrm{X}, \pi \mathrm{Y}, \pi \mathrm{X}, \pi \mathrm{Y}) = 0$$

is true $\forall \mathrm{X}, \mathrm{Y} \in \chi(M)$. If we write vector field $\mathrm{Y} + \mathrm{W}$ instead of the vector field Y in the equation then we get the following equation.

$$(B - T)(\pi \mathrm{X}, \pi \mathrm{Y} + \pi \mathrm{W}, \pi \mathrm{X}, \pi \mathrm{Y} + \pi \mathrm{W}) = 0$$

Although, the equation

$$\begin{aligned}(B-T)(\pi X,\pi Y+\pi W,\pi X,\pi Y+\pi W) &= (B-T)(\pi X,\pi Y,\pi X,\pi Y)+\\ &\quad (B-T)(\pi X,\pi Y,\pi X,\pi W)+(B-T)(\pi X,\pi W,\pi X,\pi Y)\\ &\quad +(B-T)(\pi X,\pi W,\pi X,\pi W)\\ &= (B-T)(\pi X,\pi Y,\pi X,\pi W)+(B-T)(\pi X,\pi W,\pi X,\pi Y)\\ &= 2(B-T)(\pi X,\pi Y,\pi X,\pi W)\end{aligned}$$

is true for all $X, Y, W \in \chi(M)$ we get

$(B-T)(\pi X,\pi Y,\pi X,\pi W)$ =Now in this equation when we write the vector field $X+Z$ instead of the vector field X then we get the following

$$(B-T)(\pi X+\pi Z,\pi Y,\pi X+\pi Z,\pi W)=0.$$

Although we have the following result,

$$\begin{aligned}(B-T)(\pi X+\pi Z,\pi Y,\pi X+\pi Z,\pi W) &= (B-T)(\pi X,\pi Y,\pi X,\pi W)\\ &\quad +(B-T)(\pi X,\pi Y,\pi Z,\pi W)+(B-T)(\pi Z,\pi Y,\pi X,\pi W)\\ &\quad +(B-T)(\pi Z,\pi Y,\pi Z,\pi W)\\ &= (B-T)(\pi X,\pi Y,\pi Z,\pi W)+(B-T)(\pi Z,\pi Y,\pi X,\pi W)\\ &= (B-T)(\pi X,\pi Y,\pi Z,\pi W)-(B-T)(\pi X,\pi W,\pi Y,\pi Z)\end{aligned}$$

From the equation above we see that

$$(B-T)(\pi X,\pi Y,\pi Z,\pi W)=(B-T)(\pi X,\pi W,\pi Y,\pi Z)$$

is true. Since

$$\begin{aligned}3(B-T)(\pi X,\pi Y,\pi Z,\pi W) &= (B-T)(\pi X,\pi Y,\pi Z,\pi W)+(B-T)(\pi X,\pi Z,\pi W,\pi Y)\\ &\quad +(B-T)(\pi X,\pi W,\pi Y,\pi Z)\end{aligned}$$

and from the third condition in the Definition 4.3 we have $(B-T)(\pi X,\pi Y,\pi Z,\pi W)=0$ $\forall X,Y,Z,W \in \chi(M)$. On the other hand we have

$$(B-T)(\pi X,\pi Y,\pi Z,\pi W)=(B-T)(\pi Z,\pi W,\pi X,\pi Y)=-(B-T)(\pi W,\pi Z,\pi X,\pi Y).$$

From these results we conclude that $(B-T)(\pi W,\pi Z,\pi X,\pi Y)=0$ is true $\forall X,Y,Z,W \in \chi(M)$ and we reach the following result.

$$B(\pi W,\pi Z,\pi X,\pi Y)=T(\pi W,\pi Z,\pi X,\pi Y) \qquad \forall X,Y,Z,W \in \chi(M).$$

