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Üç Boyutlu Riemannian Heisenberg Grubunda Paralel Faktorable Yüzeylerin Bazı Karakterizasyonları

Some New Characterizations of Parallel Factorable Surface in Riemannian Three Dimensional Heisenberg Group

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Abstract
In this paper, we give some properties of parallel factorable surface which is obtained by group operations in Riemannian three dimensional Heisenberg Group H_3. Then, we obtain some characterizetions of parallel factorable surface according to Levi- Civita connections of H_3.

Keywords: *Factorable surface, Heisenberg group, parallel surface, Riemannian metric*

Öz
Bu çalışmada üç boyutlu Riemannian Heisenberg grubunda grup işlemiyle elde edilen paralel faktörlenebilen yüzeylerin bazı özellikleri verildi. Daha sonra H_3 te Levi- Civita konneksiyonlarına göre paralel faktörlenebilen yüzeylerin bazı karakterizasyonları elde edildi.

Anahtar kelimeler: *Faktörlenebilen yüzey, Heisenberg grup, paralel yüzey, Riemannian metrik*

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1. Introduction

A surface S in the Euclidean 3-space is r(u,v)={x(u,v),y(u,v),z(u,v)}.
A surface of factorable S in E^3 can be given as z=f(x)g(y) or y=f(x)g(z) or x=f(y)g(z), where f and g are smooth functions on some interval of $\mathbb{R}$.

In Yu and Liu (2007), a classification and some fundamental formulas were given for factorable surfaces in the Euclidean space and in the Minkowski space. In Meng and Liu (2009), factorable surfaces in 3- dimensional Minkowski space studied and some classification of such surfaces are given.

A surface M^r is parallel to M if points of M^r are at a constant distance along the normal from the surface M. So, there are infinite numbers of parallel surfaces (Yoon, 2008).

Kızıltuğ and Yaylı studied timelike curves on timelike parallel surfaces in Minkowski 3-spaces E_1^3. They find images of timelike curve which lie on timelike surface in E_1^3. Then, they gave some characterization for its image curve in (Kızıltuğ and Yaylı, 2012). Unluturk and Ekici (2003), obtain parallel surfaces of timelike ruled surfaces which are developable are timelike ruled Weingarten surfaces. Then, some properties of that kind parallel surfaces are obtained in Minkowski 3-space in (Unluturk and Ekici, 2003). Safiulina investigate the existence and geometry of such two-dimensional Riemannian submanifolds (surfaces) and she gives their complete classification. Moreover, it is shown that in E_{s}^{n} with s>0 do exist not totally geodesic minimal semiparallel space-like surfaces in (Safiulina, 2001). Calvaruso and Van der Veken completely classified surfaces with parallel second fundamental form in all non-symmetric homogeneous Lorentzian three manifolds in (Calvaruso and Veken, 2010). In Yılmaz (2010), they investigate singularities of all parallel surfaces to a given regular surface.

2. Method

The Heisenberg group $Heis_3$ is defined as R^3 with the group operation

$$(x,y,z)*(x_1,y_1,z_1)=x+x_1,y+y_1z+z_1+\frac{1}{2}(xy_1-x_1y \qquad (1)$$

The left invariant Riemannian metric given by

$$g=ds^2=dx^2+dy^2+\left(dz+\frac{1}{2}(ydx-xdy)\right)^2 \qquad (2)$$

Then we have for the left invariant Riemann metric g:

$$e_1=\frac{\partial}{\partial x}-\frac{y}{2}\frac{\partial}{\partial z},e_2=\frac{\partial}{\partial y}+\frac{x}{2}\frac{\partial}{\partial z},e_3=\frac{\partial}{\partial z} \qquad (3)$$

These vector fields are dual to the coframe

$$w^1=dx,w^2=dy,w^3=dz+\frac{y}{2}dx-\frac{x}{2}dy \qquad (4)$$

Then, Levi- Civita connections are

$$2\nabla_{e_i}e_j=\begin{bmatrix}0 & e_3 & -e_2\\ -e_3 & 0 & e_1\\ -e_2 & e_1 & 0\end{bmatrix} \qquad (5)$$

also, we have the Heisenberg bracket relations.

$$[e_1,e_2]=e_3,[e_3,e_1]=[e_2,e_3]=0 \qquad (6)$$

In this paper, we used Riemannian metric and Levi- Civita connections for obtain mean curvature.

