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Some Integral Inequalities for s-Convex Functions

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Abstract

In the paper, by virtue of an integral identity and the Hölder inequality for integrals, the authors establish some new inequalities of the Hermite-Hadamard type for s-convex functions, derive some new inequalities of common convex functions, and apply these new results to construct some inequalities for special means.

1. INTRODUCTION

The following definitions are well known in the literature.

Definition 1.1

A function $f : I \subseteq R \to R$ is said to be convex if $f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda) f(y)$ holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 [1]

Let $s \in (0,1]$ be a real number. A function $f : R_0 \to R$ is said to be s-convex (in the second sense) if $f(\lambda x+(1-\lambda)y) \le \lambda^s f(x)+(1-\lambda)^s f(y)$ holds for all $x, y \in I$ and $\lambda \in [0,1]$.

In recent years, the following Hermite--Hadamard inequalities for s-convex functions have been proved.

Theorem 1.1 [2]

Let $f : I \subseteq R_0 \to R$ be differentiable on I° and $a,b \in I$ with $a<b$. If $|f'(x)|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then

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$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^q+|f'(b)|^q\right]^{1/q}.$$

Theorem 1.2 [3]

Let $f:I\subseteq R_0\to R$ be differentiable on I°, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'(x)|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{b-a}{2}\int_a^b f(x)dx\right|\le\frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p}$$

$$\times\left\{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[|f'(b)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\}$$

for $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.3 [4]

Let $f:I\subseteq R_0\to R$ be differentiable on I°, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'(x)|$ is s-convex on $[a,b]$, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{(s-4)\,6^{s+1}+2\times5^{s+2}-2\times3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\left[|f'(a)|+|f'(b)|\right]$$

for some $s\in(0,1]$.

There have been more Hermite-Hadamard type inequalities in, for example, [5, 6, 7, 8, 9, 10] and closely related references therein.

In this paper, by virtue of an integral identity and the Hölder inequality for integrals, we will establish some new integral inequalities of the Hermite-Hadamard type for s-convex functions, derive some new inequalities for common convex functions,and apply these new inequalities to construct some inequalities for special means.

2. A LEMMA

Before stating our main results, we need a lemma.

Lemma 2.1

Let $f:I\subseteq R\to R$ be a differentiable function on I°, and $a,b\in I$ with $a<b$. If $f'\in L_1([a,b])$, then

$$\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx$$

$$=\frac{b-a}{4}\int_0^1\left[\left(\frac{4}{5}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)+\left(\frac{1}{5}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)\right]dt.$$

Proof. By integration by parts, we have

$$\int_0^1\left(\frac{4}{5}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)dt$$

$$=-\frac{2}{b-a}\left[\left(\frac{4}{5}-t\right)f\left(ta+(1-t)\frac{a+b}{2}\right)\Bigg|_0^1+\int_0^1 f\left(ta+(1-t)\frac{a+b}{2}\right)dt\right]$$

$$=-\frac{2}{b-a}\left[-\frac{1}{5}f(a)-\frac{4}{5}f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_0^1 f\left(ta+(1-t)\frac{a+b}{2}\right)dt$$

$$=\frac{2}{b-a}\left[\frac{1}{5}f(a)+\frac{4}{5}f\left(\frac{a+b}{2}\right)\right]-\frac{4}{(b-a)^2}\int_a^{(a+b)/2} f(x)dx$$

and

$$\int_0^1\left(\frac{1}{5}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)dt$$

$$=-\frac{2}{b-a}\left[\left(\frac{1}{5}-t\right)f\left(t\frac{a+b}{2}+(1-t)b\right)\Bigg|_0^1+\int_0^1 f\left(t\frac{a+b}{2}+(1-t)b\right)dt\right]$$

$$=-\frac{2}{b-a}\left[-\frac{4}{5}f\left(\frac{a+b}{2}\right)-\frac{1}{5}f(b)\right]-\frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)b\right)dt$$

$$=\frac{2}{b-a}\left[\frac{4}{5}f\left(\frac{a+b}{2}\right)+\frac{1}{5}f(b)\right]-\frac{4}{(b-a)^2}\int_{(a+b)/2}^b f(x)dx.$$

Lemma 2.1 is thus proved.

3. INEQUALITIES OF THE HERMITE-HADAMARD TYPE FOR s-CONVEX FUNCTIONS

Now we are in a position to establish some new inequalities of the Hermite-Hadmard type for s-convex functions.

