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EVALUATING THE PERFORMANCE OF BAYESIAN QUANTILE STRUCTURAL EQUATION MODELING WITH VARYING SAMPLE SIZES AND ERROR DISTRIBUTIONS: A SIMULATION STUDY

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ABSTRACT

Structural Equation Modeling (SEM) is commonly used to examine relationships among latent variables, but it relies on strong assumptions such as normality of error terms and focuses solely on mean effects. These assumptions are not always met in practice. In contrast, Quantile Structural Equation Modeling (QSEM) does not assume normality, and allows for the interpretation of effects at different quantile levels of the outcome variable's distribution, enabling a more detailed understanding of heterogeneous relationships.

This study evaluates the performance of BQSEM through simulation using various sample sizes (100, 500, 1000, and 3000) and error term distributions (normal, t, log-normal, and beta). Quantile levels were set at 0.1, 0.5, and 0.9. Results show that BQSEM provides more consistent and reliable estimates than Bayesian SEM (BSEM) when error terms deviate from normality. All analyses were performed using Markov Chain Monte Carlo (MCMC) sampling techniques, specifically implemented through R and WinBUGS software, to estimate posterior distributions in the Bayesian framework.

Keywords: Bayesian Approach, Quantile Regression, Markov Chain Monte Carlo, Simulation

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FARKLI ÖRNEKLEM BÜYÜKLÜKLERİ VE HATA DAĞILIMLARI ALTINDA BAYESYEN KANTİL YAPISAL EŞİTLİK MODELLEMESİNİN PERFORMANSININ DEĞERLENDİRİLMESİ: BİR SİMÜLASYON ÇALIŞMASI

ÖZ

Yapısal Eşitlik Modellemesi, gizil değişkenler arasındaki ilişkileri incelemede yaygın olarak kullanılan bir yöntemdir; ancak hata terimlerinin normal dağıldığı varsayımı gibi güçlü kabullere dayanmakta ve çoğunlukla yalnızca ortalama etkileri dikkate almaktadır. Bu varsayımlar uygulamada her zaman sağlanamamaktadır. Buna karşılık, Kantil Yapısal Eşitlik Modellemesi normal dağılım varsayımına ihtiyaç duymamakta ve bağımlı değişkenin dağılımının farklı kantil düzeylerinde etkilerin incelenmesine olanak tanıyarak ilişkilerin heterojen yapısını daha ayrıntılı biçimde ortaya koymaktadır.

Bu çalışmada, Bayesyen Kantil Yapısal Eşitlik Modellemesinin performansı; farklı örneklem büyüklükleri (100, 500, 1000 ve 3000) ile farklı hata terimi dağılımları (normal, t, log-normal ve beta) altında simülasyon yöntemiyle değerlendirilmiştir. Analizlerde 0,1; 0,5 ve 0,9 kantil düzeyleri esas alınmıştır. Elde edilen bulgular, hata terimlerinin normal dağılımdan saptığı durumlarda Bayesyen Kantil Yapısal Eşitlik Modellemesinin, Bayesyen Yapısal Eşitlik Modellemesine kıyasla daha tutarlı ve güvenilir tahminler sunduğunu göstermektedir. Tüm analizler, Bayesyen çerçevede sonsal dağılımlarını tahmin etmek amacıyla, özellikle R ve WinBUGS yazılımları kullanılarak Markov Zinciri Monte Carlo örnekleme teknikleri ile gerçekleştirilmiştir.

Anahtar Kelimeler: Bayesyen Yaklaşım, Kantil Regresyonu, Markov Zinciri Monte Carlo, Simülasyon

1. INTRODUCTION

SEM is a multivariate statistical technique widely used to investigate causal relationships between observed and latent variables, particularly in the fields of behavioral science, education, health, and economics. Latent variables represent abstract concepts, such as quality of life, happiness, depression, and intelligence, that cannot be measured directly but are inferred from observed variables. These variables are modeled using observed indicators obtained from surveys, tests, and other measurement instruments (Schumacker and Lomax, 2010). SEM consists of two main components: the measurement model, which assesses the effects of latent variables on observed variables, and the structural model, which analyses the relationships between latent variables.

The primary objective of SEM is to assess the fit of the data to the specified theoretical framework and to test the hypotheses formulated by the researcher. In this context, appropriate models are developed to assess the relationships between latent-latent, observed-latent, and observed-observed variables. When examining the historical development of SEM, it can be seen that this approach has progressed through several stages, including regression models, path models with observed variables, confirmatory factor analysis, and structural equation models (Bollen, 1989; Schumacker and Lomax, 2010).

SEM is widely used because of its ability to analyze complex structures and account for measurement errors. In classical (frequentist) SEM approaches, analyses are based on the covariance matrix obtained from observed data. However, the normality assumption plays an important role in these analyses and may not be met in cases of small sample sizes or missing observations (Song and Lee, 2012). In addition, the computation of the covariance matrix can be challenging in complex models. To address these difficulties, with the development of MCMC algorithms, the Bayesian approach has become widely used, especially in complex models. BSEM, performed on raw data, has a wide range of applications in this context.

In parallel, the focus of this study, QSEM is based on the quantile regression approach. Developed by Koenker and Bassett (1978), quantile regression has gained increasing interest since the 2000s in fields such as economics, medicine, and ecology. The primary objective of quantile regression is to assess the effects of explanatory variables on the dependent variable across different percentiles (quantiles). While in classical linear regression, the conditional distribution is represented by a single curve, in quantile regression, the different percentile points of the distribution are explained by multiple curves, allowing for more detailed results. Wang (2016) adapted the quantile regression approach to latent variable models, introducing QSEM to the literature. QSEM is a method that estimates the effects of exogenous latent variables on endogenous latent variables at different quantile points, with its most specific case being the median regression curve, which explains the 50th percentile value.

Among the most commonly used estimation methods in Structural Equation Modeling (SEM) are Maximum Likelihood (MLE), Least Squares (LS), and Generalized Least Squares (GLS). However, applying these methods requires the assumption of normality to be satisfied.

Since this assumption is frequently violated in real-world data, alternative estimation methods such as Weighted Least Squares (WLS), Robust Maximum Likelihood (RML), Robust Weighted Least Squares (RWLS), and Partial Least Squares (PLS) have been developed within classical SEM approaches. Additionally, in cases where the assumption of normality is not met, methods such as BSEM and QSEM also offer viable solutions. QSEM provides more flexible assumptions regarding error terms, making it more reliable than classical SEM in the presence of extreme and outlier values.

Rather than being considered an alternative to SEM, QSEM should be regarded as a complementary method. This approach provides a more appropriate analytical framework compared to classical SEM, particularly in cases where the normality assumption is violated, outliers are present, or the effects of exogenous variables on endogenous variables need to be examined in greater detail across different quantiles. In this study, the Bayesian estimation method, which is also applicable to complex models, is employed to estimate QSEM.

2. LITERATURE REVIEW

QSEM is a technique developed as an extension of quantile regression. Quantile regression, which examines the effects of the explanatory variable(s) on the dependent variable in detail, holds a significant place in the literature due to its flexibility in application. Despite the popularity of quantile regression in various fields, its application in statistical models involving latent variables, particularly in SEM, remains highly limited. Dunson, Watson, and Taylor (2003) employed the Bayesian median regression technique for latent variable models. In their study on a two-stage quantile regression approach for SEM, Chen and Portnoy (1996) referred to their model as SEM, but it is evident that their model was, in fact, a simultaneous equations model. Thus, their study cannot be considered a QSEM application. Similarly, Ma and Koenker (2006) estimated a model established with observed variables in their research titled "Quantile Regression Method for Recursive SEM". Xu and Zhu (2019) examined a quantile regression model that does not include latent variables under the QSEM framework. Since these studies do not involve latent variables, the established models are not considered classical SEM. In this regard, Wang's (2016) dissertation stands out as the first study in QSEM that incorporates latent variables.

Although various methods have been developed to analyze quantile regression, the Bayesian estimation method has gained considerable attention (Yu and Moyeed, 2001; Zhang and Tang, 2017). The Bayesian estimation method for quantile regression was first introduced by Yu and Moyeed (2001). The primary challenge in Bayesian estimation of quantile regression lies in the difficulty of defining the likelihood function. Various approaches have been developed in the literature to address this issue. The first and most widely used approach is the pseudo-likelihood function generated using the Asymmetric Laplace Distribution (ALD), introduced by Koenker and Bassett (1978). This approach in quantile regression has been adapted for latent variable models, specifically in QSEM and random-effects varying coefficient models, in Wang's (2016) dissertation. It is evident that nearly all studies examining QSEM, a relatively novel technique in the literature, have employed Bayesian estimation methods.

In his dissertation, Wang (2016) extensively examined the theoretical structure of QSEM. He applied the Classical BQSEM, Bayesian regularized QSEM, and quantile regression techniques for longitudinal data on different datasets and supported his findings with simulations. To estimate Classical BQSEM, he employed the Bayesian quantile regression approach based on ALD. For Bayesian estimation, Gibbs sampling, one of the MCMC techniques, was utilized. Simulation studies conducted with sample sizes of 500 and 1000 to evaluate the performance of these techniques revealed that BQSEM performs remarkably well when error terms deviate from normality. In the second part of his study, simulations and real data applications were conducted on Bayesian regularized QSEM. In the simulation study, BLasso and BaLasso estimators were compared under different distribution types. In the final section, Bayesian and classical approaches were compared through simulation studies under various distributional assumptions of error terms in longitudinal data.

In the context of the ALD-based BQSEM framework, Wang (2016) evaluated BQSEM performance through a simulation study involving 9 latent variables and 1 control variable under two sample size conditions ($n = 500$ and $n = 1000$). Building upon this simulation framework, the present study extends the evaluation of BQSEM by considering a more complex model structure with 12 latent variables and 1 control variable, together with a broader range of sample size scenarios ($n = 100$, 500, 1000, and 3000) and different distributional conditions to provide a more comprehensive evaluation of BQSEM performance.

In their study, Wang, Feng, and Song (2016) conducted simulation and real data applications of the BQSEM model based on ALD, similar to the first section of Wang's (2016) dissertation. Similarly, Feng, Wang, Lu, and Song (2017) applied simulations and real data analysis to Bayesian regularized QSEM, as examined in the second part of Wang's (2016) study.

Shafeeq and Mohamed (2022) employed the ALD approach used by Wang (2016) for posterior inference in Bayesian QSEM. They evaluated the model's performance through a simulation study, particularly for cases with small sample sizes (n = 25). The results demonstrated that the estimated values remained consistent even in small samples.

Kim and Choi (2019) conducted a simulation study to assess the posterior consistency of parametric and nonparametric BSEM and parametric BQSEM. Unlike other BQSEM studies, they examined the relationships between latent variables and the effects of control variables on the quantiles of error distributions.

Another approach for deriving the likelihood function in Bayesian quantile regression is the Bayesian empirical likelihood method, developed by Yang and He (2012). Zhang and Tang (2017) adapted this method to BQSEM to estimate parameters and latent variables. This approach eliminates the need for specific assumptions about latent variables and random errors. The study employed MCMC techniques, combining Gibbs sampling and the Metropolis-Hastings algorithm. The effectiveness of this approach under different error term distributions was examined through a simulation study, which demonstrated that BQSEM provided less biased results than classical methods.

A review of the literature indicates that research on QSEM remains limited and has substantial potential for further development. This study employs the ALD approach in a simulation study for BQSEM, distinguishing itself from previous studies by considering a more complex model and a broader range of sample sizes.

3. QUANTILE STRUCTURAL EQUATION MODELING

The classical SEM includes measurement equations to explain latent variables by utilizing more than one observed variable. Furthermore, structural equations, essentially regression models based on the mean, have been used to investigate how exogenous latent variables affect the outcomes of interest. In other words, the classical SEM does not merely provide a comprehensive analysis of the relationship between latent variables since it is built solely on the mean value. In QSEM, conditional quantile values of the endogenous latent variable are examined when exogenous latent variables and covariate variables are considered. Thus, the effect of exogenous variables can be compared for each level of the endogenous latent variable (Lee, 2007; Wang, 2016).