Theorem 4.2 Let B and T be two tensors of type (0,4) and satisfy the all conditions from 1 to 5 in Definition 4.3. In this case, if the following equation is satisfied $\forall \overline{X},\overline{Y} \in D$

$$B(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y})=T(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y}) \tag{4.4}$$

then the following equation is true $\forall X,Y,Z,W \in \chi(M)$

$$B(\pi W, \pi Z, \pi X, \pi Y) = T(\pi W, \pi Z, \pi X, \pi Y) \tag{4.5}$$

Proof. Assume that $B(\pi\overline{X}, \pi\overline{Y}, \pi\overline{X}, \pi\overline{Y}) = T(\pi\overline{X}, \pi\overline{Y}, \pi\overline{X}, \pi\overline{Y})$ is true $\forall \overline{X}, \overline{Y} \in D$. Also we know that for all $X, Y \in \chi(M)$ we can write

$$X = \overline{X} + \omega_\pi(X)\xi$$
$$Y = \overline{Y} + \omega_\pi(Y)\xi$$

where $\overline{X}, \overline{Y} \in D$. Since B and T satisfy the all conditions from 1 to 5 in Definition 4.3 then it is clear that the tensor $B - T$ satisfies the all conditions too. So we have the following equation,

$$(B-T)(\pi X, \pi Y, \pi X, \pi Y) = (B-T)(\pi\overline{X} + \omega_\pi(X)\xi, \pi\overline{Y} + \omega_\pi(Y)\xi, \pi\overline{X} + \omega_\pi(X)\xi, \pi\overline{Y} + \omega_\pi(Y)\xi).$$

From the fifth condition we get $(B-T)(\pi X, \pi Y, \pi X, \pi Y) = (B-T)(\pi\overline{X}, \pi\overline{Y}, \pi\overline{X}, \pi\overline{Y})$. On the other hand from the assumption we can say that $(B-T)(\pi\overline{X}, \pi\overline{Y}, \pi\overline{X}, \pi\overline{Y}) = 0$ is true. Hence $\forall X, Y \in \chi(M)$ we have $(B-T)(\pi X, \pi Y, \pi X, \pi Y) = 0$ which means that the the following equation is true.

$$B(\pi X, \pi Y, \pi X, \pi Y) = T(\pi X, \pi Y, \pi X, \pi Y).$$

From the Theorem 4.1. we can say that $B(\pi W, \pi Z, \pi X, \pi Y) = T(\pi W, \pi Z, \pi X, \pi Y)$ $\forall X, Y, Z, W \in \chi(M)$ is true.

Theorem 4.3 Let B and T be two tensors of type (0,4) and satisfy the all conditions from 1 to 5 in Definition 4.3. In this case, if the following equation is satisfied $\forall \overline{X} \in D$

$$B(\pi\overline{X}, \phi_\pi\overline{X}, \pi\overline{X}, \phi_\pi\overline{X}) = T(\pi\overline{X}, \phi_\pi\overline{X}, \pi\overline{X}, \phi_\pi\overline{X}) \tag{4.6}$$

then the following equation is true $\forall \overline{X}, \overline{Y} \in D$.

$$B(\pi\overline{X}, \pi\overline{Y}, \pi\overline{X}, \pi\overline{Y}) = T(\pi\overline{X}, \pi\overline{Y}, \pi\overline{X}, \pi\overline{Y}) \tag{4.7}$$

Proof. Assume that

$$B(\pi\overline{X}, \phi_\pi\overline{X}, \pi\overline{X}, \phi_\pi\overline{X}) = T(\pi\overline{X}, \phi_\pi\overline{X}, \pi\overline{X}, \phi_\pi\overline{X})$$

is true for all $\overline{X} \in D$. So we have $(B-T)(\pi\overline{X}, \phi_\pi\overline{X}, \pi\overline{X}, \phi_\pi\overline{X}) = 0$. Now in this equation let's write $\overline{X} + \overline{Y}$ instead of $\overline{X}$ where $\overline{Y} \in D$. From the assumption if we open the following equation

$$I := (B-T)(\pi\overline{X} + \pi\overline{Y}, \phi_\pi(\overline{X} + \overline{Y}), \pi\overline{X} + \pi\overline{Y}, \phi_\pi(\overline{X} + \overline{Y})) = 0$$

for all $\overline{X}, \overline{Y} \in D$ then we get the following.