Theorem 3.4

Let $f: I \subseteq R_0 \to R$ be differentiable on I°, $a,b \in I$ with $a<b$, and $f' \in L_1([a,b])$. If $|f'(x)|^q$ is an s-convex functions on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then

$$\left|\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{17}{50}\right)^{1-1/q}\left[\frac{1}{5^{s+2}(s+1)(s+2)}\right]^{1/q}$$

$$\times\left\{\left[\left(2\cdot 4^{s+2}+5^{s+1}(s-3)\right)|f'(a)|^q+\left(5^{s+1}(4s+3)+2\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$

$$\left.+\left[\left(5^{s+1}(4s+3)+2\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(2\cdot 4^{s+2}+5^{s+1}(s-3)\right)|f'(b)|^q\right]^{1/q}\right\}.$$

Proof. Since $|f'(x)|^q$ is an s-convex function on $[a,b]$, from Lemma 2.1 and Hölder's integral inequality, we have

$$\left|\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left[\int_0^1\left|\frac{4}{5}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt+\int_0^1\left|\frac{1}{5}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt\right]$$

$$\leq \frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{4}{5}-t\right|dt\right)^{1-1/q}\left[\int_0^1\left|\frac{4}{5}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right]^{1/q}\right.$$

$$\left.+\left(\int_0^1\left|\frac{1}{5}-t\right|dt\right)^{1-1/q}\left[\int_0^1\left|\frac{1}{5}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right]^{1/q}\right\}$$

$$\leq \frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{4}{5}-t\right|dt\right)^{1-1/q}\left[\int_0^1\left|\frac{4}{5}-t\right|\left(t^s|f'(a)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right]^{1/q}\right.$$

$$\left.+\left(\int_0^1\left|\frac{1}{5}-t\right|dt\right)^{1-1/q}\left[\int_0^1\left|\frac{1}{5}-t\right|\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s|f'(b)|^q\right)dt\right]^{1/q}\right\}$$

$$=\frac{b-a}{4}\left(\frac{17}{50}\right)^{1-1/q}\left[\frac{1}{5^{s+2}(s+1)(s+2)}\right]^{1/q}$$

$$\times\left\{\left[\left(2\cdot 4^{s+2}+5^{s+1}(s-3)\right)|f'(a)|^{q}+\left(5^{s+1}(4s+3)+2\right)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{1/q}\right.$$

$$\left.+\left[\left(5^{s+1}(4s+3)+2\right)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\left(2\cdot 4^{s+2}+5^{s+1}(s-3)\right)|f'(b)|^{q}\right]^{1/q}\right\}.$$

The proof is completed.

Corollary 3.1

Under the assumptions of Theorem 3.4, if $s=1$, then

$$\left|\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{17(b-a)}{200}\left(\frac{1}{85}\right)^{1/q}$$

$$\times\left\{\left[26|f'(a)|^{q}+59\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{1/q}+\left[59\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+26|f'(b)|^{q}\right]^{1/q}\right\}.$$

Corollary 3.2

Under the assumptions of Theorem 3.4, if $q=s=1$, then

$$\left|\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{500}\left[13|f'(a)|+59\left|f'\left(\frac{a+b}{2}\right)\right|+13|f'(b)|\right].$$

Theorem 3.5

Let $f:I\subseteq R_0\to R$ be differentiable on I°, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'(x)|^q$ is an s-convex functions on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then

$$\left|\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{4}\left[\frac{(q-1)\left(4^{(2q-1)/(q-1)}+1\right)}{5^{(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}$$

$$\times\left\{\left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right]^{1/q}+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}\right]^{1/q}\right\}.$$

Proof. Since $|f'(x)|^q$ is an s-convex function on $[a,b]$, from Lemma 2.1 and Hölder's integral inequality, we have

$$\left|\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{b-a}{4}\left[\int_0^1\left|\frac{4}{5}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt+\int_0^1\left|\frac{1}{5}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt\right]$$

$$\le\frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{4}{5}-t\right|^{q/(q-1)}dt\right)^{1-1/q}\left[\int_0^1\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right]^{1/q}\right.$$

$$\left.+\left(\int_0^1\left|\frac{1}{5}-t\right|^{q/(q-1)}dt\right)^{1-1/q}\left[\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right]^{1/q}\right\}$$

$$\le\frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{4}{5}-t\right|^{q/(q-1)}dt\right)^{1-1/q}\left[\int_0^1\left(t^s|f'(a)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right]^{1/q}\right.$$

$$\left.+\left(\int_0^1\left|\frac{1}{5}-t\right|^{q/(q-1)}dt\right)^{1-1/q}\left[\int_0^1\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s|f'(b)|^q\right)dt\right]^{1/q}\right\}$$

$$=\frac{b-a}{4}\left[\frac{(q-1)\left(4^{(2q-1)/(q-1)}+1\right)}{5^{(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}$$

$$\times\left\{\left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right]^{1/q}+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}\right]^{1/q}\right\}.$$

