Çankaya University Journal of Science and Engineering
Volume 15, No. 2 (2018) 076–087
Date Received: 26/01/2018
Date Accepted: 10/05/2018

Generalized Ricci solitons on twisted products

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Abstract: An h-almost Ricci soliton is a generalization of the Ricci soliton. In this paper we study Ricci solitons and h-almost Ricci solitons on twisted(and warped) product manifolds. First, we obtain some results about Ricci solitons on twisted products. Then we generalize them to h- almost Ricci solitons.

Keywords: Ricci soliton, h-almost Ricci soliton, concurrent vector field, twisted product, warped product, Robertson-Walker space-time.

1. Introduction

A pseudo Riemannian manifold (M,g) is an h-almost Ricci soliton if there exist a vector field X, a soliton function $\lambda : M \to \mathbb{R}$ and a function $h : M \to \mathbb{R}$ satisfying

$$Ric + \frac{h}{2} L_X g = \lambda(x) g \tag{1}$$

where Ric,L stand respectively, for the Ricci tensor and the Lie derivative, it is denoted by (M,g,X,h,λ).

In the equation (1)

1) if λ is constant it is called an h-Ricci soliton,

2) if $h = 1$ it is called an almost Ricci soliton and

3) if $h = 1$ and λ is constant it is called a Ricci soliton.

When the vector field $X = \nabla f$ for some smooth functions $f : M \to \mathbb{R}$, we say $(M,g,\nabla f,h,\lambda)$ is a gradient h-almost Ricci soliton with potential function f.

The concept of Ricci solitons was introduced by Richard Hamilton [9], which are natural generalizations of Einstein manifolds. Ricci solitons are self-similar solutions to the Ricci flow and possible singularity models of the Ricci flow. They also are fixed points of the Ricci flow and critical points of Perelmans λ-entropy and μ-entropy.[[8], [2], [3]]

Almost Ricci solitons and h-almost Ricci solitons have been introduced by S.Pigola et al. in [11] and J.N.Gomes et al. in [7], respectively.

Now, we remind definitions of objects we need througout the paper:

ISSN 2564 - 7954

Let M_1 and M_2 be two pseudo Riemannian manifolds equipped with pseudo Riemannian metrics g_1 and g_2, respectively, and let f be a positive smooth function on $M = M_1 \times M_2$. The twisted product $M = M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ equipped with the metric g given by $g = \pi^*(g_1) + f^2\sigma^*(g_2)$. where $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ are natural projections on M_1 and M_2, respectively. Here, * denotes the pull-back operator on tensors and simply we write it as $g = g_1 + f^2 g_2$. In particular, when f is a positive function on M_1, the twisted product $M_1 \times_{f\circ\pi} M_2$ becomes a warped product $M_1 \times_f M_2$. If $f = 1$, then $M_1 \times_1 M_2 = M_1 \times M_2$ is the usual Cartesian product manifold.[[10], [4]]

A Lorentzian manifold $(M = I \times N, g)$ of dimension $n \geq 3$ is a generalized Robertson-Walker (GRW) spacetime if the metric takes the form

$$g = -dt^2 + f(t)^2 g_N \tag{2}$$

where t is the time and g_N is the pseudo Riemannian metric on the manifold N and I is an open interval of the real line. A natural generalization brings from GRW spacetimes to twisted spacetimes, where the function f in the equation (2) is $f : M \to \mathbb{R}$[4].

A vector field $\zeta \in \chi(M)$ on a manifold (M, g) with metric g is called a conformal vector field if

$$L_\zeta g = \rho g$$

where ρ is a smooth real-valued function defined on M [6]. If ρ is non-zero constant or zero, ζ is called homothetic or Killing respectively.

We also say a vector field ζ on a pseudo Riemannian manifold M is concurrent if for any vector field $X \in \chi(M)$,

$$\nabla_X \zeta = X$$

Concurrent vector fields are homothetic vector fields with factor $\rho = 2$ [5]. S.Shenawy has studied Ricci solitons on warped products in [13] and has investigated some conditions on a warped product, in this paper we are inspired by his work and investigate h-almost Ricci solitons on warped products. We also analyse Ricci solitons and h-almost Ricci solitons on twisted products. In section 2, we first remind some propositions on twisted product manifolds, conformal and concurrent vector fields. Section 3 is devoted to analysing Ricci solitons on twisted product manifolds. In section 4, we first investigate existence of concurrent vector fields on h-almost Ricci solitons, next, we prove some theorems about The existence of h-almost Ricci solitons on twisted product and warped product manifolds. Finally, in section 5 we study h-almost Ricci solitons on generalized Robertson-Walker space times.