$y_i = (y_{i1}, \dots, y_{ip})^T$ (px1) is the ith vector of observed variables with sample size n, $\omega_i = (\omega_{i1}, \dots, \omega_{iq})^T$ (qx1) is the vector of latent variables (q<p). The measurement model showing the relationship between yi and ω_i is as follows:

$$y_i = \mathrm{A}c_i + \Lambda\omega_i + \varepsilon_i \tag{1}$$

The matrix of unknown coefficients with the dimensions (pxr_1) and (pxq) are A and Λ, respectively; c_i is the vector of fixed variables with the dimension of (r_1x1) and ε_i is the vector of error terms with the dimension (px1). In the classical SEM, ε_i shows a multivariate normal distribution with zero mean. The conditional mean of y_i is assumed to be a linear combination of the latent variable ω_i and the constant variable c_i, with normally distributed errors. Biased results are obtained in cases where the normality assumption is not met. In those cases, a more robust estimation method should be used. QSEM considers the conditional median value of y_i instead of the mean value.

As expressed in Equation (1) (where c_i and ω_i are in a linear relationship), in QSEM, the error term ε_i is assumed to have a distribution with zero median. The number of latent variables to be analyzed and the Λ matrix are based on the structure or prior knowledge of the researcher.

In Equation (1) $\omega_i = (\eta_i^T, \xi_i^T)^T$ to be, η_i the vector of endogenous latent variables with the dimension of (q_1x1), ξ_i is the vector of exogenous latent variables with the dimension of (q_2x1). The primary aim of SEMs is to analyze the behavior of η_i in light of the information contained in ξ_i.

In classical SEM, when η_i, ξ_i and covariate variable d_i (r_2x1) are given, the conditional mean of η_i is as follows.

$$E(\eta_i|\xi_i, d_i) = \mathrm{B}d_i + \Gamma\xi_i \tag{2}$$

Here, B (q_1xr_2) and Γ (q_1xq_2) are matrices of unknown coefficients. This conditional mean does not provide the exact definition of the relationship between the latent variables. Conditional quantile values provided that different quantile (τ) values of η_i are $(0 < \tau < 1)$ can be considered to obtain more comprehensive results. The conditional quantile value of $\theta_\tau(\eta_i|\xi_i, d_i)$ can be obtained as follows;

$$\theta_\tau(\eta_i|\xi_i, \mathrm{d}_i) = \mathrm{B}_\tau d_i + \Gamma_\tau\xi_i \tag{3}$$

Here, the values of B_τ and Γ_τ are obtained differently for each quantile value. Thus, the structural model for QSEM can be expressed as follows:

$$\eta_i = \mathrm{B}_\tau d_i + \Gamma_\tau\xi_i + \delta_i \tag{4}$$

The measurement model describing QSEM is given in Equation (1) and the structural model in Equation (4). The distribution of δ_i has not been indicated in QSEM unlike the classical

SEM. The only assumption in QSEM is that the τ-quantile of δ_i must be 0 for Equation (3) to be valid. While the Equation differs for different quantile values in the structural model, the measurement model is limited to a median value. This makes the application of quantile regression redundant for the measurement component, as the primary role of the measurement model is to establish the relationship between observed variables and latent factors (Wang, 2016; Wang, Feng and Song, 2016; Zhang and Tang, 2017). However, since the present study is based on a simulation design, measurement parameters were evaluated across different quantile levels to assess the estimation performance of the proposed approach. This evaluation is specific to the simulation framework, where true parameter values are known, and does not suggest that the measurement model should be separately estimated at each quantile level in empirical QSEM applications.

In conclusion, while the measurement model considered in QSEM is limited to a median regression model, a model can be constructed for different quantile values in the structural model. Thus, it is possible to affirm that QSEM gives more comprehensive results than the classical SEM. The Bayesian estimation method, which is often preferred in the literature, has been utilized for QSEM estimation in the present study.

The likelihood function, which is an indispensable component of the Bayesian approach, cannot be determined because the distribution of the error term δ which is in the structural model of QSEM is not specified. The ALD is being used to improve the pseudo-likelihood function in the quantile regression and QSEM (Yu and Moyeed, 2001; Wang, 2016).

To simplify the Gibbs sampling method in the Bayesian approach, it is possible to have a normal distribution provided that yi ~ ALD(μ,σ,τ) is as indicated below. Accordingly, y is distributed as follows:

$$y = \mu + \kappa_1 e + \sqrt{\kappa_2 \sigma e}\varsigma \tag{5}$$

Here $\kappa_1 = \frac{1-2\tau}{\tau(1-\tau)}$, $\kappa_2 = \frac{2}{\tau(1-\tau)}$ and $\varsigma_i \sim N(0,1)$ are shown. e is exponentially distributed with the scale parameter σ, and ς is independent of e. In other words, the y variable is enhanced by latent variable e, thus y shows a normal distribution with the conditional distribution mean being $\mu+\kappa_1 e$ and variance $\kappa_2 \sigma e$. This enhancement makes it possible for β which is in prior distribution to distribute normally.

For BQSEM, the posterior distribution of β is shown as follows (Feng et al., 2017; Wang, 2016; Wang, Feng and Song, 2016; Yu and Moyeed, 2001).

$$P(\beta|y) \propto P(\beta) exp\left\{-\frac{\sum_{i=1}^{n} \rho_\tau(y_i - x_i^\tau \beta)}{\sigma}\right\} \tag{6}$$

ALD (independent of other basic distributions) has been used to model error terms in both Equations for the Bayesian estimation. For median regression modeling in Equation (1), the kth component of the error term ε_i is considered as AL(0, σ_{yk}, 0.5). δ_i, on the other hand, is set as AL(0, σ_η, τ) for the quantile regression model τ considered in Equation (4). Variables e_{yik} and $e_{\eta i}$ variables are addressed to express ε_{ik} and δ_i.

Provided that θ_y and θ_ω stand for the measuring model (1) and the structural model (4), respectively, and $\theta = (\theta_y, \theta_\omega)$ is the vector of unknowns, BQSEM is shown in the following way (i = 1,2,…,n):

$$\begin{aligned}
&(y_i|\eta_i, \xi_i, \theta_y, e_{yi}) \;\widetilde{bsz}\; N_p(\mathrm{A}c_i + \Lambda\omega_i, \psi_i)\\
&(\eta_i|\xi_i, \theta_\omega, e_{\eta i}) \;\widetilde{bsz}\; N(B_\tau d_i + \Gamma_\tau \xi_i + \kappa_1 e_{\eta i}, \kappa_2 \sigma_\eta e_{\eta i})\\
&\xi_i \;\widetilde{iid}\; N_{q_2}(0, \phi) \qquad (7)\\
&e_{yik} \;\widetilde{iid}\; \text{Üstel}(\sigma_{yk})\\
&e_{\eta_i} \;\widetilde{iid}\; \text{Ü}stel(\sigma_n)\\
&\theta \sim P(\theta)
\end{aligned}$$

Where $e_{yi} = (e_{yi1}, \ldots, e_{yip})^T$, $\Psi_i = \mathrm{diag}(8\sigma_{y1}e_{yi1}, \ldots, 8\sigma_{yp}e_{yip})$ and P(θ) is the prior distribution of θ. Let $\Lambda_y = (A, \Lambda)$ and $\Lambda_{\omega\tau} = (B_\tau, \Gamma_\tau)$. The kth row of Λ_y is Λ_{yk}^T, for k = 1, 2, ..., p.

Conjugate prior distributions utilized for BSEM are given in the following forms:

$$\begin{aligned}
&\Lambda_{yk} \sim N_{r_1+q}(\Lambda_{0yk}, \mathrm{H}_{0yk})\\
&\sigma_{yk}^{-1} \sim Gamma(\alpha_{0yk}, \beta_{0yk})\\
&\phi^{-1} \sim Wishart(R_0, \rho_0) \qquad (8)\\
&\Lambda_{\omega\tau} \sim N_{r_2+q_2}(\Lambda_{0\omega}, H_{0\omega})\\
&\sigma_\eta^{-1} \sim Gamma(\alpha_{0\sigma}, \beta_{0\sigma})
\end{aligned}$$

The conjugate prior distribution, which is an information-containing prior distribution, has its own parameters, called hyperparameters (Lee, 2007). Here, α_{0yk}, β_{0yk}, $\beta_{0\sigma}$, ρ_0, Λ_{0yk}, $\Lambda_{0\omega}$ and the positive definite matrices H_{0yk}, $H_{0\omega}$, R_0 are hyperparameters compiled from studies in the literature or expert knowledge.

Bayesian parameter estimations are obtained by drawing samples from the posterior distribution P(Ω,θ|Y,C,D,e_η) for latent variables and iterations of the parameters. The latent variables matrix $\Omega = (\omega_1, \ldots, \omega_n)$ indicates $Y = (y_1, \ldots, y_n)$, $C = (c_1, \ldots, c_n)$, $D = (d_1, \ldots, d_n)$ and $e_\eta = (e_{\eta 1}, \ldots, e_{\eta n})^T$ matrices. Because of the difficulty of obtaining the finite distribution, the Gibbs sampling method is utilized to iteratively generate each component from the full conditional posterior distribution. The estimates of θ obtained by the Bayesian method are determined as the sample mean of the random observations generated by the simulation (Wang, 2016; Wang, Feng and Song, 2016; Shafeeq and Mohamed, 2022).

In this study, conjugate prior distributions have been used as shown above. Furthermore, the Gibbs sampling method of Bayesian inference and the MCMC algorithms were used to estimate the parameters.

4. SIMULATION STUDY

In this study, a simulation was conducted to increase the use of QSEM and to enhance its understanding in the national literature. In the simulation, the Bayesian estimates of the classical SEM and QSEM (BSEM and BQSEM) were compared according to different distribution types and sample sizes, and the results obtained were interpreted. All analyses were performed using the R program and the WinBUGS package.

4.1. Simulation Design

The main objective of the simulation study conducted to evaluate the empirical performance of BQSEM is to determine the coefficient estimates ($b_{1\tau}$, $\gamma_{1\tau}$, $\gamma_{2\tau}$, $\gamma_{3\tau}$) in the structural model under different quantile values and compare them with the theoretical values. The simulation design used for this purpose is shown in Figure 1.

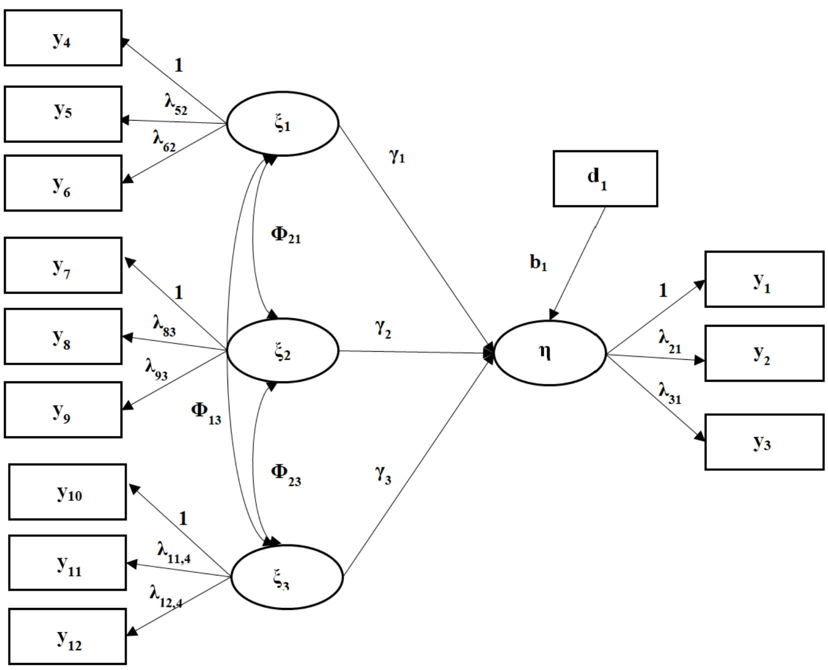

Figure 1. Simulation Design

In the simulation design presented in Figure 1, one endogenous latent variable (η), three exogenous latent variables (ξ_1, ξ_2, and ξ_3), and one control variable (d_1) are included in the model. Each latent variable is explained by three observed variables. That is, the parameters are set as p = 12, q = 4, q_1 = 1, q_2 = 3, and $r_1 = r_2 = 1$. The yi dataset will be generated using the following measurement model (9) and structural model (10).

$$y_i = \mathrm{A}c_i + \Lambda\omega_i + \varepsilon_i \tag{9}$$

$$\eta_i = b_1 d_i + \gamma_1\xi_{i1} + \gamma_2\xi_{i2} + \gamma_3\xi_{i3} + \delta_i \tag{10}$$

Here, $y_i = (y_{i1}, y_{i2}, \ldots, y_{i,12})^T$, $A = (a_1, a_2, \ldots, a_{12})$, $c_i = c_{i1}$, $\varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{i,12})^T$, $\xi_{i1} = (\xi_{i1}, \xi_{i2}, \xi_{i3})^T$, and $\omega_i = (\eta_i, \xi_i^T)^T$'dir. The Λ matrix, which represents the factor loadings, is given below:

$$\Lambda^T = \begin{pmatrix} 1 & \lambda_{21} & \lambda_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \lambda_{52} & \lambda_{62} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_{83} & \lambda_{93} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_{11,4} & \lambda_{12,4} \end{pmatrix}_{(4x12)}$$

In this matrix, the values of 1 and 0 are fixed, while λ_{jk} represents the unknown parameters. The true values of the unknown parameters considered in the model are as follows:

$A = (a_1, a_2, \ldots, a_{12})^T = (1, 1, \ldots, 1)^T$,

$$\Lambda^T = \begin{pmatrix} 1 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.8 \end{pmatrix}_{(4x12)},$$

$b_1 = 0.8$, $\gamma_1 = 0.5$, $\gamma_2 = 0.8$ and $\gamma_3 = 1.2$,

$\phi_{11} = \phi_{22} = \phi_{33} = 1$ and $\phi_{12} = \phi_{13} = \phi_{23} = 0.4$

The constant c_i and each of the control variables d_i were independently generated from a standard normal distribution N(0, 1), while the exogenous latent variables ξ_i were generated from a normal distribution N(0, Φ).