$$0 = I = 4(B-T)\left(\pi\overline{X}, \phi_\pi\overline{Y}, \pi\overline{X}, \phi_\pi\overline{Y}\right) + 2(B-T)\left(\pi\overline{X}, \phi_\pi\overline{X}, \pi\overline{Y}, \phi_\pi\overline{Y}\right)$$
$$+4(B-T)(\pi\overline{X}, \phi_\pi\overline{X}, \pi\overline{X}, \phi_\pi\overline{Y}) + 4(B-T)(\pi\overline{Y}, \phi_\pi\overline{Y}, \pi\overline{Y}, \phi_\pi\overline{X})$$

If we put the vector field $\overline{X}-\overline{Y}$ instead of the vector field $\overline{X}$ where $\overline{Y}\in D$ then similarly we get

$$II:=(B-T)(\pi\overline{X}-\pi\overline{Y},\phi_\pi(\overline{X}-\overline{Y}),\pi\overline{X}-\pi\overline{Y},\phi_\pi(\overline{X}-\overline{Y}))=0$$

and

$$0=II:=4(B-T)(\pi\overline{X},\phi_\pi\overline{Y},\pi\overline{X},\phi_\pi\overline{Y})+2(B-T)(\pi\overline{X},\phi_\pi\overline{X},\pi\overline{Y},\phi_\pi\overline{Y})$$
$$-4(B-T)(\pi\overline{X},\phi_\pi\overline{X},\pi\overline{X},\phi_\pi\overline{Y})-4(B-T)(\pi\overline{Y},\phi_\pi\overline{Y},\pi\overline{Y},\phi_\pi\overline{X})$$

From these equations I and II we get

$$2(B-T)(\pi\overline{X},\phi_\pi\overline{Y},\pi\overline{X},\phi_\pi\overline{Y})+(B-T)(\pi\overline{X},\phi_\pi\overline{X},\pi\overline{Y},\phi_\pi\overline{Y})=0$$

If we use the conditions and (1), (3) and (4) we can easily see that

$$(B-T)\big(\pi\overline{X},\phi_\pi\overline{X},\pi\overline{Y},\phi_\pi\overline{Y}\big)-(B-T)\big(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y}\big)-(B-T)\big(\pi\overline{X},\phi_\pi\overline{Y},\pi\overline{X},\phi_\pi\overline{Y}\big)=0$$

is true. If we subtract these side by side we reach the following equation.

$$3(B-T)(\pi\overline{X},\phi_\pi\overline{Y},\pi\overline{X},\phi_\pi\overline{Y})+(B-T)(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y})=0$$

If we write the vector field $\phi_\pi\overline{Y}$ instead of the vector field $\overline{Y}$ in this equation we get the following equation.

$$3(B-T)(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y})+(B-T)(\pi\overline{X},\phi_\pi\overline{Y},\pi\overline{X},\phi_\pi\overline{Y})=0$$

If we subtract the last two equations side by side we reach the following equation.

$$(B-T)(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y})+(B-T)(\pi\overline{X},\phi_\pi\overline{Y},\pi\overline{X},\phi_\pi\overline{Y})=0$$

As a result $(B-T)(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y})$ is equal to zero. Thus we get

$$B(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y})=T(\pi\overline{X},\pi\overline{Y},\pi\overline{X},\pi\overline{Y})$$

is true $\forall\overline{X},\overline{Y}\in D$.

Example 4.1 Let $(M,\phi_\pi,\omega_\pi,\pi,g,\xi)$ be a $(2n+1)$ −dimensional sliced contact metric manifold and the unit vector field $X\in\chi(M)$ is perpendicular to the characteristic vector field ξ. Also if we define the folllowing (0,4) tensors B and B_0 then it is clear that they are tensors $4-linear$.

$$B(W,Z,\pi X,\pi Y)=R(W,Z,\pi X,\pi Y)-\frac{3}{4}(g(\pi Y,Z)g(\pi X,W)-g(\pi X,Z)g(\pi Y,W))$$
$$+\tfrac{1}{4}(\omega_\pi(X)\omega_\pi(Z)g(\pi Y,W)-\omega_\pi(Y)\omega_\pi(Z)g(\pi X,W)$$
$$+\omega_\pi(Y)\omega_\pi(W)g(\pi X,Z)-\omega_\pi(X)\omega_\pi(W)g(\pi Y,Z)$$
$$+g(\phi_\pi Y,Z)g(\phi_\pi X,W)-g(\phi_\pi X,Z)g(\phi_\pi Y,W)$$
$$+2g(X,\phi_\pi Y)g(\phi_\pi Z,W))$$