2. Preliminaries

Let $(M,g)=(M_1\times_f M_2, g_1+f^2g_2,\nabla)$ be a twisted product manifold where ∇ is the Levi-Civita connection of the metric tensor g and (M_i,∇_i,g_i), $i=1,2$ are two pseudo Riemannian manifolds equipped with pseudo Riemannian metric g_i and the Levi-Civita connections ∇_i. In this paper all manifolds are assumed to be pseudo Riemannian manifolds unless otherwise stated. Now, we have the following two propositions [4].

Proposition 1. Let $(M_1\times_f M_2,g)$ be a twisted product manifold with function $f\geq 0$ on $M_1\times M_2$. Then
1) $\nabla_{X_1}Y=\nabla^1_{X_1}Y_1\in\chi(M_1)$
2) $\nabla_{X_1}Y_2=\nabla_{Y_2}X_1=\frac{X_1(f)}{f}Y_2$
3) $\nabla_{X_2}Y_2=X_2(\ln f)Y_2+Y_2(\ln f)X_2-\frac{1}{f}g_2(X_2,Y_2)\nabla^2 f-fg_2(X_2,Y_2)\nabla^1 f$
$+\nabla^2_{X_2}Y_2$
for all $X=X_1+X_2, Y=Y_1+Y_2\in\chi(M)$ where $X_i,Y_i\in\chi(M_i)$, $i=1,2$. ∇f is the gradient of f.

Proposition 2. Let $(M_1\times_f M_2,g)$ be a twisted product manifold with function $f\geq 0$ on $M_1\times M_2$ and $dim(M_2)=n_2$. Then
1) $Ric(X_1,Y_1)=Ric^1(X_1,Y_1)+\frac{n_2}{f}Hess^1(f)(X_1,Y_1)$,
2) $Ric(X_1,Y_2)=Ric(Y_2,X_1)=(n_2-1)[Y_2X_1\ln(f)]$,
3) $Ric(X_2,Y_2)=Ric^2(X_2,Y_2)-f^*g_2(X_2,Y_2)$,
for all $X_i,Y_i\in\chi(M_i)$, $i=1,2$ where $f^*=f\triangle^1 f+(n_2-1)|\nabla^1 f|^2$. Here, $\triangle f$ and ∇f are the Laplacian and the gradient of f, respectively.

Immediately, we can apply above Propositions about warped product manifolds and obtain[10, 4]

Corollary 1. Let $(M_1\times_f M_2,g)$ be a warped product manifold with function $f\geq 0$ on M_1. Then
1) $\nabla_{X_1}Y=\nabla^1_{X_1}Y_1\in\chi(M_1)$
2) $\nabla_{X_1}Y_2=\nabla_{Y_2}X_1=\frac{X_1(f)}{f}Y_2$
3) $\nabla_{X_2}Y_2=-fg_2(X_2,Y_2)\nabla f+\nabla^2_{X_2}Y_2$
for all $X=X_1+X_2, Y=Y_1+Y_2\in\chi(M)$ where $X_i,Y_i\in\chi(M_i)$, $i=1,2$. ∇f is the gradient of f.

Corollary 2. Let $(M_1\times_f M_2,g)$ be a warped product manifold with function $f\geq 0$ on M_1 and $dim(M_2)=n_2$. Then
1) $Ric(X_1,Y_1)=Ric^1(X_1,Y_1)-\frac{n_2}{f}H^f(X_1,Y_1)$,
2) $Ric(X_1,Y_2)=0$,
3) $Ric(X_2,Y_2)=Ric^2(X_2,Y_2)-f^*g_2(X_2,Y_2)$,
for all $X_i,Y_i\in\chi(M_i)$, $i=1,2$ where $f^*=f\triangle f+(n_2-1)|\nabla f|^2$. Here, $\triangle f$ and ∇f are the Laplacian and the gradient of f, respectively.

Now, let $(M,g) = (I \times_f F, g = -dt^2 + f^2 g_F, \nabla)$ be a Robertson-Walker space time, hence we have

Corollary 3. Let $(M = I \times_f F^m, g)$ be a Robertson-Walker space-time, where $g = -dt^2 + f^2 g_F$, $m = dim(F)$ and $f : I \to \mathbb{R}$ then
1) $\nabla_{\partial_t} \partial_t = 0$,
2) $\nabla_{\partial_t} V = \nabla_V \partial_t = (lnf)^{'} V$,
3) $< \nabla_W V, \partial_t > = < V, W > (lnf)^{'}$.