To assess the effectiveness of the proposed simulation study, the model was repeated for different sample sizes and quantile (tau - τ) values. The sample sizes were set as n =100, 500, 1000, and 3000, while the quantile values were chosen as τ = 0.1, 0.5, and 0.9.

In classical SEM, it is assumed that the error terms of the measurement and structural models (ε_i and δ_i) follow a multivariate normal distribution. However, since this assumption does not hold in QSEM, different distribution types were considered to evaluate its effectiveness. The error terms ε_i and δ_i were assumed to follow different distributions:

(I) ε_i and $\delta_i \sim N(0, 0.3)$: Follow a normal distribution similar to classical SEM.

(II) ε_i and $\delta_i \sim t(2)$: Follow a t-distribution with heavy tails, assessing QSEM's performance in the presence of outliers.

(III) $\varepsilon_i \sim N(0, 0.3)$ and $\delta_i \sim$ Log-Normal(0, 0.3) : Evaluate QSEM's performance when data exhibits skewness.

(IV) $\varepsilon_i \sim N(0, 0.3)$ and $\delta_i \sim$ Beta(0.3, 0.3) : Assess QSEM's performance in U-shaped distributions that deviate from normality.

To visualize the structure of the structural model error terms (δ) under the distributions given above, histogram plots based on randomly generated data in R for n = 1000 are presented in Figure 2.

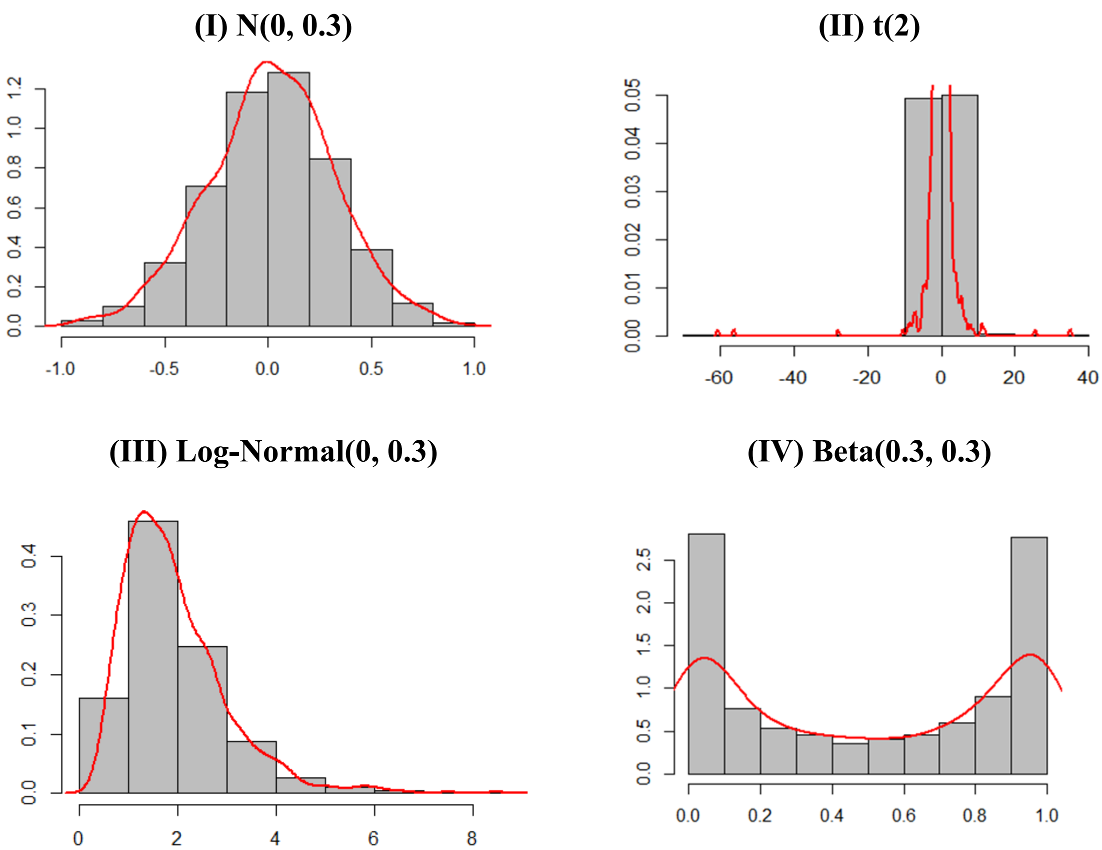

Figure 1. Histogram Plots

As seen in Figure 2, Figure (I) represents a classical normal distribution, Figure (II) exhibits a heavy-tailed distribution, Figure (III) shows a right-skewed distribution, and Figure (IV) deviates from normality with a U-shaped distribution. In the simulation study, error terms will be generated according to all these distributions.

4.2. Prior Distributions

The conjugate prior distributions for the measurement and structural models are provided below. The prior mean values for A and the factor loading matrix Λ_{0yk} in the measurement model are given as follows:

$a_1 = a_2 = \ldots = a_{12} = 1.2$

$$\Lambda^T = \begin{pmatrix} 1 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.8 \end{pmatrix}_{(4x12)}$$

Additionally H_{0yk} is considered a diagonal matrix with diagonal elements equal to 100^2. The prior mean values for the structural model at each τ level are given by $\Lambda_{\omega\tau}$ as follows:

$b_1 = 0.8$

$\gamma_1 = \gamma_2 = \gamma_3 = 0.6$

The Bayesian method generally provides robust results under different prior specifications. When different values are assigned to the hyperparameters in the prior distributions to assess the sensitivity of the estimates, similar results are obtained. Therefore, alternative prior distribution strategies have not been developed.

4.3. Evaluation of Simulation Results

Determining the number of Markov chains and iterations is crucial for the convergence of the established model. The Gibbs sampling method within the MCMC algorithm was employed to obtain the posterior distribution. Three parallel chains were run with different initial values. Each chain consisted of 8,000 iterations, with the first 2,000 iterations discarded as burn-in. No thinning was applied because autocorrelation levels were acceptable.

To evaluate the performance of the method, 100 independent datasets were generated for the entire simulation design. For each quantile level, both BQSEM and, additionally, classical BSEM were applied. To compare these models, the bias and root mean square (RMS) of each parameter estimate were calculated, and the differences between them were examined.

The formulas for bias and RMS are provided below (Song and Lee, 2012):

$$Bias = M\left(\hat{\theta}(r)\right) - \theta_0(r) \tag{11}$$

$$RMS = \left[\frac{\sum_{j=1}^{t}\left[\hat{\theta}_j(r) - \theta_0(r)\right]^2}{n}\right]^{\frac{1}{2}} = \left[\frac{\sum_{j=1}^{100}\left[\hat{\theta}_j(r) - \theta_0(r)\right]^2}{100}\right]^{\frac{1}{2}} \tag{12}$$

The notations used in these formulas are defined as follows:

$\theta(r)$: The rth element of θ.

$\theta_0(r)$: The true value of $\theta(r)$.

$\hat{\theta}_j(r)$: The estimate of $\theta(r)$ in the jth iteration.

$M(\hat{\theta}_j(r))$: The mean of the parameter estimates for $\theta(r)$.

In the RMS formula, t represents the number of replications. Since 100 replications were conducted in this study, t is set to 100.

5. RESULTS

5.1. MCMC Convergence Diagnostics

Before proceeding to the analysis of the simulation results, convergence of the MCMC chains was assessed using trace plots and Geweke diagnostics. Since the simulation study involves multiple sample sizes, error distributions, quantile levels, and model parameters, reporting convergence diagnostics for all simulation conditions would substantially increase the length of the manuscript. Therefore, representative convergence results are presented only for (n=1000), which corresponds to a moderate sample size and provides a representative illustration of the MCMC behavior. For ease of presentation, trace plots are reported only for BSEM and BQSEM at the median quantile level (tau=0.5).

Three independent MCMC chains were generated for each parameter. The Geweke diagnostic was computed separately for each chain, and the maximum absolute Z-statistic across the three chains was reported as a conservative summary measure. The Geweke diagnostic results were generally consistent with the graphical assessment. Convergence was assessed using the standard Geweke diagnostic criterion, where absolute Z-scores exceeding 1.96 and 2.58 may indicate potential differences between the early and late portions of the chain at the 5% and 1% significance levels, respectively (Geweke, 1992). Visual inspection of the trace plots suggested that the chains mixed reasonably well and fluctuated around stable regions, indicating satisfactory convergence behavior overall, although minor deviations were observed for a small number of parameters under certain simulation conditions. While a limited number of parameters produced relatively large maximum absolute Geweke (Z)-statistics across chains, the majority of the diagnostic results did not indicate systematic convergence problems. Taken together, the graphical and numerical diagnostics suggest that the MCMC estimation procedure achieved adequate convergence for the purposes of the simulation study, and the resulting posterior summaries are appropriate for comparative evaluation.

Representative trace plots are provided in Appendix A, whereas the corresponding Geweke diagnostic results are reported in Appendix B.

5.2. Simulation Results

This section presents a detailed analysis of the results from the simulation study, which was conducted to enhance the applicability of QSEM, a relatively new method in the literature, and to evaluate the efficiency of the obtained parameter estimates. Since the estimation in QSEM was performed using the Bayesian approach, the results are presented as BQSEM. Additionally, the parameter estimates obtained from BQSEM were compared with those from BSEM.

Since the parameter estimates in the structural model vary across different quantile levels in BQSEM, the estimates for b, γ_1, γ_2, and γ_3 are presented first. The parameter estimates of the structural model for different sample sizes in BSEM and BQSEM are provided in Tables 1, 2, 3, and 4. These tables present the bias and RMS values of the parameter estimates obtained for each estimation method. The key point to consider is that the estimation method with smaller absolute bias and RMS values demonstrates better performance. Table 1 presents the bias and RMS values of the estimated structural model parameters for all distribution types when the sample size is n = 100.

