

A Note on the Serret-Frenet Frame for Spacelike Curves

Selcan KOCABAŞ

Matematik Bölümü, Fen Edebiyat Fakültesi, Kırklareli Üniversitesi, Kırklareli, Türkiye

✉: selcan.kocabas@gmail.com  0000-0003-3416-9608

Geliş (Received): 01.04.2026

Düzeltilme (Revision): 23.05.2026

Kabul (Accepted): 27.06.2026

ABSTRACT

In this study, a system of equations is introduced to characterize spacelike curves with timelike principal normal vector field in three-dimensional Minkowski space. The formulation expresses the Serret-Frenet equations with respect to the curvature κ and a rotation angle θ around the osculating plane, providing a structured representation of the Serret-Frenet system through these quantities. Moreover, the cases in which κ or θ are constant are analyzed explicitly, and the expressions for the tangent vector field and torsion are obtained. Finally, illustrative examples are presented together with their graphical representations.

Keywords: angle of rotation, Minkowski space, hyperbolic functions, osculating plane, Serret-Frenet frame, spacelike curves

Uzay-Benzeri Eğrilerin Serret-Frenet Çatısı Üzerine Analiz

ÖZ

Bu çalışmada, üç boyutlu Minkowski uzayında zaman-benzeri asal normal vektör alanına sahip uzay-benzeri eğrileri karakterize eden bir denklem sistemi ortaya konulmuştur. Geliştirilen formülasyon, Serret-Frenet elemanlarının κ eğrilik fonksiyonu ve θ dönme açısı cinsinden sistematik bir gösterimini sağlamaktadır. Ayrıca, κ veya θ değerlerinin sabit olduğu durumlar açık bir biçimde incelenmiş; teğet vektör alanı ile burulma fonksiyonuna ilişkin ifadeler elde edilmiştir. Son olarak, elde edilen sonuçlar grafik gösterimleriyle birlikte örneklendirilmiştir.

Anahtar Kelimeler: dönme açısı, Minkowski uzayı, hiperbolik fonksiyonlar, oskütatör düzlem, Serret-Frenet çatısı, uzay benzeri eğriler

INTRODUCTION

Curves in three-dimensional Minkowski space play a fundamental role in Lorentzian geometry and mathematical physics, particularly in the study of relativistic motion and spacetime structure [2, 5, 9]. Unlike the Euclidean case, Minkowski space is endowed with a Lorentzian metric, which gives rise to a natural causal classification of curves as timelike, spacelike, or lightlike (null), each possessing distinct geometric properties and physical interpretations [3, 4, 6]. Among these, spacelike curves are closely related to the causal structure of spacetime and appear naturally in both special and general relativity, particularly in discussions of simultaneity and initial value formulations [15, 17].

The Serret-Frenet equations provide a framework for the description of the local geometry of curves through intrinsic quantities such as curvature and torsion. Initially developed for the purpose of modelling Euclidean geometry, these equations have been extended to the Minkowski geometry setting, where they are adapted in order to account for the Lorentzian inner product and the associated orthonormal frame of the curve [7, 13].

In this context, this study aims to adapt Ruffa's (2008) approach, where the Serret-Frenet elements were obtained in Euclidean space via rotation matrices and the rotation angle around the osculating plane, to Minkowski 3-space [16]. However, due to the Lorentzian metric, the classical trigonometric structure is replaced by hyperbolic functions, which leads to a characteristic difference, particularly for spacelike curves [1, 10, 11]. Although a similar approach can be applied to timelike curves, their different causal character requires a separate analysis. Accordingly, this paper reformulates Ruffa's method for spacelike curves within the framework of the metric signature and causal structure of Minkowski geometry.

This study explores a system of equations defined in terms of curvature and the rotation angle of the osculating plane for spacelike curves in three dimensional Minkowski space. By analyzing the Serret-Frenet equations under these conditions, we investigate special cases such as constant curvature or constant angle. The tangent vector field and torsion are deduced using these findings, providing new insights into the geometry of curves in Minkowski space..

PRELIMINARIES

Let V be a real vector space equipped with a non-degenerate symmetric bilinear form g . The matrix representation of g with respect to an orthonormal basis $e_1, \dots, e_n$ is diagonal, satisfying

$$g(e_j, e_j) = \delta_{ij} \varepsilon_j, \text{ where } \varepsilon_j = g(e_j, e_j) = \pm 1.$$

Whenever convenient, we shall order the vectors in an orthogonal basis so that the negative signs-if any-come first in the so-called signature $(\varepsilon_1, \dots, \varepsilon_n)$. Taking these signs into account orthonormal expansion is still available [16].

Motivated by this convention, we consider the Minkowski three space $\mathbb{E}_1^3 = (\mathbb{R}^3, \langle, \rangle)$ as a real vector space endowed with the Lorentzian metric;

$$\langle u, v \rangle = u_1 v_1 - u_2 v_2 + u_3 v_3$$

$$u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \quad u, v \in \mathbb{E}_1^3.$$

The metric symbol " $\langle, \rangle$ " is referred to as the Lorentzian metric [12].

In accordance with O'Neill's formulation, we use the metric signature $(+, -, +)$, in order to obtain more clear and concise results from computations; all causal characterizations are made with respect to this metric throughout the paper [12, 13].

The vector product $u \times v$ in $\mathbb{E}_1^3$ is defined in accordance with the chosen Lorentzian metric structure as:

$$u \times v = \begin{vmatrix} i & -j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \quad (1)$$

According to the definition of the Lorentz metric, a vector u is classified into three different causal characters: it is spacelike vector if $\langle u, u \rangle > 0$ or $u = 0$; a timelike vector if $\langle u, u \rangle < 0$ and a lightlike (null) vector if $\langle u, u \rangle = 0$ and $u \neq 0$ [13].

