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ON IRREGULAR COLORINGS OF DOUBLE WHEEL GRAPH FAMILIES

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Abstract. An assignment of colors to the vertices of a graph, so that no two adjacent vertices get the same color is called a proper coloring. An irregular coloring of a graph is a proper vertex coloring that distinguishes vertices in the graph either by their own colors or by the colors of their neighbours. In this paper, we investigate the irregular chromatic number for the middle graph, total graph, central graph and line graph of double wheel graph.

1. Introduction

A (proper) *coloring* [5] of a graph G is a function $c : V(G) \to N$ having the same property that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G, where N is the set of positive integers. A k - coloring of G uses k colors. The chromatic number $\chi(G)$ of G is the minimum positive integer k for which there is a k - coloring of G.

For a positive integer k and a proper coloring $c : V(G) \to 1, 2, \ldots, k$ of the vertices of a graph G, the ***color code*** [5] of a vertex v of G is the ordered $(k+1)$-tuple $code(v) = (a_0, a_1, \ldots a_k)$ where a_0 is the color assigned to v (that is $c(v) = a_0$) and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored, the color i.

A proper coloring c is an ***irregular coloring*** [1] if two distinct vertices have distinct color code. *i.e.*, for every pair of vertices u and w, $code(u) \neq code(w)$ whenever
$c(u) = c(w)$. Thus, an irregular coloring distinguishes each vertex from each of other vertex by its color or by its color code.

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Since every irregular graph of a graph G is a proper coloring of G, it follows that $\chi(G) \leq \chi_{ir}(G)$. The neighbourhood of a vertex u in a graph G is $N(u) = \{v \in V(G) : uv \in E(G)\}$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The **middle graph** [2,8] of G, denoted by $M(G)$ is defined as follows: The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:

(i) x, y are in $E(G)$ and x, y are adjacent in G.
(ii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The **total graph** [2,8] of G, denoted by $T(G)$ is defined as follows: The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:

(i) x, y are in $V(G)$ and x is adjacent to y in G.
(ii) x, y are in $E(G)$, x, y are adjacent in G.
(iii) x is in $V(G)$ and y is in $E(G)$, and x, y are incident in G.

The **central graph** [7] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The **line graph** [3] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

A **double wheel graph** DW_n [6] of size N can be composed of $2C_n + K_1$ i.e., it consists of two cycle of size N, where the vertices of the two cycles are all connected to a common hub. Let $V(DW_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and

$$E(DW_n) = \{p_i p_{i+1}, p_n p_1, q_i q_{i+1}, q_n q_1 : 1 \leq i \leq n-1\} \cup \{vp_i, vq_i : 1 \leq i \leq n\}$$

In [5], Mary Radcliffe and Ping Zhang established sharp upper and lower bounds for the irregular chromatic number of a disconnected graph in terms of the irregular chromatic numbers of its components. Irregular chromatic number of some classes of disconnected graphs are determined.

It is shown that if G is a nontrivial graph of order n, then $\sqrt[2]{n} \leq \chi_{ir}(G) + \chi_{ir}(\bar{G}) \leq 2n, n \leq \chi_{ir}(G)\chi_{ir}(\bar{G}) \leq n^2$, and each bound in these inequalities is sharp. In [4], Radcliffe and Zhang found a bound for the irregular chromatic number of a graph on n vertices and they discussed as follows,

Let c be a (proper) coloring of the vertices of a nontrivial graph G and let u and v be two vertices of G then

(1) If $c(u) \neq c(v)$, then $code(u) \neq code(v)$.
(2) If $d(u) \neq d(v)$, then $code(u) \neq code(v)$.
(3) If c is irregular and $N(u) = N(v)$, then $c(u) \neq c(v)$.

2. Irregular coloring of middle, total, central and line graph of double wheel graph

Theorem 2.1. *For any double wheel graph DW_n, the irregular chromatic number of its middle wheel graph is $\chi_{ir}(M(DW_n)) = 2n+1 \ \forall \ n \geq 3$.*

Proof. Let the vertices of $M(DW_n)$ is $\{v, p_i, q_i : 1 \leq i \leq n\} \cup \{r_i, s_i : 1 \leq i \leq n\} \cup \{t_i, x_i : 1 \leq i \leq n\}$. By the definition of middle graph r_i, s_i, t_i and x_i subdivides the edges of vp_i, vq_i, p_ip_{i+1} and q_iq_{i+1} for $(1 \leq i \leq n)$ and the degrees of vertices are $d(v) = 2n$, $d(s_i) = d(r_i) = 2n+3$, $d(q_i) = d(p_i) = 3$ and $d(x_i) = d(t_i) = 6$ for $(1 \leq i \leq n)$.

Assign the following irregular $(2n+1)$-coloring to $V(M(DW_n))$

- For $1 \leq i \leq n$, assign the color c_i to r_i.
- For $1 \leq i \leq n$, assign the color c_{n+i} to s_i.
- For $1 \leq i \leq n-1$, assign the color c_{i+1} to p_i and q_i.
- For n, assign the color c_1 to p_n and q_n.
- For $1 \leq i \leq n-1$, assign the color $c_{(n+1)+i}$ to t_i.
- For n, assign the color c_{n+1} to t_n.
- For $1 \leq i \leq n-1$, assign the color c_{n+i} to x_{i+1}.
- Assign the color c_{2n+1} to v and x_1.