Table 1. Parameter Estimates of BSEM and BQSEM (*n=100*)

		n=100							
		BSEM		BQSEM					
		-		τ =0.1		τ =0.5		τ =0.9	
Case		**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
(I) N(0, 0.3)	b_1	0.0057	0.0843	-0.1468	0.1489	-0.1576	0.1583	-0.1438	0.1454
	γ_1	0.0250	0.0947	-0.0535	0.1703	-0.0554	0.1673	-0.0197	0.1561
	γ_2	-0.0349	0.1055	-0.0833	0.1813	-0.0311	0.1441	0.0575	0.0727
	γ_3	-0.1086	0.1500	-0.0427	0.0830	-0.0548	0.0840	-0.0832	0.1134
(II) t(2)	b_1	-0.2597	0.5006	0.0338	0.2724	0.0104	0.1693	-0.0646	0.1050
	γ_1	-0.2700	0.3726	-0.0943	0.3797	-0.0900	0.3039	-0.0698	0.4898
	γ_2	-0.5685	0.6515	0.5114	0.5685	0.3481	0.3954	0.5466	0.5590
	γ_3	-0.8069	0.9259	-0.4705	0.5030	-0.5443	0.5523	-0.4431	0.4939
(III) LN(0, 0.3)	b_1	0.0181	0.0746	-0.1317	0.1352	-0.1448	0.1524	-0.1200	0.1551
	γ_1	0.0631	0.1556	-0.0351	0.1711	-0.0543	0.1648	-0.0225	0.2090
	γ_2	-0.0442	0.0949	-0.1013	0.1891	-0.0562	0.1455	0.0705	0.0755
	γ_3	-0.1362	0.1602	-0.0319	0.0836	-0.0338	0.0741	-0.0303	0.0831
(IV) Beta(0.3, 0.3)	b_1	-0.0010	0.0739	0.0201	0.1480	0.0370	0.1346	0.0747	0.1353
	γ_1	0.0228	0.0912	0.0337	0.0415	0.0143	0.0413	-0.0047	0.0642
	γ_2	-0.0638	0.1082	0.0289	0.0650	0.0343	0.0699	0.0556	0.0655
	γ_3	-0.1138	0.1417	-0.0112	0.0911	0.0455	0.1572	0.0737	0.2138

When examining Table 1, it is noticeable that in Case II, where ε_i and δ_i follow a t(2) distribution, BQSEM exhibits lower bias and RMS values. This indicates that BQSEM provides more robust estimates in the presence of outliers in the error terms. For n = 100, a clearer conclusion can be drawn regarding the t-distribution, but for other distributional assumptions, making definitive comments is more challenging due to higher variability. These findings become even more evident at larger sample sizes (n = 500, 1000, and 3000), for which the bias and RMS values of the estimated structural model parameters are reported in Table 2, Table 3, and Table 4 under all distributional assumptions.

Table 2. Parameter Estimates of BSEM and BQSEM (*n=500*)

		n=500							
		BSEM		BQSEM					
		-		τ =0.1		τ =0.5		τ =0.9	
Case		**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
(I) N(0, 0.3)	b_1	0.0144	0.0872	0.0287	0.0444	0.0205	0.0286	0.0309	0.0351
	γ_1	0.0241	0.0944	-0.0401	0.0509	-0.0168	0.0298	-0.0058	0.0296
	γ_2	-0.0443	0.1117	-0.0295	0.0768	-0.0537	0.0924	-0.0585	0.1019
	γ_3	-0.0898	0.1328	-0.0312	0.0579	-0.0420	0.0702	-0.0237	0.0685
(II) t(2)	b_1	-0.2120	0.4342	0.0510	0.1096	0.0017	0.0817	-0.0234	0.0966
	γ_1	-0.2798	0.3891	0.0088	0.2947	0.0157	0.2081	0.0651	0.3174
	γ_2	-0.5252	0.5979	0.2853	0.3202	0.0914	0.1170	0.0696	0.1619
	γ_3	-0.8646	0.9331	0.2917	0.3152	-0.0459	0.0817	0.5266	0.5298
(III) LN(0, 0.3)	b_1	-0.0240	0.0944	0.0218	0.0379	0.0194	0.0304	0.0383	0.0504
	γ_1	0.0319	0.1065	-0.0295	0.0439	-0.0147	0.0253	-0.0261	0.0520
	γ_2	-0.0461	0.1270	-0.0428	0.0812	-0.0574	0.0993	-0.0420	0.1162
	γ_3	-0.0829	0.1375	-0.0414	0.0724	-0.0483	0.0751	0.0416	0.0956
(IV) Beta(0.3, 0.3)	b_1	-0.0126	0.0865	0.0283	0.0333	0.0167	0.0267	0.0097	0.0314
	γ_1	0.0249	0.0792	0.0082	0.0318	0.0137	0.0270	0.0139	0.0288
	γ_2	-0.0436	0.1019	-0.0148	0.0208	-0.0265	0.0306	-0.0296	0.0338
	γ_3	-0.1406	0.1645	0.0177	0.0369	0.0128	0.0302	0.0104	0.0222

Table 3. Parameter Estimates of BSEM and BQSEM (*n=1000*)

		n=1000							
		BSEM		BQSEM					
		-		τ =0.1		τ =0.5		τ =0.9	
Case		**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
(I) N(0, 0.3)	b_1	-0.0177	0.1022	0.0202	0.0324	0.0165	0.0240	0.0110	0.0187
	γ_1	0.0182	0.0966	-0.0230	0.0268	0.0046	0.0223	0.0317	0.0373
	γ_2	-0.0348	0.1065	0.0558	0.0568	0.0329	0.0358	0.0261	0.0331
	γ_3	-0.1097	0.1420	-0.0047	0.0211	-0.0201	0.0251	-0.0108	0.0219
(II) t(2)	b_1	-0.2161	0.4166	0.0428	0.0848	0.0183	0.0797	0.0008	0.0891
	γ_1	-0.2761	0.3848	0.0069	0.2604	0.0369	0.1078	-0.1080	0.3119
	γ_2	-0.4831	0.5672	0.1078	0.1867	0.0015	0.0690	0.2233	0.3167
	γ_3	-0.8471	0.9206	0.2603	0.3331	-0.1396	0.1538	0.3020	0.3281
(III) LN(0, 0.3)	b_1	0.0192	0.1023	0.0261	0.0314	0.0186	0.0264	0.0137	0.0233
	γ_1	0.0210	0.1130	-0.0092	0.0211	0.0079	0.0254	0.0247	0.0403
	γ_2	-0.0588	0.1071	0.0402	0.0412	0.0278	0.0332	0.0467	0.0570
	γ_3	-0.1006	0.1449	-0.0236	0.0290	-0.0248	0.0316	0.0474	0.0564
(IV) Beta(0.3, 0.3)	b_1	0.0143	0.0855	0.0142	0.0170	0.0183	0.0222	0.0183	0.0263
	γ_1	0.0217	0.0915	0.0213	0.0280	0.0267	0.0315	0.0421	0.0442
	γ_2	-0.0428	0.0937	0.0001	0.0261	-0.0053	0.0195	-0.0109	0.0159
	γ_3	-0.1287	0.1598	-0.0386	0.0440	-0.0356	0.0394	-0.0311	0.0342

Table 4. Parameter Estimates of BSEM and BQSEM (*n=3000*)

		n=3000							
		BSEM		**BQSEM**					
		-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
Case		**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
(I) N(0, 0.3)	b_1	0.0073	0.0853	0.0109	0.0135	0.0095	0.0095	0.0092	0.0096
	γ_1	0.0039	0.0946	-0.0049	0.0203	-0.0072	0.0236	-0.0065	0.0279
	γ_2	-0.0487	0.1020	0.0172	0.0316	0.0148	0.0242	0.0216	0.0252
	γ_3	-0.0938	0.1410	0.0177	0.0225	-0.0020	0.0115	0.0261	0.0272
(II) t(2)	b_1	-0.2335	0.4782	0.0061	0.0319	0.0067	0.0242	-0.0240	0.0396
	γ_1	-0.2530	0.3831	-0.0910	0.2094	-0.0157	0.0837	-0.0994	0.2583
	γ_2	-0.5390	0.6089	0.1098	0.1935	-0.0113	0.1015	0.0873	0.2538
	γ_3	-0.8342	0.9334	0.5156	0.5243	0.0527	0.0932	0.6223	0.6648
(III) LN(0, 0.3)	b_1	-0.0139	0.0921	0.0093	0.0133	0.0083	0.0102	0.0072	0.0096
	γ_1	0.0142	0.1219	-0.0041	0.0184	-0.0043	0.0274	-0.0248	0.0383
	γ_2	-0.0292	0.1080	0.0067	0.0317	0.0072	0.0204	0.0316	0.0348
	γ_3	-0.1043	0.1543	-0.0029	0.0108	-0.0112	0.0133	0.1040	0.1068
(IV) Beta(0.3, 0.3)	b_1	0.0044	0.0712	-0.0104	0.0218	-0.0070	0.0165	-0.0087	0.0135
	γ_1	0.0160	0.0800	0.0126	0.0254	0.0126	0.0234	0.0113	0.0238
	γ_2	-0.0396	0.0974	-0.0034	0.0293	-0.0035	0.0306	0.0013	0.0291
	γ_3	-0.1264	0.1497	-0.0111	0.0128	-0.0210	0.0219	-0.0137	0.0143

The results obtained from Tables 2, 3, and 4, which illustrate cases where the sample sizes are 500, 1000, and 3000, are interpreted below:

- Despite the absence of distributional assumptions for the error terms in BQSEM, traditional BSEM requires the normality assumption for error terms. The simulation study reveals that when εi and δi follow a normal distribution, i.e., N(0, 0.3) (Case I), the bias and RMS values obtained for both models are quite similar. Although one might expect Bayesian estimation in classical SEM to perform better than BQSEM under the normality assumption, the closeness of the results suggests that both methods yield comparable performance. Wang (2016) applied BQSEM using the ALD approach in a relatively less complex simulation study and similarly found that BSEM and BQSEM performed similarly when error terms followed a normal distribution. Likewise, Shafeeq and Mohamed (2022) compared model performances under small sample sizes using the ALD approach and reported that BSEM and BQSEM exhibited similar performances under the normality assumption. Furthermore, Zhang and Tang (2017) compared the empirical likelihood approach with the ALD approach for implementing BQSEM under the normality assumption of error terms and found that the empirical likelihood approach outperformed ALD. Overall, when the normality assumption holds, BSEM and BQSEM exhibit similar performances.

- When the error terms ε_i and δ_i follow a t(2) distribution (Case II), BQSEM yields smaller bias and RMS values compared to BSEM. This indicates that BQSEM performs better than BSEM in the presence of outliers and extreme values.
- When the structural model error term δ_i follows a Log-Normal(0, 0.3) distribution (Case III) and a Beta(0.3, 0.3) distribution (Case IV), the bias and RMS values obtained in BQSEM are smaller than those in BSEM. Particularly, examining the RMS values reveals that BQSEM provides more accurate estimates. Under the assumption that error terms follow a Log-Normal or Beta distribution, BQSEM produces more unbiased estimates and demonstrates superior performance.
- Given that the error terms in the structural model exhibit skewness, contain outliers, or follow a U-shaped distribution (i.e., significantly deviating from normality) under the assumption of t-distribution, Log-Normal, or Beta distribution, classical SEM fails to meet its assumption requirements. While BSEM offers advantages over classical SEM estimation methods when normality is lacking, its performance is not particularly robust compared to BQSEM in cases of normality deviations. Therefore, even with the application of Bayesian estimation in classical SEM when deviating from normality, BQSEM still yields more unbiased results. These findings are consistent with previous studies, including those by Wang (2016), Wang, Feng, and Song (2016), Zhang and Tang (2017), and Shafeeq and Mohamed (2022), which showed that BQSEM outperforms BSEM when structural model error terms deviate from normality (e.g., t-distribution, Log-Normal, Beta distribution). Thus, it can be concluded that this study's findings align with existing literature.
- The bias and RMS values obtained for the t-distribution in BSEM are larger than those obtained for Log-Normal and Beta distributions. This suggests that BSEM performs poorly in the presence of extreme values as well as skewed or U-shaped distributions.
- For all considered error term distributions, increasing the sample size reduces the absolute values of most bias and RMS estimates. In other words, larger sample sizes lead to more unbiased estimates, which is an expected result. Similar conclusions were drawn by Wang (2016), Wang, Feng, and Song (2016), and Zhang and Tang (2017), who emphasized that parameter estimates become more unbiased as sample sizes increase.

Following the simulation study, the bias and RMS values for all estimated parameters (both measurement and structural model parameters) are presented in Tables 5, 6, 7, and 8 for the case where the sample size is 1000. Since the results obtained for other sample sizes show similar patterns, only the interpretations for the sample size of 1000 are provided here. The detailed results for the other sample sizes are presented in Appendices C–N.

Table 5 presents the results under the assumption that measurement and structural model error terms follow a normal distribution (Case I). Given that classical SEM assumes normality in error terms, it is expected that Bayesian estimation in classical SEM would yield better performance. However, when bias and RMS values are examined for each quantile level, the results are found to be quite similar. This suggests that when the normality assumption holds, BSEM and BQSEM yield comparable results.