Corollary 4. Let $(M = I \times_f F^m, g)$ be a Robertson-Walker space-time where $g = -dt^2 + f^2 g_F$ and $f : I \to \mathbb{R}$ then
1) $Ric(\partial_t, \partial_t) = -m\frac{f^{''}}{f}$,
2) $Ric(\partial_t, X) = 0$,
3) $Ric(X,Y) = Ric^F(X,Y) - [ff^{''} + 2(f^{'})^2] g_F$.

Let ζ be a vector field, then

$$(L_\zeta g)(X,Y) = g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta) \tag{3}$$

for any vector fields $X, Y \in \chi(M)$. Hence, $\zeta \in \chi(M)$ is a Killing vector field if and only if $g(\nabla_X \zeta, X) = 0$ for any vector field $X \in \chi(M)$.

Proposition 3. Let $\zeta \in \chi(M_1 \times_f M_2)$ be a vector field on the twisted product manifold $M_1 \times_f M_2$ with function f. Then for any vector field $X \in \chi(M_1 \times_f M_2)$,$X = X_1 + X_2$ we have

$$g(\nabla_X \zeta, X) = g_1(\nabla^1_{X_1} \zeta_1, X_1) + f^2 g_2(\nabla^2_{X_2} \zeta_2, X_2) + f[\zeta_1(f) + \zeta_2(f)] \|X_2\|^2 \tag{4}$$

Proof.

$$\begin{aligned}
g(\nabla_X \zeta, X) &= g(\nabla_{X_1+X_2} \zeta_1 + \zeta_2, X_1 + X_2) \\
&= g(\nabla_{X_1} \zeta_1, X_1) + g(\nabla_{X_1} \zeta_1, X_2) + g(\nabla_{X_2} \zeta_1, X_1) + g(\nabla_{X_2} \zeta_1, X_2) \\
&+ g(\nabla_{X_1} \zeta_2, X_1) + g(\nabla_{X_1} \zeta_2, X_2) + g(\nabla_{X_2} \zeta_2, X_1) + g(\nabla_{X_2} \zeta_2, X_2) \\
&= g(\nabla_{X_1} \zeta_1, X_1) + g(\frac{\zeta_1(f)}{f} X_2, X_2) + g(\frac{X_1(f)}{f} \zeta_2, X_2) \\
&- g(f g_2(\zeta_2, X_2) \nabla^1 f, X_1) + g(\zeta_2(\ln f) X_2 + X_2(\ln f) \zeta_2 \\
&- \frac{1}{f} g_2(\zeta_2, X_2) \nabla^2 f + \nabla^2_{X_2} \zeta_2, X_2) \\
&= g_1(\nabla_{X_1} \zeta_1, X_1) + f\zeta_1(f) g_2(X_2, X_2) + f X_1(f) g_2(\zeta_2, X_2) \\
&- f X_1(f) g_2(\zeta_2, X_2) + f\zeta_2(f) g_2(X_2, X_2) + f X_2(f) g_2(\zeta_2, X_2) \\
&- f X_2(f) g_2(\zeta_2, X_2) + f^2 g_2(\nabla^2_{X_2} \zeta_2, X_2) \\
&= g_1(\nabla_{X_1} \zeta_1, X_1) + f[\zeta_1(f) + \zeta_2(f)] g_2(X_2, X_2) + f^2 g_2(\nabla^2_{X_2} \zeta_2, X_2)
\end{aligned}$$

■

Immediately, from 4 we obtain

Proposition 4. Let $\zeta = (\zeta_1, \zeta_2) \in \chi(M_1 \times_f M_2)$ be a vector field on the twisted product manifold $M_1 \times_f M_2$ with function f where $\zeta_1(f) = -\zeta_2(f)$. Then ζ is a Killing vector field if and only if ζ_i is a Killing vector field on M_i, $i = 1,2$.

The Lie derivative on a twisted product is expressed as follows

Proposition 5. Let $\zeta = (\zeta_1, \zeta_2) \in \chi(M_1 \times_f M_2)$ be a vector field on the twisted product manifold $M_1 \times_f M_2$ with function f. Then

$$(L_\zeta g)(X,Y) = (L^1_{\zeta_1} g_1)(X_1,Y_1) + \{f^2(L^2_{\zeta_2} g_2) + 2f[\sum_{i=1}^{2} \zeta_i(f)]g_2\}(X_2,Y_2) \tag{5}$$

for all $X = X_1 + X_2, Y = Y_1 + Y_2 \in \chi(M)$ where $X_i, Y_i \in \chi(M_i)$, $i = 1,2$.