and

$$B_0(W,Z,\pi X,\pi Y)=\frac{1}{4}(g(\pi Y,Z)g(\pi X,W)-g(\pi X,Z)g(\pi Y,W)+\omega_\pi(X)\omega_\pi(Z)g(\pi Y,W)$$
$$-\omega_\pi(Y)\omega_\pi(Z)g(\pi X,W)+\omega_\pi(Y)\omega_\pi(W)g(\pi X,Z)$$
$$-\omega_\pi(X)\omega_\pi(W)g(\pi Y,Z)+g(\phi_\pi Y,Z)g(\phi_\pi X,W)$$
$$-g(\phi_\pi X,Z)g(\phi_\pi Y,W)+2g(X,\phi_\pi Y)g(\phi_\pi Z,W)$$

If we calculate $B(X,Y,\pi W,\pi Z)$ for all $X,Y,Z,W\in\chi(M)$ then we get the following

$$B(X,Y,\pi W,\pi Z)=R(X,Y,\pi W,\pi Z)-\frac{3}{4}(g(\pi Z,Y)g(\pi W,X)-g(\pi W,Y)g(\pi Z,X))$$
$$+\tfrac{1}{4}(\omega_\pi(W)\omega_\pi(Y)g(\pi Z,X)-\omega_\pi(Z)\omega_\pi(Y)g(\pi W,X)$$
$$+\omega_\pi(Z)\omega_\pi(X)g(\pi W,Y)-\omega_\pi(W)\omega_\pi(X)g(\pi Z,Y)$$
$$+g(\phi_\pi Z,Y)g(\phi_\pi W,X)-g(\phi_\pi W,Y)g(\phi_\pi Z,X)$$
$$+2g(W,\phi_\pi Z)g(\phi_\pi Y,X))$$

It is easy to see that $B(W,Z,\pi X,\pi Y)=B(X,Y,\pi W,\pi Z)$ is true. By similar operations we can show that the other properties are satisfied. Define K^* for te orthonormal base $\{X,Y\}$ as follows.

$$K^*(X,Y)=B(X,Y,X,Y)$$

If X and Y are two linearly independent vector fields then we can write the following.

$$K^*(X,Y)=\frac{B(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}$$

So for a plane $\Pi=sp\{X,\phi_\pi X\}$ we have

$$K^*(X,\phi_\pi X)=\frac{B(X,\phi_\pi X,X,\phi_\pi X)}{g(X,X)g(\phi_\pi X,\phi_\pi X)-g(X,\phi_\pi X)^2}$$

$$=\frac{B(X,\phi_\pi X,X,\phi_\pi X)}{g(X,X)g(\phi_\pi X,\phi_\pi X)}$$

If $\overline{X}\in D$ then

$$K^*(\overline{X},\phi_\pi\overline{X})=\frac{B(\overline{X},\phi_\pi\overline{X},\overline{X},\phi_\pi\overline{X})}{g(\overline{X},\overline{X})g(\phi_\pi\overline{X},\phi_\pi\overline{X})-g(\overline{X},\phi_\pi\overline{X})^2}$$

$$=\frac{B(\overline{X},\phi_\pi\overline{X},\overline{X},\phi_\pi\overline{X})}{g(\overline{X},\overline{X})^2}$$

From the definations of B and B_0 we have the following equations.

$$B(\overline{X},\phi_\pi\overline{X},\overline{X},\phi_\pi\overline{X})=R(\overline{X},\phi_\pi\overline{X},\overline{X},\phi_\pi\overline{X})$$

$$B_0(\overline{X},\phi_\pi\overline{X},\overline{X},\phi_\pi\overline{X})=g(\overline{X},\overline{X})^2$$