Table 5. Case (I) - Estimates of All Parameters for BSEM and BQSEM

	ε_i and $\delta_i \sim N(0, 0.3)$, n=1000							
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	0.0006	0.0311	0.0026	0.0060	0.0035	0.0077	0.0015	0.0065
λ_{31}	0.0083	0.0369	0.0043	0.0079	0.0054	0.0096	0.0034	0.0086
λ_{52}	-0.0193	0.0689	0.0158	0.0399	0.0144	0.0395	0.0148	0.0390
λ_{62}	-0.0035	0.0761	0.0165	0.0269	0.0182	0.0287	0.0181	0.0297
λ_{83}	-0.0177	0.0672	0.0221	0.0230	0.0254	0.0260	0.0272	0.0274
λ_{93}	-0.0187	0.0704	0.0156	0.0516	0.0150	0.0488	0.0149	0.0474
$\lambda_{11,4}$	-0.0299	0.0749	0.0268	0.0331	0.0302	0.0355	0.0328	0.0383
$\lambda_{12,4}$	-0.0244	0.0763	-0.0135	0.0228	-0.0125	0.0203	-0.0111	0.0184
b_1	-0.0177	0.1022	0.0202	0.0324	0.0165	0.0240	0.0110	0.0187
γ_1	0.0182	0.0966	-0.0230	0.0268	0.0046	0.0223	0.0317	0.0373
γ_2	-0.0348	0.1065	0.0558	0.0568	0.0329	0.0358	0.0261	0.0331
γ_3	-0.1097	0.1420	-0.0047	0.0211	-0.0201	0.0251	-0.0108	0.0219
ϕ_{11}	0.1525	0.2140	0.0240	0.0933	0.0232	0.0932	0.0214	0.0913
ϕ_{12}	-0.0082	0.1192	0.0077	0.0333	0.0069	0.0336	0.0055	0.0342
ϕ_{13}	0.0031	0.1082	0.0005	0.0347	-0.0011	0.0340	-0.0054	0.0326
ϕ_{22}	0.1869	0.2555	0.0257	0.0856	0.0260	0.0829	0.0250	0.0804
ϕ_{23}	0.0078	0.1071	-0.0132	0.0197	-0.0134	0.0196	-0.0164	0.0229
ϕ_{33}	0.2157	0.2635	0.0712	0.0886	0.0669	0.0837	0.0554	0.0715

Table 6. Case (II) - Estimates of All Parameters for BSEM and BQSEM

	ε_i and δ_i~ t(2), n=1000							
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	-0.1225	1.7785	0.0019	0.0225	0.0073	0.0159	0.0010	0.0126
λ_{31}	-0.1111	1.6259	0.0205	0.0223	0.0219	0.0243	0.0197	0.0219
λ_{52}	-0.0731	0.8862	-0.0091	0.0157	0.0065	0.0096	0.0064	0.0075
λ_{62}	-0.1786	0.7152	-0.0615	0.0774	-0.0537	0.0830	-0.0565	0.0873
λ_{83}	-0.3756	0.7122	-0.0253	0.0779	-0.0197	0.0741	-0.0343	0.0803
λ_{93}	-0.2272	0.7604	-0.0537	0.0581	-0.0569	0.0598	-0.0630	0.0681
$\lambda_{11,4}$	-0.3376	0.9130	-0.0494	0.0564	-0.0484	0.0550	-0.0505	0.0597
$\lambda_{12,4}$	-0.2312	0.6696	-0.0866	0.0918	-0.0936	0.0943	-0.0962	0.0973
b_1	-0.2161	0.4166	0.0428	0.0848	0.0183	0.0797	0.0008	0.0891
γ_1	-0.2761	0.3848	0.0069	0.2604	0.0369	0.1078	-0.1080	0.3119
γ_2	-0.4831	0.5672	0.1078	0.1867	0.0015	0.0690	0.2233	0.3167
γ_3	-0.8471	0.9206	0.2603	0.3331	-0.1396	0.1538	0.3020	0.3281
ϕ_{11}	2.8883	3.6925	-0.0776	0.0871	-0.0650	0.0806	-0.0780	0.0858
ϕ_{12}	-0.0892	0.4287	0.0821	0.0851	0.0998	0.1017	0.0734	0.0779
ϕ_{13}	-0.1146	0.4684	0.0311	0.0624	0.0597	0.0770	0.0238	0.0643
ϕ_{22}	3.2644	4.2556	-0.0097	0.0845	0.0190	0.0908	-0.0082	0.1000
ϕ_{23}	-0.0421	0.3940	-0.0106	0.0482	0.0402	0.0739	-0.0146	0.0731
ϕ_{33}	3.1539	3.8985	-0.0008	0.0625	0.0698	0.0716	-0.0071	0.0237

On the other hand, the results of BSEM and BQSEM, established to examine cases where the measurement and structural model error terms follow a t-distribution (Case II), indicating the presence of outliers, are presented in Table 6. It is noticeable that the bias and RMS values obtained in BSEM are larger than those obtained in BQSEM for each τ value. In other words, BSEM results exhibit greater bias compared to BQSEM results. This suggests that in SEM studies involving outliers, utilizing BQSEM provides more reliable estimates.

The estimates of all parameters for the simulation study conducted under Case (III) and Case (IV), where measurement model errors follow a normal distribution while structural model error terms deviate from normality and follow Log-Normal and Beta distributions, respectively, are presented in Table 7 and Table 8 for n = 1000. Since the measurement model error terms were generated under the normality assumption, the bias and RMS values of the BSEM and BQSEM estimates for λ and ϕ were found to be similar in both models. However, for the structural model parameters, particularly γ_1, γ_2, and γ_3 BQSEM estimates exhibited lower bias and RMS values in both Case (III) and Case (IV).

In the literature, it is recommended to use BSEM instead of classical SEM methods when error terms deviate from a normal distribution. However, the results of this simulation study

Table 7. Case (III) - Estimates of All Parameters for BSEM and BQSEM

	$\varepsilon_i \sim N(0, 0.3)$, $\delta_i \sim LN(0, 0.3)$, n=1000							
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	0.0023	0.0281	0.0020	0.0070	0.0041	0.0076	0.0021	0.0061
λ_{31}	-0.0014	0.0291	0.0047	0.0090	0.0066	0.0106	0.0041	0.0096
λ_{52}	-0.0244	0.0668	0.0165	0.0410	0.0132	0.0376	0.0147	0.0379
λ_{62}	-0.0246	0.0722	0.0189	0.0279	0.0172	0.0277	0.0195	0.0298
λ_{83}	-0.0214	0.0669	0.0217	0.0226	0.0255	0.0259	0.0305	0.0309
λ_{93}	-0.0359	0.0774	0.0136	0.0500	0.0149	0.0490	0.0168	0.0481
$\lambda_{11,4}$	-0.0347	0.0755	0.0283	0.0327	0.0310	0.0356	0.0328	0.0381
$\lambda_{12,4}$	-0.0276	0.0760	-0.0127	0.0234	-0.0117	0.0195	-0.0101	0.0177
b_1	0.0192	0.1023	0.0261	0.0314	0.0186	0.0264	0.0137	0.0233
γ_1	0.0210	0.1130	-0.0092	0.0211	0.0079	0.0254	0.0247	0.0403
γ_2	-0.0588	0.1071	0.0402	0.0412	0.0278	0.0332	0.0467	0.0570
γ_3	-0.1006	0.1449	-0.0236	0.0290	-0.0248	0.0316	0.0474	0.0564
ϕ_{11}	0.1773	0.2407	0.0225	0.0970	0.0239	0.0929	0.0169	0.0899
ϕ_{12}	-0.0084	0.1257	0.0083	0.0345	0.0077	0.0336	0.0027	0.0337
ϕ_{13}	-0.0183	0.1102	0.0011	0.0358	-0.0007	0.0337	-0.0085	0.0316
ϕ_{22}	0.1927	0.2498	0.0293	0.0868	0.0268	0.0834	0.0163	0.0775
ϕ_{23}	0.0007	0.1183	-0.0114	0.0185	0.1257	0.2239	-0.0211	0.0260
ϕ_{33}	0.1914	0.2555	0.0742	0.0915	0.0399	0.1970	0.0379	0.0597

Table 8. Case (IV) - Estimates of All Parameters for BSEM and BQSEM

	$\varepsilon_i \sim N(0, 0.3)$, $\delta_i \sim Beta(0.3, 0.3)$, n=1000							
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	-0.0027	0.0365	-0.0067	0.0075	-0.0048	0.0054	-0.0060	0.0065
λ_{31}	-0.0041	0.0353	-0.0075	0.0080	-0.0062	0.0069	-0.0082	0.0085
λ_{52}	-0.0165	0.0780	0.0113	0.0307	0.0114	0.0320	0.0127	0.0323
λ_{62}	-0.0072	0.0625	0.0261	0.0300	0.0270	0.0310	0.0293	0.0326
λ_{83}	-0.0275	0.0627	0.0243	0.0319	0.0241	0.0323	0.0230	0.0309
λ_{93}	-0.0169	0.0723	-0.0004	0.0075	-0.0015	0.0066	-0.0019	0.0049
$\lambda_{11,4}$	-0.0298	0.0729	-0.0094	0.0142	-0.0089	0.0129	-0.0092	0.0133
$\lambda_{12,4}$	-0.0246	0.0691	0.0042	0.0136	0.0032	0.0132	0.0033	0.0140
b_1	0.0143	0.0855	0.0142	0.0170	0.0183	0.0222	0.0183	0.0263
γ_1	0.0217	0.0915	0.0213	0.0280	0.0267	0.0315	0.0421	0.0442
γ_2	-0.0428	0.0937	0.0000	0.0261	-0.0053	0.0195	-0.0109	0.0159
γ_3	-0.1287	0.1598	-0.0386	0.0440	-0.0356	0.0394	-0.0311	0.0342
ϕ_{11}	0.1643	0.2260	0.0658	0.0944	0.0659	0.0964	0.0621	0.0947
ϕ_{12}	-0.0028	0.1313	-0.0103	0.0288	-0.0101	0.0290	-0.0101	0.0286
ϕ_{13}	-0.0078	0.1280	-0.0046	0.0389	-0.0037	0.0389	-0.0052	0.0383
ϕ_{22}	0.2124	0.2788	0.0538	0.0863	0.0551	0.0867	0.0571	0.0861
ϕ_{23}	0.0223	0.1304	0.0009	0.0267	0.0019	0.0264	0.0035	0.0270
ϕ_{33}	0.1999	0.2569	0.0493	0.0685	0.0502	0.0699	0.0489	0.0681

clearly demonstrate that when error terms deviate from normality, applying BQSEM instead of BSEM yields more robust and unbiased results.

In conclusion, when the structural model error term (δ) follows a normal distribution, the results of BSEM and BQSEM are found to be quite similar. However, in cases where (δ) follows a t-distribution, Log-Normal distribution, or Beta distribution—indicating deviations from normality—BQSEM produces more unbiased results compared to BSEM.

As mentioned in previous sections, in QSEM, the quantile structure applies only to the structural model parameter estimates. The λ and ϕ values obtained in the measurement model are computed solely for the median quantile. However, since this study is based on a simulation, estimates for all values are presented. The results clearly demonstrate that, except for cases where the measurement model error term (ε) follows a t-distribution, the λ and ϕ estimates in the measurement model are quite similar for both BSEM and BQSEM under other distributional assumptions. Theoretically, in QSEM, measurement model estimates are only computed for the median quantile. Therefore, when applied to real data, the measurement model results should be interpreted solely for the median quantile.

6. DISCUSSION

This study evaluates the performance of BSEM across varying sample sizes and error distributions. The simulation evidence indicates that BQSEM provides more robust parameter estimates than classical BSEM, particularly under deviations from normality.

Under normality (Case I), BQSEM and BSEM exhibit virtually indistinguishable performance in terms of bias and RMS. This finding is consistent with prior evidence on ALD-based Bayesian quantile formulations (Wang, 2016; Wang, Feng, & Song, 2016), suggesting that the quantile-based formulation does not lead to a noticeable loss of estimation accuracy when Gaussian assumptions hold. Therefore, BQSEM may be viewed not only as an alternative for non-normal settings but also as a competitive estimation approach under normal conditions.

In contrast, pronounced performance differences emerge under heavy-tailed and skewed distributions. For the t(2) scenario, BQSEM generally produces lower bias and RMS values than BSEM. These findings are consistent with the notion that quantile-based estimation is less influenced by extreme observations than mean-based Bayesian approaches when heavy-tailed error distributions are present.

A similar dominance pattern is observed under asymmetric log-normal and Beta distributions. Across both the median quantile ($\tau = 0.5$) and the lower and upper tail quantiles ($\tau = 0.1, 0.9$), BQSEM generally yields lower RMS values and improved bias control. Notably, these improvements persist at extreme quantiles, although posterior variability increases as expected. These findings suggest that the superior performance of BQSEM is not limited to the median quantile but also extends to the lower and upper tails of the conditional distribution.