Proof.

$$\begin{aligned}
(L_\zeta g)(X,Y) &= g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta) \\
&= g_1(\nabla^1_{X_1} \zeta_1, Y_1) + f^2 g_2(\nabla^2_{X_2} \zeta_2, Y_2) + f[\zeta_1(f) + \zeta_2(f)]g_2(X_2,Y_2) \\
&+ g_1(\nabla^1_{\zeta_1} Y_1, X_1) + f^2 g_2(\nabla^2_{\zeta_2} X_2, Y_2) + f[\zeta_1(f) + \zeta_2(f)]g_2(X_2,Y_2) \\
&= (L^1_{\zeta_1} g_1)(X_1,Y_1) + \{f^2(L^2_{\zeta_2} g_2) + 2f[\zeta_1(f) + \zeta_2(f)]g_2\}(X_2,Y_2)
\end{aligned}$$

■

Shenawy has proved that a concurrent vector field ζ on a warped product $(M = M_1 \times_f M_2, \zeta = \zeta_1 + \zeta_2, g)$ provides that ζ_1 is concurrent vector field[13]. Now, from (5) we obtain

Proposition 6. Let $\zeta = \zeta_1 + \zeta_2$ be a vector field on $M = (M_1 \times_f M_2, g)$ where (M, g) is a twisted product. ζ is concurrent on M if and only if ζ_1 is a concurrent vector field on M_1 and one of the following conditions holds

i) ζ_2 is a concurrent vector field on M_2 and $\zeta_1(f) = -\zeta_2(f)$.

ii) $\zeta_2 = 0$ and $\zeta_1(f) = f$.

3. Ricci solitons on twisted products

In this section, we investigate Ricci solitons on twisted products. First, we consider special cases such as: Einstein manifolds, concurrent vector fields, conformal vector fields and ..., then we analyse the general case. We assume $X = X_1 + X_2, Y = Y_1 + Y_2 \in \chi(M)$ where $X_i, Y_i \in \chi(M_i)$, $i = 1,2$.

Theorem 1. Let the twisted product $(M,g)=(M_1\times_f M_2, g_1+f^2g_2)$ be a Ricci soliton with λ and $\zeta=\zeta_1+\zeta_2$. If M_2 is Einstein then ζ_2 is conformal.

Proof. As (M,g,ζ,λ) is a Ricci soliton hence

$$Ric(X,Y)+\frac{1}{2}(L_\zeta g)(X,Y)=\lambda g(X,Y)$$

From the above equation and using Propositions 2 and 5 we obtain

$$\begin{aligned}Ric^2(X_2,Y_2)+\frac{f^2}{2}(L^2_{\zeta_2}g_2)(X_2,Y_2)&=[f\triangle^1 f+(n_2-1)|\nabla^1 f|^2]g_2(X_2,Y_2)\\&+[\lambda f^2-f(\zeta_1(f)+\zeta_2(f))]g_2(X_2,Y_2)\end{aligned}$$

Since M_2 is Einstein, hence $Ric^2=\beta_2 g_2$ for some $\beta_2\in\mathbb{R}$ and we have

$$\frac{f^2}{2}(L^2_{\zeta_2}g_2)(X_2,Y_2)=A(x)g_2(X_2,Y_2) \tag{6}$$

where $A(x)=f\triangle^1 f+(n_2-1)|\nabla^1 f|^2-f(\zeta_1(f)+\zeta_2(f))+\lambda f^2-\beta_2$. Now, equation (6) shows that ζ_2 is conformal(because $f>0$). ■

Remark 3.1. In the above Theorem, if f is a constant function and M_1 is Einstein then ζ_1 is conformal, too.

Proposition 7. Let the twisted product $(M,g)=(M_1\times_f M_2, g_1+f^2g_2)$ be a Ricci soliton with λ and $\zeta=\zeta_1+\zeta_2$. Then (M,g) is Einstein if
i) ζ_i is conformal on M_i with factor $2\rho_i$, $i=1,2$.
ii) $f\rho_1=f\rho_2+\zeta_1(f)+\zeta_2(f)$.