3. Parallel Surface of the Factorable Surface in H_3

3.1. *Parallel Surfaces of Type 1 Factorable Surface*

Let $\alpha(x)=(\alpha_1(x),\alpha_2(x),\alpha_3(x))$ and $\beta(y)=(\beta_1(y),\beta_2(y),\beta_3(y))$ be differentiable nongeodesic curves in $Heis_3$ which is endowed with left invariant Riemannian metric g. Type 1 of the factorable surface parameterized as

$$\varphi(x,y)=(\alpha_1(x),0,c)*(0,\beta_2(y),d)=(\alpha_1(x),\beta_2(y),\tfrac{1}{2}\alpha_1(x)\beta_2(y)) \qquad (7)$$

Then, the parallel surface of factorable surface $\varphi(x,y)=\alpha(x)*\beta(y)$ is defined as

$$\theta(x,y)=\varphi(x,y)+N(x,y) \qquad (8)$$

where N is the unit normal vector field of the factorable surface.

Theorem 3.1. If the parallel surface of the factorable surface of type 1 θ(x,y) in the H_3 is a minimal surface, then

$$\beta_2'(\frac{a^2}{(1+\beta_2^2)^2}+(\frac{\alpha_1}{2}-\frac{\beta_2}{(1+\beta_2^2)^{3/2}})^2+1)(-\beta_2\alpha_1''+\frac{1}{2}\alpha_1'^2_1\beta_2(\alpha_1+\frac{\beta_2(a+2)}{(1+\beta_2^2)^{3/2}})+2\alpha_1''\beta_2)$$
$$-\frac{\beta_2'}{2}(\alpha_1'(-\beta_2^{\,2}+(\alpha_1+\frac{\beta_2(a+2)}{(1+\beta_2^2)^{3/2}})(\frac{\alpha_1}{2}+\frac{\beta_2(a+1)}{(1+\beta_2^2)^{3/2}})+2(\alpha_1'-1))-\beta_2^{\,2}$$
$$-(\beta_2+\frac{\alpha_1}{2}-\frac{\beta_2}{(1+\beta_2^2)^{3/2}})(\alpha_1+\frac{\beta_2(a+2)}{(1+\beta_2^2)^{3/2}})+2(\beta_2'-1)+\alpha_1'\beta_2'(1+\frac{1}{4}\beta_2^2) \tag{9}$$
$$(-\beta_2\frac{\alpha_1}{2}-\frac{\beta_2 a}{(1+\beta_2^2)^{3/2}}+A)-\frac{a}{(1+\beta_2^2)^{3/2}}(\frac{\alpha_1}{2}-\frac{\beta_2}{(1+\beta_2^2)^{3/2}})$$
$$\alpha_1+\frac{\beta_2(a+2)}{(1+\beta_2^2)^{3/2}})+2B=0$$

where

$$A=\frac{a(-3\beta_2\beta_2'+(1+\beta_2^{\,2})\beta_2'')}{(1+\beta_2^{\,2})^{5/2}} \tag{10}$$

$$B=\frac{\beta_2'(1-2\beta_2^{\,2})+\beta_2\beta_2''(1+\beta_2^{\,2})}{(1+\beta_2^{\,2})^{5/2}} \tag{11}$$

Proof. The parameterization of the parallel factorable suface

$$\theta(x,y)=(\alpha_1(x)-\frac{a\beta_2(y)}{\sqrt{1+\beta_2(y)^2}})e_1+\beta_2(y)e_2+(\frac{1}{2}\alpha_1(x)\beta_2(y)+\frac{a}{\sqrt{1+\beta_2(y)^2}})e_3 \tag{12}$$

where a is a non-zero constant. If we take derivatives of the parallel surface of the factorable surface according to x and y, then we have

$$\theta_x=\alpha_1'(e_1+\frac{1}{2}\beta_2 e_3) \tag{13}$$

$$\theta_y=\beta_2'(\frac{a}{(1+\beta_2^{\,2})^{3/2}}e_1+e_2+(\frac{1}{2}\alpha_1+\frac{a\beta_2}{(1+\beta_2^{\,2})^{3/2}})e_3) \tag{14}$$