The proof is completed.

Corollary 3.3

Under the assumptions of Theorem 3.5, if $s=1$, then

$$\left|\frac{1}{10}\left[f(a)+8f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{4}\left[\frac{(q-1)\left(4^{(2q-1)/(q-1)}+1\right)}{5^{(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}$$

$$\times\left\{\left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{1/q}+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right]^{1/q}\right\}.$$

4. APPLICATIONS TO SPECIAL MEANS

Now we apply some new inequalities of the Hermite--Hadamard type for s-convex functions to construct some inequalities for special means.

For positive numbers $b>a>0$, define

$$A(a,b)=\frac{a+b}{2} \quad\text{and}\quad L_s(a,b)=\left[\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right]^{1/s},$$

where $s\ne 0,-1$.

Now let $f(x)=\frac{x^{s+1}}{s+1}$ for $x\ge 0$ and $0<s\le 1$, then $f'(x)=x^s$. So

$$[\lambda x+(1-\lambda)y]^s\le\lambda^s x^s+(1-\lambda)^s y^s$$

for $x,y>0$ and $\lambda\in[0,1]$. This means that $f'(x)$ for $x>0$ is an s-convex function on R_0 and

$$\frac{f(a)+f(b)}{10}=\frac{A\left(a^{s+1},b^{s+1}\right)}{5(s+1)},\ \frac{8}{10}f\left(\frac{a+b}{2}\right)=\frac{4A^{s+1}(a,b)}{5(s+1)},\ \frac{1}{b-a}\int_a^b f(x)dx=\frac{L_{s+1}^{s+1}(a,b)}{s+1}.$$

By Theorem 3.4, we obtain Theorem 4.6 below.

Theorem 4.6

Let $b>a>0$, $0<s\le 1$, and $q\ge 1$. Then

$$\left|A\left(a^{s+1},b^{s+1}\right)+4A^{s+1}(a,b)-5L_{s+1}^{s+1}(a,b)\right|\le\frac{5(b-a)}{4}\left[\frac{17(s+1)}{50}\right]^{1-1/q}\left[\frac{1}{5^{s+1}(s+2)}\right]^{1/q}$$

$$\times\left\{\left[\left(2\cdot 4^{s+2}+5^{s+1}(s-3)\right)a^{sq}+\left(5^{s+1}(4s+3)+2\right)A^{sq}(a,b)\right]^{1/q}\right.$$

$$\left.+\left[\left(5^{s+1}(4s+3)+2\right)A^{sq}(a,b)+\left(2\cdot 4^{s+2}+5^{s+1}(s-3)\right)b^{sq}\right]^{1/q}\right\}.$$

By Theorem 3.5, we can obtain Theorem 4.7 below.

Theorem 4.7

Let $b>a>0$, $0<s\le 1$, and $q>1$. Then

$$\left|A\left(a^{s+1},b^{s+1}\right)+4A^{s+1}(a,b)-5L_{s+1}^{s+1}(a,b)\right|$$

$$\le\frac{5(b-a)}{4}\left[\frac{(s+1)(q-1)\left(4^{(2q-1)/(q-1)}+1\right)}{5^{(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}\left\{\left[a^{sq}+A^{sq}(a,b)\right]^{1/q}+\left[A^{sq}(a,b)+b^{sq}\right]^{1/q}\right\}.$$

5. CONCLUSIONS

In this paper, by virtue of an integral identity in Lemma 2.1 and the famous Hölder integral inequality, we establish some new inequalities of the Hermite-Hadamard type for s-convex functions in Theorems 3.4 and 3.5, derive some new inequalities for common convex functions in Corollaries 3.1 to 3.3, and apply these new inequalities to construct some inequalities for special means in Theorems 4.6 and 4.7.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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