Proof. As (M,g,ζ,λ) is a Ricci soliton hence

$$Ric(X,Y)+\frac{1}{2}(L_\zeta g)(X,Y)=\lambda g(X,Y) \tag{7}$$

Now, (5) shows

$$\begin{aligned}Ric(X,Y)+\frac{1}{2}(L^1_{\zeta_1}g_1)(X_1,Y_1)+\frac{f^2}{2}(L^2_{\zeta_2}g_2)(X_2,Y_2)+f[\zeta_1(f)+\zeta_2(f)]g_2(X_2,Y_2)\\=\lambda g_1(X_1,Y_1)+f^2\lambda g_2(X_2,Y_2)\end{aligned}$$

Since ζ_i is conformal on M_i with factor $2\rho_i$ for $i=1,2$, hence

$$Ric(X,Y)=(\lambda-\rho_1)g_1(X_1,Y_1)+f^2(\lambda-\rho_2)g_2(X_2,Y_2)-f[\zeta_1(f)+\zeta_2(f)]g_2(X_2,Y_2)$$

Now, from part(ii) we have

$$Ric(X,Y)=(\lambda-\rho_1)g_1(X_1,Y_1)+f^2(\lambda-\rho_1)g_2(X_2,Y_2)=(\lambda-\rho_1)g(X,Y)$$

and this means that (M,g) is Einstein. ■

Remark 3.2. In Proposition 7, if we assume (M,g) is Einstein then ζ_1 is conformal on M_1.

Immediately, from previous Proposition we obtain the following Corollary

Corollary 5. Let the twisted product $(M,g)=(M_1\times_f M_2, g_1+f^2g_2)$ be a Ricci soliton with λ and $\zeta=\zeta_1+\zeta_2$ where $\zeta_1(f)=-\zeta_2(f)$. Then (M,g) is Einstein if and only if ζ_i is a Killing vector field on M_i, $i=1,2$.

Theorem 2. Let $(M_1,g_1,\zeta_1,\lambda_1)$ be a Ricci soliton and (M_2,g_2) be an Einstein manifold with factor μ. Then (M,g,ζ,λ_1) is a Ricci soliton if
i) ζ_2 is conformal with factor 2ρ,
ii) $Hess^1(f)=0$ and
iii) $(\lambda_1-\rho)f^2=f[\zeta_1(f)+\zeta_2(f)]+\mu-(n_2-1)|\nabla^1 f|^2$.

Proof. We assume $X_i\in\chi(M_i)$, $i=1,2$. According to Proposition 2 and part (ii) we have

$$\begin{aligned}
Ric(X,Y) &= Ric(X_1,Y_1)+Ric(X_2,Y_2)\\
&= Ric^1(X_1,Y_1)+\frac{n_2}{f}Hess^1(f)(X_1,Y_1)+Ric^2(X_2,Y_2)\\
&\quad -[f\triangle^1 f+(n_2-1)|\nabla^1 f|^2]g_2(X_2,Y_2)\\
&= Ric^1(X_1,Y_1)+\mu g_2(X_2,Y_2)-(n_2-1)|\nabla^1 f|^2 g_2(X_2,Y_2) \qquad (8)
\end{aligned}$$

Now, from Proposition 5 and part (i) we obtain

$$\begin{aligned}
\frac{1}{2}(L_\zeta g)(X,Y) &= \frac{1}{2}(L^1_{\zeta_1}g_1)(X_1,Y_1)+\{\frac{f^2}{2}(L^2_{\zeta_2}g_2)+f(\zeta_1(f)+\zeta_2(f))g_2\}(X_2,Y_2)\\
&= \frac{1}{2}(L^1_{\zeta_1}g_1)(X_1,Y_1)+f^2\rho g_2(X_2,Y_2)+f(\zeta_1(f)+\zeta_2(f))g_2(X_2,Y_2) \qquad (9)
\end{aligned}$$

By assumption and equations (8) and (9) we obtain

$$\begin{aligned}
Ric(X,Y)+\frac{1}{2}(L_\zeta g)(X,Y) &= Ric^1(X_1,Y_1)+\frac{1}{2}(L^1_{\zeta_1}g_1)(X_1,Y_1)\\
&\quad +[\mu-(n_2-1)|\nabla^1 f|^2+f^2\rho+f(\zeta_1(f)+\zeta_2(f))]g_2(X_2,Y_2)\\
&= \lambda_1 g_1(X_1,Y_1)+[\mu-(n_2-1)|\nabla^1 f|^2+f^2\rho]g_2(X_2,Y_2)\\
&\quad +f(\zeta_1(f)+\zeta_2(f))g_2(X_2,Y_2)
\end{aligned}$$

Now, part (iii) shows that (M,g,ζ,λ_1) is a Ricci soliton. ■

4. *h*-Almost Ricci solitons on twisted(warped) products

In this section, we investigate *h*-almost Ricci solitons on twisted and warped products. Since theorems on twisted products are similar to warped products case, (with difference that in warped product case, $\zeta_2(f)$ vanishes.) we only express theorems and corollaries on twisted products. We assume $X = X_1 + X_2, Y = Y_1 + Y_2 \in \chi(M)$ where $X_i, Y_i \in \chi(M_i)$, $i = 1,2$. Now, we prove that under what circumstance, there is a concurrent vector field on an h-almost Ricci soliton.