An orthogonal vector at each point of the surface is

$$N=\frac{1}{\rho}(-\beta_2 e_1+(\alpha_1+\frac{3\beta_2 a}{(1+\beta_2^{\,2})^{3/2}})e_2+2e_3) \tag{15}$$

where

$$\rho=\sqrt{\beta_2^{\,2}+(\alpha_1+\frac{3\beta_2 a}{(1+\beta_2^{\,2})^{3/2}})^2+4} \tag{16}$$

The coefficients of the first fundamental form are

$$E=\alpha_1'^2(1+\frac{1}{4}\beta_2^{\,2}) \tag{17}$$

$$F=\alpha_1'\beta_2'^2(\frac{a}{(1+\beta_2^{\,2})^{3/2}}+\frac{1}{2}\beta_2(\frac{\alpha_1}{2}-\frac{a\beta_2}{(1+\beta_2^{\,2})^{3/2}})) \tag{18}$$

$$G = \beta_2'^2\left(\frac{a^2}{(1+\beta_2^2)^3} + \left(\frac{1}{2}\alpha_1 + \frac{a\beta_2}{(1+\beta_2^2)^{3/2}}\right)^2 + 1\right) \tag{19}$$

Then, components of the second fundamental form are

$$h_{11} = \frac{\alpha_1'}{\rho}\left(-\beta_2\alpha_1'' + \frac{1}{2}\left(\alpha_1'^2\,\beta_2\left(\alpha_1 + \frac{\beta_2(a+2)}{(1+\beta_2^2)^{3/2}}\right) + 2\alpha_1''\beta_2\right)\right) \tag{20}$$

$$h_{12} = \frac{\alpha_1'\beta_2'}{2\rho}\left(-\beta_2^2 + \alpha_1\left(\frac{\alpha_1}{2} + \frac{\beta_2(1+a)}{(1+\beta_2^2)^{3/2}} + 2(\alpha_1'-1)\right)\right) \tag{21}$$

$$h_{21} = \frac{\beta_2'}{2\rho}\left(-\beta_2^2 - \left(\beta_2 + \frac{\alpha_1}{2} - \frac{a\beta_2}{(1+\beta_2^2)^{3/2}}\left(\alpha_1 + \frac{\beta_2(2+a)}{(1+\beta_2^2)^{3/2}} + 2(\beta_2'-1)\right)\right)\right) \tag{22}$$

$$\begin{aligned} h_{22} = {} & \frac{\beta_2'}{2\rho}\left(-\beta_2(y)\left(\frac{\alpha_1}{2} - \frac{a\beta_2}{(1+\beta_2^2)^{3/2}} + A\right) - \frac{a}{(1+\beta_2^2)^{3/2}}\left(\frac{\alpha_1}{2} - \frac{a\beta_2}{(1+\beta_2^2)^{3/2}}\right)\right. \\ & \left.\left(\alpha_1 + \frac{\beta_2(2+a)}{(1+\beta_2^2)^{3/2}} + 2B\right)\right), \end{aligned} \tag{23}$$

So, if $\theta(x,y)$ is minimal, we have the equation (17).

3.2. *Parallel Surfaces of Type 2 Factorable Surface*

Let $\alpha(x)$ and $\beta(y)$ be differentiable nongeodesic curves in $Heis_3$ which is endowed with left invariant Riemannian metric g. Type 1 of the factorable surface parameterized as

$$\varphi(x,y) = (0, \alpha_2(x), c) * (\beta_1(y), 0, -c) = \left(\beta_1(y), \alpha_2(x), \frac{1}{2}\beta_1(y)\alpha_2(x)\right) \tag{24}$$

Then, the parallel surface of factorable surface $\phi(x,y) = \alpha(x) * \beta(y)$ is defined as

$$\xi(x,y) = \left(\beta_1(y), \alpha_2(x), \frac{1}{2}\beta_1(y)\alpha_2(x)\right) + bN(x,y) \tag{25}$$

where N is the unit normal vector field of the factorable surface and b is a non-zero constant.