Sample size effects follow the expected asymptotic structure, with both frameworks benefiting from increased n. However, BQSEM demonstrates stronger finite-sample stability, particularly under t-distributed errors at $n = 100$. This suggests that quantile-based Bayesian formulations provide a practical advantage in data-constrained settings where classical asymptotics are unreliable.

Overall, the results suggest that BQSEM should not be viewed as a replacement for classical Bayesian SEM, but rather as an alternative framework that provides advantages under non-normal conditions while maintaining comparable performance under normality. Thus, BQSEM offers a flexible approach for evaluating structural relationships under different distributional settings.

7. CONCLUSION

In this study, the performance of Bayesian Quantile Structural Equation Modeling was comprehensively evaluated through a simulation study under varying sample sizes and error distributions. The results indicate that BQSEM yields more reliable and less biased estimates compared to BSEM, particularly when the assumption of normality is violated.

Under normal distribution settings, the performance difference between the two methods was negligible. However, under non-normal conditions such as t, log-normal, and beta distributions, BQSEM generally demonstrates better estimation performance than BSEM. This finding highlights the potential advantage of BQSEM under varying distributional conditions.

In addition, increasing sample size improves estimation accuracy for both methods; however, BQSEM demonstrates more stable performance, especially in small- and medium-sized samples. This suggests a practical advantage in empirical applications.

Consistent with previous findings by Wang (2016), Wang, Feng, and Song (2016), and Zhang and Tang (2017), this study confirms that BQSEM is a powerful alternative in cases where distributional assumptions are violated. Accordingly, BQSEM can be regarded as a complementary and extended framework within the SEM literature.

Future research is recommended to extend the BQSEM framework to nonlinear models, multilevel structural equation models, missing data mechanisms, and categorical variable structures. In addition, although this study provides comprehensive evidence based on simulation results, the lack of real-data applications represents a limitation. Therefore, future studies should further investigate the practical applicability and generalizability of BQSEM using empirical datasets from diverse research fields. Another limitation of this study is the potential computational burden associated with the Bayesian estimation procedure, particularly for more complex models and larger sample sizes. Furthermore, the evaluation of model performance in this study is primarily based on parameter estimation accuracy; therefore, the assessment of model fit remains limited. Future studies may address these issues by considering more efficient computational strategies and incorporating additional model fit criteria.

ETHICAL DECLARATION

In the writing process of the study titled “Evaluating the performance of Bayesian Quantile Structural Equation Modeling with varying sample sizes and error distributions: A simulation study”, there were followed the scientific, ethical and the citation rules; was not made any falsification on the collected data and this study was not sent to any other academic media for evaluation.

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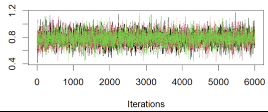
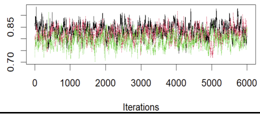
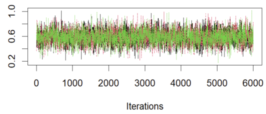
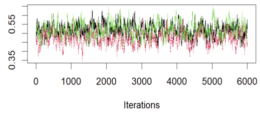
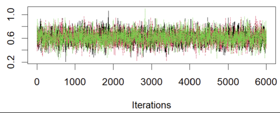
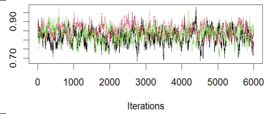
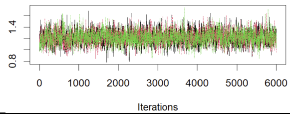
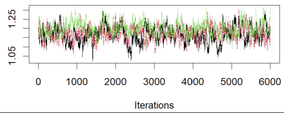
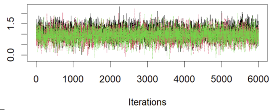
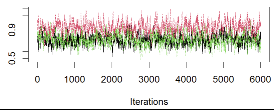
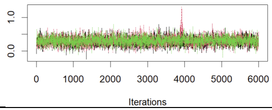
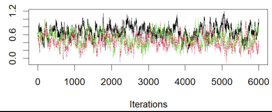
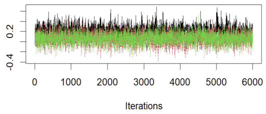
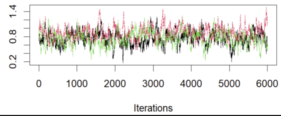
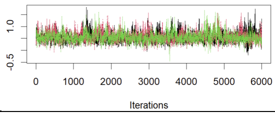
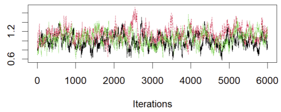
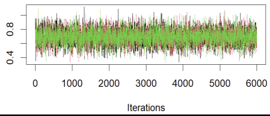
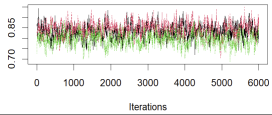
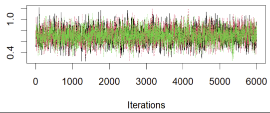
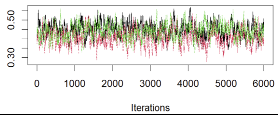
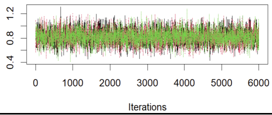
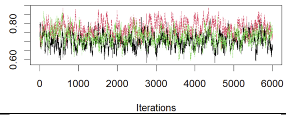
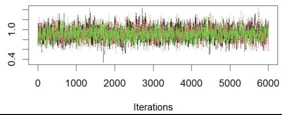
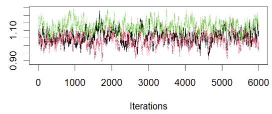
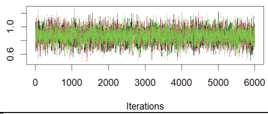
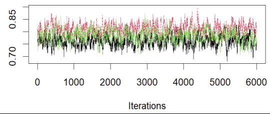
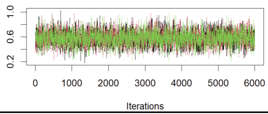
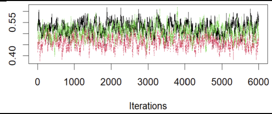
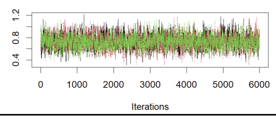
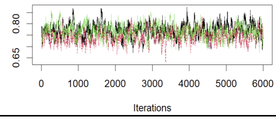
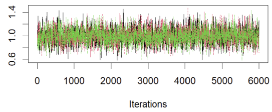
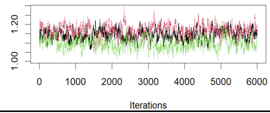
APPENDICES

Appendix A. Trace plots (n = 1000)

Case		**BSEM**	**BQSEM τ=0.5**
(I) N(0, 0.3)	b_1		
	γ_1		
	γ_2		
	γ_3		
(II) t(2)	b_1		
	γ_1		
	γ_2		
	γ_3		
(III) LN(0, 0.3)	b_1		
	γ_1		
	γ_2		
	γ_3		
(IV) Beta(0.3, 0.3)	b_1		
	γ_1		

	γ_2		
	γ_3		

Appendix B. Geweke statistics (n = 1000)

Max |Geweke Z|		**BSEM**	**BQSEM**		
Case		-	**τ =0.1**	**τ =0.5**	**τ =0.9**
(I) N(0, 0.3)	**b_1**	2.361	1.058	0.928	0.503
	γ_1	1.089	2.358	1.116	1.308
	γ_2	1.006	1.481	3.003	2.531
	γ_3	0.544	2.300	2.371	2.580
(II) t(2)	**b_1**	0.957	0.985	2.332	2.224
	γ_1	0.777	4.615	1.563	1.467
	γ_2	1.222	3.608	1.398	2.579
	γ_3	1.906	2.082	1.397	4.607
(III) LN(0, 0.3)	**b_1**	2.077	0.669	0.559	1.794
	γ_1	1.268	1.329	2.460	1.032
	γ_2	0.916	1.669	2.829	1.656
	γ_3	0.860	2.133	1.849	1.357
(IV) Beta(0.3, 0.3)	**b_1**	1.108	0.618	0.364	1.794
	γ_1	0.688	0.905	2.108	1.032
	γ_2	1.175	0.248	1.862	1.656
	γ_3	0.448	1.575	2.407	1.357

Appendix C. Case (I) - Estimates of All Parameters for BSEM and BQSEM

ε_i and δ_i ~ N(0, 0.3), n=100								
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	0.0013	0.0329	-0.0345	0.0409	-0.0225	0.0308	-0.0347	0.0410
λ_{31}	0.0001	0.0344	-0.0006	0.0319	0.0111	0.0367	-0.0010	0.0368
λ_{52}	-0.0053	0.0711	-0.0156	0.0641	-0.0028	0.0676	-0.0002	0.0626
λ_{62}	-0.0174	0.0662	-0.0214	0.0325	-0.0135	0.0320	-0.0126	0.0328
λ_{83}	-0.0138	0.0705	0.0089	0.0446	0.0045	0.0426	0.0037	0.0430
λ_{93}	-0.0163	0.0696	-0.0393	0.0648	-0.0417	0.0711	-0.0440	0.0739
$\lambda_{11,4}$	-0.0258	0.0709	0.0576	0.0636	0.0562	0.0639	0.0549	0.0643
$\lambda_{12,4}$	-0.0344	0.0772	0.0322	0.0690	0.0308	0.0672	0.0346	0.0725
b_1	0.0057	0.0843	-0.1468	0.1489	-0.1576	0.1583	-0.1438	0.1454
γ_1	0.0250	0.0947	-0.0535	0.1703	-0.0554	0.1673	-0.0197	0.1561
γ_2	-0.0349	0.1055	-0.0833	0.1813	-0.0311	0.1441	0.0575	0.0727
γ_3	-0.1086	0.1500	-0.0427	0.0830	-0.0548	0.0840	-0.0832	0.1134
ϕ_{11}	0.1643	0.2264	0.3189	0.3704	0.3034	0.3603	0.3017	0.3579
ϕ_{12}	-0.0037	0.0987	0.0030	0.0403	-0.0026	0.0367	-0.0025	0.0410
ϕ_{13}	-0.0045	0.1087	-0.0331	0.1073	-0.0330	0.1047	-0.0299	0.1029
ϕ_{22}	0.1782	0.2373	0.2860	0.3178	0.2907	0.3242	0.2902	0.3206
ϕ_{23}	-0.0003	0.1151	-0.0174	0.0539	-0.0165	0.0540	-0.0167	0.0524
ϕ_{33}	0.2046	0.2668	0.0467	0.0960	0.0526	0.1071	0.0495	0.1069

Appendix D. Case (II) - Estimates of All Parameters for BSEM and BQSEM

ε_i and $\delta_i \sim t(2)$, n=100								
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	-0.2135	1.8535	-0.0462	0.0513	-0.0069	0.0157	-0.0252	0.0378
λ_{31}	-0.3222	1.9329	-0.0170	0.1037	-0.0122	0.0392	-0.0232	0.0594
λ_{52}	-0.1516	0.7743	-0.2622	0.3883	-0.2266	0.3483	-0.2375	0.3710
λ_{62}	-0.1260	0.7050	-0.0193	0.3152	-0.0180	0.2799	-0.0259	0.2882
λ_{83}	-0.3059	0.7666	-0.1130	0.1494	-0.0879	0.1263	-0.0869	0.1328
λ_{93}	-0.2748	0.7885	-0.0449	0.1361	-0.0231	0.1146	-0.0490	0.0918
$\lambda_{11,4}$	-0.3708	0.6730	-0.2502	0.2704	-0.2290	0.2455	-0.2038	0.2236
$\lambda_{12,4}$	-0.2715	0.6700	-0.0611	0.2906	-0.0811	0.2797	-0.1298	0.2603
b_1	-0.2597	0.5006	0.0338	0.2724	0.0104	0.1693	-0.0646	0.1050
γ_1	-0.2700	0.3726	-0.0943	0.3797	-0.0900	0.3039	-0.0698	0.4898
γ_2	-0.5685	0.6515	0.5114	0.5685	0.3481	0.3954	0.5466	0.5590
γ_3	-0.8069	0.9259	-0.4705	0.5030	-0.5443	0.5523	-0.4431	0.4939
ϕ_{11}	2.9779	3.9593	0.2932	0.3174	0.3080	0.3458	0.2820	0.3107
ϕ_{12}	-0.0179	0.5883	-0.1571	0.2276	-0.1557	0.2195	-0.1687	0.2361
ϕ_{13}	-0.1056	0.4741	-0.1594	0.1708	-0.1470	0.1555	-0.1708	0.1779
ϕ_{22}	3.8059	4.8803	0.3863	0.3893	0.3801	0.3814	0.3734	0.3781
ϕ_{23}	-0.0405	0.4407	-0.0621	0.1126	-0.0634	0.1141	-0.0859	0.1225
ϕ_{33}	2.8303	3.6807	0.4521	0.5526	0.4669	0.5598	0.4490	0.5233