To prove this $(2n+1)$-coloring is an irregular coloring of $M(DW_n)$. Since $d(s_i) \neq d(x_i)$ it shows that $code(s_i) \neq code(x_i)$ for $1 \leq i \leq n, d(p_i) = d(q_i)$ and each $p_i^{'}s$ are adjacent to e_i. But $q_i^{'}s$ are not adjacent to e_i. Hence, $code(p_i) \neq code(q_i)$ for $1 \leq i \leq n$. Thus c is an irregular $(2n+1)$-coloring of $M(DW_n)$. Therefore, $\chi_{ir}(M(DW_n)) \leq 2n+1$.

Also, the vertices $\{v, r_i(1 \leq i \leq n), s_i(1 \leq i \leq n)\}$ induce a complete graph of order $(2n+1)$ in $M(DW_n)$ hence, $\chi_{ir}(M(DW_n)) \geq \chi(M(DW_n)) = 2n+1$. Therefore, $\chi_{ir}(M(DW_n)) = 2n+1$. □

Theorem 2.2. *For any double wheel graph DW_n, the irregular chromatic number of its total wheel graph is $\chi_{ir}(T(DW_n)) = 2n+1 \ \forall \ n \geq 3$.*

Proof. Let the vertices of $T(DW_n)$ is $\{v, p_i, q_i : 1 \leq i \leq n\} \cup \{r_i, s_i : 1 \leq i \leq n\} \cup \{t_i, x_i : 1 \leq i \leq n\}$. By the definition of total graph r_i, s_i, t_i and x_i subdivides the edges of vp_i, vq_i, p_ip_{i+1} and q_iq_{i+1} for $(1 \leq i \leq n)$ and the degrees of vertices are $d(v) = 4n$, $d(s_i) = d(r_i) = 2n+3$ and $d(q_i) = d(p_i) = d(x_i) = d(t_i) = 6$ for $(1 \leq i \leq n)$.

Assign the following irregular $(2n+1)$-coloring to $V(T(DW_n))$

- For $1 \leq i \leq n$, assign the color c_i to r_i and q_i.
- For $1 \leq i \leq n$, assign the color c_{n+i} to s_i and p_i.
- Assign the color c_{2n+1} to v.
- For $2 \leq i \leq n$, assign the color c_{i-1} to t_i.
- Assign the color c_n to t_1.
- For $1 \leq i \leq n-1$, assign the color c_{n+i} to x_{i+1}.

- Assign the color c_{2n} to x_1.

To prove this $(2n+1)$-coloring is an irregular coloring of $T(DW_n)$. Since $d(r_i) \neq d(q_i)$ it shows that $code(r_i) \neq code(q_i)$ for $1 \leq i \leq n$, in the same way $d(p_i) \neq d(s_i)$ shows that $code(p_i) \neq code(s_i)$ for $1 \leq i \leq n$. Since $d(s_i) = d(x_i)$ and each $s_i^{'}s$ are adjacent to r_i but $x_i^{'}s$ are not adjacent to r_i, hence $code(s_i) \neq code(x_i)$ for $1 \leq i \leq n$. Therefore, $\chi_{ir}(T(DW_n)) \leq 2n+1$.

Also, the vertices $\{v, r_i(1 \leq i \leq n), s_i(1 \leq i \leq n)\}$ induce a clique of order $(2n+1)$ in $T(DW_n)$ hence, $\chi_{ir}(T(DW_n)) \geq \chi(T(DW_n)) = 2n+1$. Therefore, $\chi_{ir}(T(DW_n)) = 2n+1$. □

Theorem 2.3. *For any double wheel graph DW_n, the irregular chromatic number of its central wheel graph is $\chi_{ir}(C(DW_n)) = 2\left\lceil \frac{n}{2} \right\rceil\ \forall\ n \geq 5$.*

Proof. Let the vertices of $C(DW_n)$ is $\{v, p_i, q_i : 1 \leq i \leq n\} \cup \{t_i, x_i : 1 \leq i \leq n\}$ By the definition of central graph r_i, s_i, t_i and x_i subdivides the edges of vp_i, vq_i, p_ip_{i+1} and q_iq_{i+1} for $(1 \leq i \leq n)$ and the degrees of vertices are $d(v) = d(q_i) = d(p_i) = 2n$ and $d(s_i) = d(r_i) = d(x_i) = d(t_i) = 2$ for $(1 \leq i \leq n)$.