Theorem 3. Let (M^n, g) be a Riemannian manifold that has a concurrent vector field ζ. Then $(M, g, \zeta, h, \lambda(x))$ is an h-almost Ricci soliton if and only if the following two conditions hold:
i) $\lambda = h$ is constant.
ii) M^n is an open part of a warped product manifold $I \times_s F$, where I is an open interval with arclength s and F is an Einstein $(n-1)$-manifold whose Ricci tensor satisfies $Ric_F = (n-2)g_F$. Here, g_F is the metric tensor of F .

Proof. The proof is similar to Theorem 3.1 in [5]. ■

The proof of following results is easy and it is similar to Ricci soliton case.

Theorem 4. Let the twisted product $(M, g) = (M_1 \times_f M_2, g_1 + f^2 g_2)$ be an h-almost Ricci soliton with $\lambda(x)$ and $\zeta = \zeta_1 + \zeta_2$. If h is a function that is nowhere zero and M_2 is Einstein then ζ_2 is conformal.

Remark 4.1. In the above Theorem, if f is a constant function and M_1 is Einstein then ζ_1 is conformal, too.

Proposition 8. Let the twisted product $(M, g) = (M_1 \times_f M_2, g_1 + f^2 g_2)$ be an h-almost Ricci soliton with $\lambda(x)$ and $\zeta = \zeta_1 + \zeta_2$. Then (M, g) is Einstein if
i) ζ_i is conformal on M_i with factor $2\rho_i$, $i = 1,2$.
ii) $fh\rho_1 = fh\rho_2 + h[\zeta_1(f) + \zeta_2(f)]$.

Proof. Since $(M, g, \zeta, \lambda(x))$ is an h-almost Ricci soliton hence

$$Ric(X,Y) + \frac{h}{2}(L_\zeta g)(X,Y) = \lambda(x) g(X,Y) \tag{10}$$

According to (5) we have

$$\begin{aligned} Ric(X,Y) + \frac{h}{2}(L^1_{\zeta_1} g_1)(X_1,Y_1) + \frac{f^2 h}{2}(L^2_{\zeta_2} g_2)(X_2,Y_2) + fh[\zeta_1(f) + \zeta_2(f)] g_2(X_2,Y_2) \\ = \lambda(x) g_1(X_1,Y_1) + f^2 \lambda(x) g_2(X_2,Y_2) \end{aligned}$$

Next, similar to Proposition 7, we prove that (M, g) is Einstein. ■

Remark 4.2. We can express Theorem 4 and Proposition 8 when $(M, g, \zeta, \lambda(x))$ is an almost Ricci soliton. In this case, the function h is eliminated from assumption.

Remark 4.3. Similar to Remark 3.2, in Proposition 4, if we assume (M, g) is Einstein and h is a function that is nowhere zero then ζ_1 is conformal on M_1.

We can obtain a Corollary from Proposition 8

Corollary 6. Let the twisted product $(M,g) = (M_1 \times_f M_2, g_1 + f^2 g_2)$ be an h-almost Ricci soliton with $\lambda(x)$ and $\zeta = \zeta_1 + \zeta_2$ where $\zeta_1(f) = -\zeta_2(f)$. Then (M,g) is Einstein if and only if ζ_i is a Killing vector field on M_i, $i = 1,2$.

Theorem 5. Let $(M_1, g_1, \zeta_1, h_1, \lambda_1(x))$ be an h_1-almost Ricci soliton and (M_2, g_2) be an Einstein manifold with factor μ. Then the twisted product $(M, g, \zeta, h_1, \lambda_1(x))$ is an h_1-almost Ricci soliton if
i) ζ_2 is conformal with factor 2ρ,
ii) $Hess^1(f) = 0$ and
iii) $(\lambda_1(x) - \rho h_1) f^2 = h_1 f[\zeta_1(f) + \zeta_2(f)] + \mu - (n_2 - 1)|\nabla f|^2$.