Appendix E. Case (III) - Estimates of All Parameters for BSEM and BQSEM

$\varepsilon_i \sim N(0, 0.3)$, $\delta_i \sim LN(0, 0.3)$, n=100								
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	0.0015	0.0334	-0.0341	0.0393	-0.0242	0.0312	-0.0342	0.0386
λ_{31}	-0.0060	0.0296	0.0000	0.0338	0.0119	0.0366	-0.0018	0.0375
λ_{52}	-0.0152	0.0687	-0.0091	0.0698	-0.0070	0.0562	0.0041	0.0554
λ_{62}	-0.0247	0.0744	-0.0165	0.0325	-0.0179	0.0284	-0.0133	0.0314
λ_{83}	-0.0180	0.0696	0.0067	0.0437	0.0071	0.0429	0.0059	0.0414
λ_{93}	-0.0209	0.0667	-0.0389	0.0671	-0.0418	0.0724	-0.0414	0.0747
$\lambda_{11,4}$	-0.0353	0.0702	0.0544	0.0625	0.0501	0.0574	0.0547	0.0642
$\lambda_{12,4}$	-0.0313	0.0682	0.0330	0.0716	0.0273	0.0708	0.0396	0.0765
b_1	0.0181	0.0746	-0.1317	0.1352	-0.1448	0.1524	-0.1200	0.1551
γ_1	0,0631	0,1556	-0.0351	0.1711	-0.0543	0.1648	-0.0225	0.2090
γ_2	-0.0442	0.0949	-0.1013	0.1891	-0.0562	0.1455	0.0705	0.0755
γ_3	-0.1362	0.1602	-0.0319	0.0836	-0.0338	0.0741	-0.0303	0.0831
ϕ_{11}	0.1740	0.2362	0.3116	0.3698	0.3109	0.3605	0.2970	0.3509
ϕ_{12}	-0.0075	0.0983	-0.0010	0.0322	0.0039	0.0429	-0.0050	0.0425
ϕ_{13}	-0.0039	0.1082	-0.0349	0.1045	-0.0275	0.1050	-0.0287	0.0999
ϕ_{22}	0.1852	0.2393	0.2870	0.3197	0.2916	0.3271	0.2817	0.3137
ϕ_{23}	0.0227	0.1236	-0.0186	0.0537	-0.0137	0.0553	-0.0204	0.0534
ϕ_{33}	0.2484	0.2918	0.0511	0.1024	0.0558	0.1065	0.0416	0.1031

Appendix F. Case (IV) - Estimates of All Parameters for BSEM and BQSEM

	$\varepsilon_i \sim N(0, 0.3)$, $\delta_i \sim Beta(0.3, 0.3)$, n=100							
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	0.0072	0.0347	-0.0516	0.0800	-0.0424	0.0759	-0.0544	0.0804
λ_{31}	0.0024	0.0334	-0.0349	0.0588	-0.0255	0.0549	-0.0377	0.0593
λ_{52}	-0.0102	0.0594	0.0619	0.0685	0.0610	0.0668	0.0657	0.0725
λ_{62}	-0.0146	0.0687	0.0103	0.0316	0.0091	0.0350	0.0090	0.0357
λ_{83}	-0.0321	0.0800	0.0460	0.0864	0.0478	0.0897	0.0522	0.0977
λ_{93}	-0.0179	0.0795	0.0464	0.0649	0.0490	0.0642	0.0492	0.0654
$\lambda_{11,4}$	-0.0159	0.0585	-0.0060	0.0490	-0.0086	0.0391	-0.0048	0.0328
$\lambda_{12,4}$	-0.0280	0.0633	0.0492	0.0687	0.0510	0.0687	0.0505	0.0763
b_1	-0.0010	0.0739	0.0201	0.1480	0.0370	0.1346	0.0747	0.1353
γ_1	0.0228	0.0912	0.0337	0.0415	0.0143	0.0413	-0.0047	0.0642
γ_2	-0.0638	0.1082	0.0289	0.0650	0.0343	0.0699	0.0556	0.0655
γ_3	-0.1138	0.1417	-0.0112	0.0911	0.0455	0.1572	0.0737	0.2138
ϕ_{11}	0.1602	0.2168	0.1352	0.1581	0.1368	0.1620	0.1317	0.1559
ϕ_{12}	0.0108	0.1157	0.0392	0.0727	0.0374	0.0722	0.0369	0.0701
ϕ_{13}	0.0066	0.1079	-0.1315	0.1611	-0.1288	0.1601	-0.1292	0.1578
ϕ_{22}	0.2006	0.2731	0.1473	0.1937	0.1439	0.1900	0.1410	0.1873
ϕ_{23}	0.0176	0.1179	-0.0543	0.0650	-0.0573	0.0661	-0.0630	0.0691
ϕ_{33}	0.2277	0.2742	-0.0361	0.1008	-0.0379	0.0974	-0.0413	0.0921

Appendix G. Case (I) - Estimates of All Parameters for BSEM and BQSEM

	ε_i and $\delta_i \sim N(0, 0.3)$, n=500							
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	-0.0087	0.0299	0.0018	0.0019	0.0057	0.0058	0.0011	0.0014
λ_{31}	-0.0052	0.0337	0.0008	0.0080	0.0056	0.0100	0.0022	0.0095
λ_{52}	-0.0136	0.0721	-0.0275	0.0312	-0.0293	0.0328	-0.0284	0.0326
λ_{62}	-0.0156	0.0737	0.0110	0.0305	0.0106	0.0294	0.0101	0.0270
λ_{83}	-0.0190	0.0768	-0.0385	0.0407	-0.0392	0.0409	-0.0399	0.0419
λ_{93}	-0.0216	0.0749	-0.0170	0.0349	-0.0174	0.0335	-0.0182	0.0322
$\lambda_{11,4}$	-0.0371	0.0810	-0.0284	0.0498	-0.0268	0.0498	-0.0230	0.0503
$\lambda_{12,4}$	-0.0325	0.0805	-0.0392	0.0433	-0.0375	0.0406	-0.0362	0.0390
b_1	0.0144	0.0872	0.0287	0.0444	0.0205	0.0286	0.0309	0.0351
γ_1	0.0241	0.0944	-0.0401	0.0509	-0.0168	0.0298	-0.0058	0.0296
γ_2	-0.0443	0.1117	-0.0295	0.0768	-0.0537	0.0924	-0.0585	0.1019
γ_3	-0.0898	0.1328	-0.0312	0.0579	-0.0420	0.0702	-0.0237	0.0685
ϕ_{11}	0.1906	0.2500	0.0230	0.0321	0.0279	0.0360	0.0291	0.0359
ϕ_{12}	0.0067	0.1182	-0.0091	0.0417	-0.0069	0.0397	-0.0060	0.0381
ϕ_{13}	0.0007	0.1099	-0.0260	0.0542	-0.0254	0.0531	-0.0258	0.0512
ϕ_{22}	0.2018	0.2513	0.1163	0.1265	0.1208	0.1299	0.1219	0.1290
ϕ_{23}	0.0093	0.1226	-0.0174	0.0267	-0.0151	0.0261	-0.0154	0.0260
ϕ_{33}	0.1890	0.2522	0.0938	0.1319	0.0924	0.1334	0.0841	0.1301

Appendix H. Case (II) - Estimates of All Parameters for BSEM and BQSEM

	ε_i and $\delta_i \sim t(2)$, n=500							
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	-0.2579	1.9025	-0.0177	0.0361	-0.0005	0.0314	-0.0122	0.0250
λ_{31}	-0.1605	1.9048	-0.0112	0.0202	0.0039	0.0193	-0.0012	0.0174
λ_{52}	-0.1789	0.9056	-0.1672	0.1729	-0.1639	0.1682	-0.1554	0.1580
λ_{62}	-0.2739	0.7270	-0.1293	0.1370	-0.1235	0.1312	-0.1156	0.1258
λ_{83}	-0.2304	0.7122	-0.0043	0.0721	0.0146	0.0917	0.0048	0.0930
λ_{93}	-0.3160	0.6590	-0.0236	0.0751	-0.0003	0.0698	-0.0029	0.0594
$\lambda_{11,4}$	-0.3473	0.7337	-0.1055	0.1366	-0.0859	0.1157	-0.0674	0.0995
$\lambda_{12,4}$	-0.1300	0.8903	-0.0536	0.0807	-0.0461	0.0654	-0.0212	0.0667
b_1	-0.2120	0.4342	0.0510	0.1096	0.0017	0.0817	-0.0234	0.0966
γ_1	-0.2798	0.3891	0.0088	0.2947	0.0157	0.2081	0.0651	0.3174
γ_2	-0.5252	0.5979	0.2853	0.3202	0.0914	0.1170	0.0696	0.1619
γ_3	-0.8646	0.9331	0.2917	0.3152	-0.0459	0.0817	0.5266	0.5298
ϕ_{11}	2.5785	3.0742	0.0943	0.1247	0.0878	0.1017	0.0557	0.0742
ϕ_{12}	-0.0705	0.3809	-0.0560	0.0776	-0.0477	0.0811	-0.0748	0.1030
ϕ_{13}	-0.1206	0.4450	-0.0753	0.1229	-0.0600	0.1179	-0.1131	0.1501
ϕ_{22}	3.3073	4.0524	-0.0459	0.0840	-0.0482	0.1014	-0.0587	0.1045
ϕ_{23}	-0.1025	0.3860	-0.0425	0.0947	-0.0191	0.0753	-0.0589	0.1023
ϕ_{33}	2.5916	3.2585	-0.0217	0.1249	0.0185	0.1137	-0.0761	0.1415

Appendix I. Case (III) - Estimates of All Parameters for BSEM and BQSEM

	$\varepsilon_i \sim N(0, 0.3)$, $\delta_i \sim LN(0, 0.3)$, n=500							
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	0.0001	0.0309	0.0034	0.0038	0.0070	0.0071	0.0016	0.0023
λ_{31}	-0.0024	0.0298	-0.0003	0.0081	0.0049	0.0093	0.0021	0.0087
λ_{52}	-0.0141	0.0745	-0.0292	0.0319	-0.0286	0.0314	-0.0279	0.0324
λ_{62}	-0.0144	0.0697	0.0114	0.0277	0.0118	0.0273	0.0110	0.0292
λ_{83}	-0.0124	0.0725	-0.0380	0.0400	-0.0381	0.0398	-0.0389	0.0404
λ_{93}	-0.0237	0.0758	-0.0151	0.0356	-0.0166	0.0344	-0.0154	0.0326
$\lambda_{11,4}$	-0.0310	0.0723	-0.0264	0.0499	-0.0256	0.0500	-0.0228	0.0520
$\lambda_{12,4}$	-0.0356	0.0738	-0.0388	0.0439	-0.0351	0.0391	-0.0299	0.0320
b_1	-0.0240	0.0944	0.0218	0.0379	0.0194	0.0304	0.0383	0.0504
γ_1	0.0319	0.1065	-0.0295	0.0439	-0.0147	0.0253	-0.0261	0.0520
γ_2	-0.0461	0.1270	-0.0428	0.0812	-0.0574	0.0993	-0.0420	0.1162
γ_3	-0.0829	0.1375	-0.0414	0.0724	-0.0483	0.0751	0.0416	0.0956
ϕ_{11}	0.1679	0.2223	0.0259	0.0331	0.0266	0.0329	0.0258	0.0347
ϕ_{12}	0.0083	0.1157	-0.0091	0.0425	-0.0080	0.0402	-0.0095	0.0409
ϕ_{13}	0.0000	0.1227	-0.0261	0.0533	-0.0257	0.0540	-0.0301	0.0534
ϕ_{22}	0.2088	0.2677	0.1161	0.1284	0.1177	0.1278	0.1131	0.1220
ϕ_{23}	0.0144	0.1309	-0.0190	0.0271	-0.0164	0.0276	-0.0236	0.0326
ϕ_{33}	0.2222	0.2896	0.0939	0.1347	0.0902	0.1338	0.0668	0.1206