Let the coloring function $C : V\left(C\left(DW_n\right)\right) \longrightarrow \left\{c_1, c_2, c_3 \cdots c_{2\lceil \frac{n}{2} \rceil}\right\}$ as follows:

$$\begin{aligned} C(v) &= c_1 \\ C(x_i) &= c_1\ for\ (1 \leq i \leq n) \\ C(t_i) &= \end{aligned}$$

For n is odd,

$$\begin{aligned} C(p_{2i-1}) &= c_i\ for\ (1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil) \\ C(q_{2i-1}) &= c_{\lceil \frac{n}{2} \rceil + i}\ for\ (1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil) \\ C(p_{2i}) &= c_i\ for\ (1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor) \\ C(q_{2i}) &= c_{\lceil \frac{n}{2} \rceil + i}\ for\ (1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor) \\ C(r_{2i-1}) &= c_{i+1}\ for\ (1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil) \\ C(r_{2i}) &= c_{i+2}\ for\ (1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor) \\ C(s_{2i-1}) &= c_{\lceil \frac{n}{2} \rceil + 1 + i}\ for\ (1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1) \\ C(s_{2i}) &= c_{\lceil \frac{n}{2} \rceil + 2 + i}\ for\ (1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1) \\ C(s_i) &= c_2\ for\ i = n-1, n \end{aligned}$$

For n is even,

$$C(p_{2i-1}) = c_i \ for \ (1 \leq i \leq \frac{n}{2})$$
$$C(q_{2i-1}) = c_{\frac{n}{2}+i} \ for \ (1 \leq i \leq \frac{n}{2})$$
$$C(p_{2i}) = c_i \ for \ (1 \leq i \leq \frac{n}{2})$$
$$C(q_{2i}) = c_{\frac{n}{2}+i} \ for \ (1 \leq i \leq \frac{n}{2})$$
$$C(r_{2i-1}) = c_{i+1} \ for \ (1 \leq i \leq \frac{n}{2})$$
$$C(r_{2i}) = c_{i+2} \ for \ (1 \leq i \leq \frac{n}{2})$$
$$C(s_{2i-1}) = c_{\frac{n}{2}+1+i} \ for \ (1 \leq i \leq \frac{n}{2}-1)$$
$$C(s_{2i}) = c_{\frac{n}{2}+2+i} \ for \ (1 \leq i \leq \frac{n}{2}-2)$$
$$C(s_i) = c_2 \ for \ i = n-1, n-2$$
$$C(s_n) = c_3$$

The above coloring function $C$is an irregular coloring, it is clear from the observation. Thus $\chi_{ir}\left(C\left(DW_n\right)\right) \leq 2\left\lceil \frac{n}{2} \right\rceil$.

To prove $\chi_{ir}\left(C\left(DW_n\right)\right) \geq 2\left\lceil \frac{n}{2} \right\rceil$, The vertices $\left\{p_{2i-1}, q_{2i-1} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil\right\}$of central graph $DW_n$are induce a complete graph of order $2\left\lceil \frac{n}{2} \right\rceil$, and so $\chi_{ir}\left(C\left(DW_n\right)\right) \geq 2\left\lceil \frac{n}{2} \right\rceil$. Hence $\chi_{ir}\left(C\left(DW_n\right)\right) = 2\left\lceil \frac{n}{2} \right\rceil$. □

Theorem 2.4. *For any double wheel graph DW_n, the irregular chromatic number of its line wheel graph is $\chi_{ir}(L(DW_n)) = 2n \ \forall \ n \geq 3$.*

Proof. Let the vertices of $L(DW_n)$ is $\{r_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\} \cup \{t_i : 1 \leq i \leq n\} \cup \{x_i : 1 \leq i \leq n\}$, where t_i is the vertex corresponding to the edge $p_i p_{i+1}$ of DW_n, $(1 \leq i \leq n-1)$, $t_n = p_n p_1$, r_i is the vertex corresponding to the edge vp_i of $DW_n(1 \leq i \leq n)$, x_i is the vertex corresponding to the edge $q_i q_{i+1}$ of DW_n $(1 \leq i \leq n-1)$, $x_n = q_n q_1, s_i$ is the vertex corresponding to the edge vq_i of DW_n $(1 \leq i \leq n)$ and the degrees of vertices are $d(r_i) = d(s_i) = 2n+1$ and $d(t_i) = d(x_i) = 4$ for $(1 \leq i \leq n)$.

Assign the following irregular $2n$-coloring to $V(L(DW_n))$

- For $1 \leq i \leq n$, assign the color c_i to r_i and x_i.
- For $1 \leq i \leq n$, assign the color c_{n+i} to s_i and t_i.

To prove this $2n$-coloring is an irregular coloring of $T(DW_n)$. Since $d(r_i) \neq d(x_i)$ it follows that $code(r_i) \neq code(x_i)$ for $1 \leq i \leq n$ in the same sense $d(s_i) \neq d(t_i)$ it follows that $code(s_i) \neq code(t_i)$ for $1 \leq i \leq n$. Therefore, $\chi_{ir}(L(DW_n)) \leq 2n$. By the definition of line graph the vertices $r_i : (1 \leq i \leq n)$, $s_i(1 \leq i \leq n)$ induce a clique of order $2n$ in $L(DW_n)$.Hence $\chi_{ir}(L(DW_n)) \geq \chi(L(DW_n)) = 2n$. Therefore, $\chi_{ir}(L(DW_n)) = 2n$. □

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