Proof. Similar to Theorem 2 for $X_i \in \chi(M_i)$, $i = 1,2$ we have

$$Ric(X,Y) = Ric^1(X_1,Y_1) + \mu g_2(X_2,Y_2) - (n_2 - 1)|\nabla^1 f|^2 g_2(X_2,Y_2) \tag{11}$$

We also obtain from Proposition 5 and part (i)

$$\begin{aligned}\frac{h_1}{2}(L_\zeta g)(X,Y) &= \frac{h_1}{2}(L^1_{\zeta_1} g_1)(X_1,Y_1) + h_1\{\frac{f^2}{2}(L^2_{\zeta_2} g_2) + f(\zeta_1(f) + \zeta_2(f))g_2\}(X_2,Y_2) \\ &= \frac{h_1}{2}(L^1_{\zeta_1} g_1)(X_1,Y_1) + h_1 f^2 \rho g_2(X_2,Y_2) + h_1 f(\zeta_1(f) + \zeta_2(f))g_2(X_2,Y_2)\end{aligned}$$

From assumption, equation (11) and last equation, we have

$$\begin{aligned}Ric(X,Y) + \frac{h_1}{2}(L_\zeta g)(X,Y) &= Ric^1(X_1,Y_1) + \frac{h_1}{2}(L^1_{\zeta_1} g_1)(X_1,Y_1) \\ &+ \{\mu - (n_2 - 1)|\nabla^1 f|^2 + h_1 f^2 \rho + h_1 f[\zeta_1(f) + \zeta_2(f)]\} g_2(X_2,Y_2) \\ &= \lambda_1(x) g_1(X_1,Y_1) + [\mu - (n_2 - 1)|\nabla^1 f|^2 + h_1 f^2 \rho] g_2(X_2,Y_2) \\ &+ h_1 f(\zeta_1(f) + \zeta_2(f)) g_2(X_2,Y_2)\end{aligned}$$

Now, by part (iii) we obtain $(M, g, \zeta, h_1, \lambda_1(x))$ is an h_1-almost Ricci soliton. ■

Theorem 6. Let the twisted product $(M,g) = (M_1 \times_f M_2, g_1 + f^2 g_2)$ be a Ricci soliton with λ and $\zeta = \zeta_1 + \zeta_2$ then
i) $(M_2, g_2, \zeta_2, h_2, \lambda_2(x))$ is an h_2-almost Ricci soliton where $h_2 = f^2$ and $\lambda_2(x) = \lambda f^2 + f\triangle^1 f + (n_2 - 1)|\nabla^1 f|^2 - f[\zeta_1(f) + \zeta_2(f)]$.
ii) (M_1, g_1, ζ_1) is a Ricci soliton if $Hess^1_f = 0$.

Proof. We assume (M, ζ, g) is a Ricci soliton hence $Ric(X,Y) + \frac{1}{2}(L_\zeta g)(X,Y) = \lambda g(X,Y)$ and we have
$\mathrm{Ric}^1(X_1,Y_1) + \frac{1}{2}(L^1_{\zeta_1} g_1)(X_1,Y_1) = [\lambda g_1 + \frac{n_2}{f} Hess^1_f](X_1,Y_1)$
$Ric^2(X_2,Y_2) + \frac{f^2}{2}(L^2_{\zeta_2} g_2)(X_2,Y_2) = [f\triangle^1 f + (n_2 - 1)|\nabla^1 f|^2$

$-f(\zeta_1(f)+\zeta_2(f))+\lambda f^2]g_2(X_2,Y_2)$
Now, the above equations complete the proof. ■

Theorem 7. Let the twisted product $(M,g)=(M_1\times_f M_2, g_1+f^2g_2)$ be an h-almost Ricci soliton with $\lambda(x)$ and $\zeta=\zeta_1+\zeta_2$ then
1) $(M_2,g_2,h_2,\zeta_2,\lambda_2(x))$ is an h_2-almost Ricci soliton where $h_2=f^2h$ and $\lambda_2(x)=f\triangle^1 f+(n_2-1)|\nabla^1 f|^2-fh(\zeta_1(f)+\zeta_2(f))+\lambda(x)f^2$.
2) $(M_1,g_1,\zeta_1,h_1,\lambda_1(x))$ is an h_1-almost Ricci soliton where $h_1=h$ and $\lambda_1(x)=\lambda(x)$ if $Hess^1_f=0$.