Appendix J. Case (IV) - Estimates of All Parameters for BSEM and BQSEM

	$\varepsilon_i \sim N(0, 0.3)$, $\delta_i \sim Beta(0.3, 0.3)$, n=500							
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	0.0052	0.0311	-0.0061	0.0193	-0.0005	0.0196	-0.0026	0.0183
λ_{31}	-0.0019	0.0345	-0.0107	0.0135	-0.0070	0.0113	-0.0109	0.0125
λ_{52}	-0.0180	0.0687	-0.0248	0.0495	-0.0259	0.0512	-0.0271	0.0500
λ_{62}	-0.0114	0.0786	-0.0247	0.0405	-0.0239	0.0405	-0.0245	0.0410
λ_{83}	-0.0341	0.0770	0.0026	0.0290	0.0015	0.0291	0.0005	0.0228
λ_{93}	-0.0335	0.0757	0.0396	0.0449	0.0383	0.0439	0.0376	0.0450
$\lambda_{11,4}$	-0.0300	0.0826	0.0414	0.0473	0.0425	0.0480	0.0444	0.0501
$\lambda_{12,4}$	-0.0252	0.0743	0.0051	0.0293	0.0064	0.0288	0.0096	0.0314
b_1	-0.0126	0.0865	0.0283	0.0333	0.0167	0.0267	0.0097	0.0314
γ_1	0.0249	0.0792	0.0082	0.0318	0.0137	0.0270	0.0139	0.0288
γ_2	-0.0436	0.1019	-0.0148	0.0208	-0.0265	0.0306	-0.0296	0.0338
γ_3	-0.1406	0.1645	0.0177	0.0369	0.0128	0.0302	0.0104	0.0222
ϕ_{11}	0.2106	0.2832	0.0850	0.0936	0.0866	0.0956	0.0879	0.0950
ϕ_{12}	0.0126	0.1341	0.0051	0.0231	0.0065	0.0249	0.0073	0.0230
ϕ_{13}	0.0158	0.1368	-0.0305	0.0308	-0.0305	0.0309	-0.0289	0.0295
ϕ_{22}	0.2101	0.2620	0.0519	0.0708	0.0548	0.0732	0.0554	0.0779
ϕ_{23}	0.0312	0.1351	0.0024	0.1014	0.0033	0.1007	0.0031	0.1013
ϕ_{33}	0.2247	0.2817	0.0726	0.1250	0.0730	0.1244	0.0712	0.1247

Appendix K. Case (I) - Estimates of All Parameters for BSEM and BQSEM

	ε_i and $\delta_i \sim N(0, 0.3)$, n=3000							
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	-0.0077	0.0341	0.0015	0.0056	0.0027	0.0067	0.0025	0.0062
λ_{31}	-0.0010	0.0303	-0.0028	0.0060	-0.0019	0.0060	-0.0021	0.0059
λ_{52}	-0.0214	0.0703	0.0188	0.0211	0.0187	0.0218	0.0186	0.0215
λ_{62}	-0.0247	0.0711	0.0170	0.0196	0.0162	0.0196	0.0156	0.0182
λ_{83}	-0.0304	0.0814	0.0318	0.0340	0.0342	0.0364	0.0336	0.0357
λ_{93}	-0.0283	0.0764	0.0174	0.0185	0.0197	0.0209	0.0184	0.0195
$\lambda_{11,4}$	-0.0268	0.0821	-0.0041	0.0157	-0.0045	0.0155	-0.0022	0.0141
$\lambda_{12,4}$	-0.0358	0.0728	0.0072	0.0253	0.0072	0.0241	0.0102	0.0230
b_1	0.0073	0.0853	0.0109	0.0135	0.0095	0.0095	0.0092	0.0096
γ_1	0.0039	0.0946	-0.0049	0.0203	-0.0072	0.0236	-0.0065	0.0279
γ_2	-0.0487	0.1020	0.0172	0.0316	0.0148	0.0242	0.0216	0.0252
γ_3	-0.0938	0.1410	0.0177	0.0225	-0.0020	0.0115	0.0261	0.0272
ϕ_{11}	0.1539	0.2333	-0.0171	0.0217	-0.0143	0.0211	-0.0153	0.0215
ϕ_{12}	-0.0249	0.1274	0.0038	0.0071	0.0047	0.0078	0.0032	0.0066
ϕ_{13}	-0.0178	0.1171	-0.0069	0.0236	-0.0041	0.0234	-0.0080	0.0247
ϕ_{22}	0.1823	0.2520	0.0046	0.0207	0.0031	0.0211	0.0019	0.0186
ϕ_{23}	-0.0035	0.1090	0.0078	0.0084	0.0103	0.0105	0.0058	0.0070
ϕ_{33}	0.1847	0.2432	0.0382	0.0522	0.0445	0.0577	0.0336	0.0493

Appendix L. Case (II) - Estimates of All Parameters for BSEM and BQSEM

	ε_i and δ_i~ t(2), n=3000							
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	-0.3331	1.9302	0.0041	0.0090	-0.0024	0.0076	0.0055	0.0128
λ_{31}	-0.3271	2.0529	-0.0160	0.0179	-0.0185	0.0200	-0.0143	0.0221
λ_{52}	-0.3122	1.3052	0.0174	0.0687	0.0077	0.0574	0.0102	0.0601
λ_{62}	-0.1695	0.6932	0.0126	0.0188	0.0152	0.0195	0.0116	0.0133
λ_{83}	-0.3496	0.6593	-0.0198	0.0238	-0.0211	0.0303	-0.0228	0.0362
λ_{93}	0.1287	4.4466	-0.0409	0.0431	-0.0342	0.0364	-0.0351	0.0388
$\lambda_{11,4}$	-0.3989	0.6876	0.0218	0.0277	0.0272	0.0400	0.0093	0.0271
$\lambda_{12,4}$	-0.1881	0.6640	0.0058	0.0145	0.0162	0.0281	0.0021	0.0063
b_1	-0.2335	0.4782	0.0061	0.0319	0.0067	0.0242	-0.0240	0.0396
γ_1	-0.2530	0.3831	-0.0910	0.2094	-0.0157	0.0837	-0.0994	0.2583
γ_2	-0.5390	0.6089	0.1098	0.1935	-0.0113	0.1015	0.0873	0.2538
γ_3	-0.8342	0.9334	0.5156	0.5243	0.0527	0.0932	0.6223	0.6648
ϕ_{11}	3.2311	4.1331	-0.2336	0.2369	-0.2168	0.2180	-0.2337	0.2354
ϕ_{12}	0.0584	0.5224	0.0017	0.0034	0.0208	0.0247	-0.0016	0.0131
ϕ_{13}	-0.0309	0.4875	-0.0407	0.0463	-0.0096	0.0277	-0.0468	0.0540
ϕ_{22}	3.6967	4.6221	-0.1318	0.1353	-0.1014	0.1064	-0.1414	0.1468
ϕ_{23}	-0.0520	0.4804	-0.0175	0.0246	0.0279	0.0309	-0.0260	0.0372
ϕ_{33}	2.7960	3.4988	-0.2179	0.2234	-0.1565	0.1699	-0.2293	0.2392

Appendix M. Case (III) - Estimates of All Parameters for BSEM and BQSEM

	ε_i ~ N(0, 0.3), δ_i ~ LN(0, 0.3), n=3000							
	BSEM		BQSEM					
	-		τ =0.1		τ =0.5		τ =0.9	
	Bias	RMS	Bias	RMS	Bias	RMS	Bias	RMS
λ_{21}	0.0046	0.0377	0.0018	0.0058	0.0030	0.0067	0.0029	0.0063
λ_{31}	-0.0007	0.0328	-0.0024	0.0054	-0.0015	0.0055	-0.0014	0.0045
λ_{52}	-0.0270	0.0621	0.0177	0.0201	0.0181	0.0213	0.0175	0.0207
λ_{62}	-0.0073	0.0673	0.0158	0.0186	0.0161	0.0195	0.0154	0.0181
λ_{83}	-0.0164	0.0716	0.0333	0.0351	0.0335	0.0357	0.0352	0.0370
λ_{93}	-0.0197	0.0762	0.0186	0.0191	0.0196	0.0205	0.0204	0.0217
$\lambda_{11,4}$	-0.0350	0.0720	-0.0037	0.0154	-0.0041	0.0145	-0.0012	0.0135
$\lambda_{12,4}$	-0.0423	0.0736	0.0063	0.0243	0.0078	0.0233	0.0106	0.0228
b_1	-0.0139	0.0921	0.0093	0.0133	0.0083	0.0102	0.0072	0.0096
γ_1	0.0142	0.1219	-0.0041	0.0184	-0.0043	0.0274	-0.0248	0.0383
γ_2	-0.0292	0.1080	0.0067	0.0317	0.0072	0.0204	0.0316	0.0348
γ_3	-0.1043	0.1543	-0.0029	0.0108	-0.0112	0.0133	0.1040	0.1068
ϕ_{11}	0.1484	0.2000	-0.0140	0.0197	-0.0140	0.0212	-0.0159	0.0223
ϕ_{12}	-0.0212	0.1219	0.0055	0.0081	0.0056	0.0085	0.0009	0.0060
ϕ_{13}	-0.0180	0.1235	-0.0044	0.0236	-0.0041	0.0239	-0.0114	0.0267
ϕ_{22}	0.1732	0.2376	0.0049	0.0193	0.0048	0.0204	-0.0067	0.0199
ϕ_{23}	0.0129	0.1250	0.0114	0.0117	0.0117	0.0120	-0.0016	0.0041
ϕ_{33}	0.2067	0.2660	0.0435	0.0555	0.0449	0.0587	0.0179	0.0465

Appendix N. Case (IV) - Estimates of All Parameters for BSEM and BQSEM

$\varepsilon_i \sim N(0, 0.3)$, $\delta_i \sim Beta(0.3, 0.3)$, n=3000								
	BSEM		**BQSEM**					
	-		**τ =0.1**		**τ =0.5**		**τ =0.9**	
	Bias	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**	**Bias**	**RMS**
λ_{21}	-0.0050	0.0314	-0.0062	0.0080	-0.0057	0.0074	-0.0066	0.0080
λ_{31}	-0.0062	0.0315	-0.0001	0.0003	0.0003	0.0006	0.0000	0.0007
λ_{52}	-0.0087	0.0728	0.0129	0.0173	0.0134	0.0171	0.0135	0.0173
λ_{62}	-0.0144	0.0724	0.0184	0.0207	0.0189	0.0207	0.0187	0.0203
λ_{83}	-0.0152	0.0751	0.0120	0.0130	0.0124	0.0136	0.0122	0.0136
λ_{93}	-0.0328	0.0702	0.0084	0.0124	0.0073	0.0119	0.0075	0.0123
$\lambda_{11,4}$	-0.0464	0.0833	0.0010	0.0033	-0.0002	0.0046	-0.0006	0.0046
$\lambda_{12,4}$	-0.0367	0.0716	0.0076	0.0139	0.0058	0.0136	0.0048	0.0113
b_1	0.0044	0.0712	-0.0104	0.0218	-0.0070	0.0165	-0.0087	0.0135
γ_1	0.0160	0.0800	0.0126	0.0254	0.0126	0.0234	0.0113	0.0238
γ_2	-0.0396	0.0974	-0.0034	0.0293	-0.0035	0.0306	0.0013	0.0291
γ_3	-0.1264	0.1497	-0.0111	0.0128	-0.0210	0.0219	-0.0137	0.0143
ϕ_{11}	0.1736	0.2383	-0.0078	0.0421	-0.0085	0.0416	-0.0079	0.0402
ϕ_{12}	-0.0071	0.1091	0.0050	0.0153	0.0053	0.0162	0.0054	0.0158
ϕ_{13}	0.0052	0.1137	0.0032	0.0271	0.0044	0.0278	0.0045	0.0266
ϕ_{22}	0.1961	0.2590	0.0210	0.0264	0.0219	0.0285	0.0215	0.0276
ϕ_{23}	0.0310	0.1200	0.0155	0.0274	0.0175	0.0278	0.0174	0.0277
ϕ_{33}	0.2400	0.2916	0.0410	0.0461	0.0461	0.0511	0.0458	0.0498