Theorem 3.2. If the parallel surface of the factorable surface of type 2 $\xi(x,y)$ in the H_3 is a minimal surface, then

$$
\begin{aligned}
&\alpha_2'^2\beta_1'((1+\frac{\beta_1^2}{4})(\frac{b}{(1+\beta_1^2)^{3/2}}(\beta_1''+\frac{1}{2}(\frac{\alpha_2}{2}-\frac{b\beta_1}{(1+\beta_1^2)^{3/2}})(\frac{\alpha_2}{2}-\frac{\beta_1(b+1)}{(1+\beta_1^2)^{3/2}})\\
&+(\frac{1}{4}\beta_1(\frac{\alpha_2}{2}-\frac{b\beta_1}{(1+\beta_1^2)^{3/2}})(\frac{b\beta_1}{(1+\beta_1^2)^{3/2}}-1)-\frac{3b\beta_1\beta_1'}{(1+\beta_1^2)^{5/2}}-\frac{\beta_1}{2}))\\
&-\alpha_2'\beta_1^2(1+\frac{\beta_1}{2}(\frac{\alpha_2}{2}-\frac{b\beta_1}{(1+\beta_1^2)^{3/2}}))((\frac{\alpha_2}{2}-\frac{b\beta_1}{(1+\beta_1^2)^{3/2}})(\frac{\alpha_2}{2}-\frac{\beta_1(b+1)}{(1+\beta_1^2)^{3/2}})\\
&-\frac{3b\beta_1\beta_1'}{2(1+\beta_1^2)^{5/2}}+b(\frac{3\beta_1\beta_1'^2-(1+\beta_1^2)\beta_1''}{(1+\beta_1^2)^{5/2}}+\frac{1}{2})+\frac{\alpha_2'}{2}(\frac{\alpha_2}{2}-\frac{b\beta_1}{(1+\beta_1^2)^{3/2}})(\frac{\alpha_2}{2}-\frac{\beta_1(b+1)}{(1+\beta_1^2)^{3/2}})\\
&+\frac{1}{2}\beta_1(\alpha_2''-\frac{1}{4}\alpha_2'\beta_1)+\frac{1}{2}(\beta_1\alpha_2''+\alpha_2'\beta_1'(\frac{b}{(1+\beta_1^2)^{3/2}}+1)))\\
&+\beta_1'^2\alpha_2'^2(1+\frac{b^2}{(1+\beta_1^2)^3}+(\frac{\alpha_2}{2}-\frac{b\beta_1}{(1+\beta_1^2)^{3/2}})^2)((\frac{\alpha_2}{2}-\frac{\beta_1(b+1)}{(1+\beta_1^2)^{3/2}})+\beta_1'=0.
\end{aligned}
\tag{26}
$$

Proof. The proof can be shown similarly to Theorem 1.

Example 3.3. Let the factorable surface given with

$$\varphi(x,y)=\cos y e_1+\sin x e_2+\frac{1}{2}\sin x\cos y e_3 \quad (27)$$

and the unit normal vector fied of this surface is

$$
\begin{aligned}
&N=-\frac{\sin x}{\sqrt{\cos^2 y+\sin^2 x+4}}e_1-\frac{\cos y}{\sqrt{\cos^2 y+\sin^2 x+4}}e_2+\frac{2}{\sqrt{\cos^2 y+\sin^2 x+4}}e_3.\\
&\cos y e_1+\sin x e_2+\frac{1}{2}\sin x\cos y e_3
\end{aligned}
\tag{28}
$$

Then, the parallel factorable surface of type 2 is

$$
\begin{aligned}
&\xi(x,y)=(\cos y-\frac{\sin x}{\sqrt{\cos^2 y+\sin^2 x+4}})e_1+(\sin x-\frac{\cos y}{\sqrt{\cos^2 y+\sin^2 x+4}})e_2\\
&+(\frac{\sin x\cos y}{2}+\frac{2}{\sqrt{\cos^2 y+\sin^2 x+4}}e_3.
\end{aligned}
\tag{29}
$$

4. Conclusion

In this paper, we deduce conditions of minimal parallel factorable surface in three dimensional Riemannian Heisnberg group.

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