Proof. The proof is similar to Theorem 6 except that $(M,g,\zeta,h,\lambda(x))$ is an h-almost Ricci soliton i.e $Ric(X,Y)+\frac{h}{2}(L_\zeta g)(X,Y)=\lambda(x)g$ and
$\mathrm{Ric}^1(X_1,Y_1)+\frac{h}{2}(L^1_{\zeta_1}g_1)(X_1,Y_1)=[\lambda(x)g_1+\frac{n_2}{f}Hess^1_f](X_1,Y_1)$
$Ric^2(X_2,Y_2)+\frac{f^2h}{2}(L^2_{\zeta_2}g_2)(X_2,Y_2)=[f\triangle^1 f+(n_2-1)|\nabla^1 f|^2$
$-fh(\zeta_1(f)+\zeta_2(f))+\lambda(x)f^2]g_2(X_2,Y_2)$ ■

5. Generalized Robertson-Walker space-time as Ricci soliton

In this section, we apply the above results for generalized Robertson-Walker space-time. In two following corollaries, we consider M as a generalized Robertson-Walker space-time $M=I\times_f N^m$ with $g=-dt^2+f^2g_N$ and investigate Ricci solitons on them. First, we assume M is a twisted product manifold, i.e. the positive function f is as $f:I\times N\to\mathbb{R}$, next, we assume M is warped product manifold, i.e. the positive function f is as $f:I\to\mathbb{R}$, where I is an open interval and (N,g_N) is a Riemannian manifold. Since $Ric^1(\partial_t,\partial_t)=(L^1_{\zeta_1}g_1)(\partial_t,\partial_t)=0$, hence we have the following corollaries :

Corollary 7. Let (M,g) be a generalized Robertson-Walker space-time $M=I\times_f N^m$ with $g=-dt^2+f^2g_N$ and positive function $f:I\times N\to\mathbb{R}$ where I is an open interval and (N,g_N) is a Riemannian manifold. If (M,g,ζ) is a steady Ricci soliton then $Hess^1(f)=0$.

Proof. According to the metric $g=-dt^2+f^2g_N$, we use equation of (4) on (∂_t,∂_t) and obtian

$$[\lambda g_1+\frac{n_2}{f}Hess^1_f](\partial_t,\partial_t)=0$$

since (M,g,ζ) is a steady Ricci soliton($\lambda=0$), hence we have $Hess^1(f)=0$. ■

Corollary 8. Let (M,g) be a generalized Robertson-Walker space-time $M=I\times_f N^m$ with $g=-dt^2+f^2g_N$ and positive function $f:I\to\mathbb{R}$ where I is an open interval and (N,g_N) is a pseudo Riemannian manifold. If (M,g,ζ) is a Ricci soliton then
i) if (M,g,ζ) is steady then $f(t)=at+b$ where $a,b\in\mathbb{R}$.
ii) if (M,g,ζ) is expanding or shrinking then $f(t)=ae^{-\frac{\lambda}{m}t}+b$ where $a,b\in\mathbb{R}$.

Proof. Similar to proof of last Corollary, we apply Corollary 4 and equation (4) on (∂_t, ∂_t) and obtain

$$m\frac{f''}{f} = \lambda$$

Now, we can solve the above differential equation according to sign of λ. ∎

Remark 5.1. In Corollary 8, we note that the constants of a, b in part (i) and (ii) are chosen so that $f > 0$.

Remark 5.2. If Robertson-Walker space-time $(M = I \times_f N^m, g = -dt^2 + f^2 g_N, h, \lambda(x))$ is an h-almost Ricci soliton where $f : I \to \mathbb{R}$ then from (4) we obtain $\lambda = \lambda(t)$ and can find f from the equation $mf'' + \lambda(t) f = 0$.

According to the first part of equation (1), we know that λ in an h-Ricci soliton is constant, hence using Theorem 7 and equation of (4), we can express Corollaries 7 and 8, when (M, g, ζ, h) is an h-Ricci soliton.

Corollary 9. Let (M, g) be a generalized Robertson-Walker space-time $M = I \times_f N^m$ with $g = -dt^2 + f^2 g_N$ and positive function $f : I \times N \to \mathbb{R}$ where I is an open interval and (N, g_N) is a pseudo Riemannian manifold. If (M, g, ζ, h) is a steady h-Ricci soliton then $Hess^1(f) = 0$.

Corollary 10. Let (M, g) be a generalized Robertson-Walker space-time $M = I \times_f N^m$ with $g = -dt^2 + f^2 g_N$ and positive function $f : I \to \mathbb{R}$ where I is an open interval and (N, g_N) is a pseudo Riemannian manifold. If (M, g, ζ, h) is an h-Ricci soliton then
i) if (M, g, ζ, h) is steady then $f(t) = at + b$ where $a, b \in \mathbb{R}$.
ii) if (M, g, ζ, h) is expanding or shrinking then $f(t) = ae^{-\frac{\lambda}{m}t} + b$ where $a, b \in \mathbb{R}$.

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