Here if we use the fact $\|\overline{X}\| = \|\phi_\pi \overline{X}\|$ then we get

$$K^*(\overline{X}, \phi_\pi \overline{X}) = \frac{R(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X})}{g(\overline{X}, \overline{X})^2}$$

$$= \frac{R(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X})}{\|\overline{X}\|^4}$$

$$= R(\frac{\overline{X}}{\|\overline{X}\|}, \frac{\phi_\pi \overline{X}}{\|\phi_\pi \overline{X}\|}, \frac{\overline{X}}{\|\overline{X}\|}, \frac{\phi_\pi \overline{X}}{\|\phi_\pi \overline{X}\|})$$

$$= K(\frac{\overline{X}}{\|\overline{X}\|}, \frac{\phi_\pi \overline{X}}{\|\phi_\pi \overline{X}\|}).$$

We know that $\left\{\frac{\overline{X}}{\|\overline{X}\|}, \frac{\phi_\pi \overline{X}}{\|\phi_\pi \overline{X}\|}\right\}$ is orthonormal. If the ϕ_π −sectional curvature of the space is equal to c then we have

$$K^*(\overline{X}, \phi_\pi \overline{X}) = K(\frac{\overline{X}}{\|\overline{X}\|}, \frac{\phi_\pi \overline{X}}{\|\phi_\pi \overline{X}\|}) = c$$

So we have

$$\frac{B(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X})}{B_0(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X})} = K^*(\overline{X}, \phi_\pi \overline{X}) = c$$

Then we get the following equation.

$$B(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X}) = cB_0(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X})$$

If we say $T(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X}) = (cB_0)(\overline{X}, \phi_\pi \overline{X}, \overline{X}, \phi_\pi \overline{X})$ the $4-linear$ tensor $T \equiv cB_0$ satisfies all the conditions. From the Theorem 4.2 we see that $\forall \overline{X}, \overline{Y} \in D$

$$B(\overline{X}, \overline{Y}, \overline{X}, \overline{Y}) = (cB_0)(\overline{X}, \overline{Y}, \overline{X}, \overline{Y})$$

So from the Theorem 4.2 we say that $\forall X, Y, Z, W \in \chi(M)$ we have

$$B(W, Z, X, Y) = (cB_0)(W, Z, X, Y).$$

At the end we get the following.

$$R(W, Z, \pi X, \pi Y) - \frac{3}{4}(g(\pi Y, Z)g(\pi X, W) - g(\pi X, Z)g(\pi Y, W))$$

$$+\frac{1}{4}(\omega_\pi(X)\omega_\pi(Z)g(\pi Y, W) - \omega_\pi(Y)\omega_\pi(Z)g(\pi X, W) + \omega_\pi(Y)\omega_\pi(W)g(\pi X, Z)$$

$$-\omega_\pi(X)\omega_\pi(W)g(\pi Y,Z)+g(\phi_\pi Y,Z)g(\phi_\pi X,W)-g(\phi_\pi X,Z)g(\phi_\pi Y,W)$$
$$+\,2g(X,\phi_\pi Y)g(\phi_\pi Z,W))$$

$$=\frac{c}{4}(g(\pi Y,Z)g(\pi X,W)-g(\pi X,Z)g(\pi Y,W)+\omega_\pi(X)\omega_\pi(Z)g(\pi Y,W)$$
$$-\omega_\pi(Y)\omega_\pi(Z)g(\pi X,W)+\omega_\pi(Y)\omega_\pi(W)g(\pi X,Z)-\omega_\pi(X)\omega_\pi(W)g(\pi Y,Z)$$
$$+g(\phi_\pi Y,Z)g(\phi_\pi X,W)-g(\phi_\pi X,Z)g(\phi_\pi Y,W)+2g(X,\phi_\pi Y)g(\phi_\pi Z,W))$$

From these we get

$$R(\pi X,\pi Y)Z=\frac{c+3}{4}(g(\pi Y,Z)\pi X-g(\pi X,Z)\pi Y)+\frac{c-1}{4}(\omega_\pi(X)\omega_\pi(Z)\pi Y$$
$$-\omega_\pi(Y)\omega_\pi(Z)\pi X+g(\pi X,Z)\omega_\pi(Y)\xi-g(\pi Y,Z)\omega_\pi(X)\xi$$
$$+g(\phi_\pi Y,Z)\phi_\pi X-g(\phi_\pi X,Z)\phi_\pi Y+\;2g(X,\phi_\pi Y)\phi_\pi Z).$$

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