Let the curve $r(s)$ be a unit speed curve in the space $\mathbb{E}_1^3$. If $r'(s)$ is a spacelike, timelike and lightlike (null), then $r(s)$ is spacelike, timelike and lightlike (null) at s , respectively [4].

Let $r = r(s)$ be a spacelike curve in $\mathbb{E}_1^3$, parametrized by the arc length s , so that its tangent vector $T = r'(s)$ satisfies $\langle T, T \rangle = 1$. When T is spacelike, there are two possible causal configurations for the normal and binormal vector fields: one of them must be timelike and the other spacelike. In this study, we assume that the normal vector N is timelike and the binormal vector B is spacelike. Accordingly,

$$\langle T, T \rangle = 1, \quad \langle N, N \rangle = -1, \quad \langle B, B \rangle = 1$$

Under this causal assumption, the Serret-Frenet vector frame $\{T, N, B\}$ of a spacelike curve r satisfies the following derivative equations

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2)$$

where T, N, B, κ and τ are the spacelike tangent, timelike normal, spacelike binormal vector fields as well as the curvature and the torsion of the curve $r(s)$, respectively [12].

We emphasize that equation (2) corresponds specifically to the case where T and B are spacelike and N is timelike. If a different causal configuration were chosen, appropriate sign changes would appear in the Serret-Frenet coefficients.

Additionally, in the Lorentzian setting, the curvature must be defined accordingly. The quantity $|T'|$ is not directly interpreted as the square root of $\langle T', T' \rangle$, since the Lorentzian inner product is indefinite. In particular, if N is timelike and normalized by $\langle N, N \rangle = -1$, then $\langle T', T' \rangle = \kappa^2 \langle N, N \rangle = -\kappa^2$. Hence $\kappa^2 = |\langle T', T' \rangle|$, and therefore the curvature becomes as [4],

$$\kappa = \sqrt{|\langle T', T' \rangle|} = \sqrt{\left| \left\langle \frac{dT}{ds}, \frac{dT}{ds} \right\rangle \right|} = \left\| \frac{dT}{ds} \right\|.$$

It is shown here that a set of equations can be developed to describe a spacelike curve in the space with the curvature κ and the angle of rotation θ of the osculating plane [16].

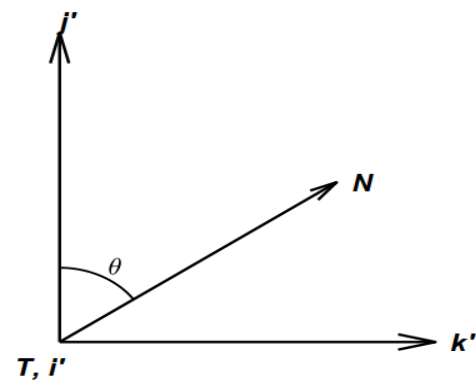


Figure 1. The local coordinate system i', j' and k' are spacelike, timelike and spacelike vectors as well as spacelike vector T is normal to the plane including j' and k' .

METHODOLOGY

Let $\{i', j', k'\}$ denote the standard orthonormal basis of $\mathbb{E}_1^3$ with the signature $(+, -, +)$, where i' and k' are spacelike and j' is timelike. Consider a spacelike curve $r(s)$ whose tangent vector is chosen as $T = i'$ (Figure 1). Since T is spacelike and the principal normal vector N is assumed to be timelike, the vector N lies in the osculating plane spanned by j' and k' . In Minkowski

space, rotations in a plane containing a timelike direction are not Euclidean rotations but Lorentz boosts. Consequently, such transformations are parametrized by hyperbolic functions rather than circular trigonometric functions [12].

For this reason, the timelike normal vector field $\mathbf{N}$ is expressed as

$$\mathbf{N} = -\cosh\theta\mathbf{j}' + \sinh\theta\mathbf{k}' \quad (3)$$

where θ denotes the hyperbolic angle in the osculating plane. This representation can be interpreted as obtaining the principal normal vector from the standard basis vectors via a Lorentzian rotation in the $(\mathbf{j}', \mathbf{k}')$ -plane. A direct computation shows that

$$\langle \mathbf{N}, \mathbf{N} \rangle = -\cosh^2\theta + \sinh^2\theta = -1$$

which confirms that $\mathbf{N}$ is indeed timelike, as required by our causal configuration.

The curvature κ and the angle of rotation θ of the osculating plane $(\mathbf{T}, \mathbf{N})$ characterize the curve. Assume that the plane determined by the tangent vector $\mathbf{T}$ and and the global coordinate vector $\mathbf{j}$ is normal to $\mathbf{k}'$ (Figure 2).

Given that

$$\mathbf{T} = iT_i + jT_j + kT_k,$$

then $T_i^2 - T_j^2 + T_k^2 = 1$. Using the definition of the Lorentzian vector product consistent with this signature in (1), we obtain

$$\mathbf{T} \times \mathbf{j} = (iT_i + jT_j + kT_k) \times \mathbf{j} = iT_k - kT_i.$$

Since $T_i^2 - T_j^2 + T_k^2 = 1$, it is found that $|\mathbf{T} \times \mathbf{j}| = \sqrt{1 + T_j^2}$. Thus the vectors $\mathbf{k}'$ and $\mathbf{j}'$ are calculated as

$$\mathbf{k}' = \frac{\mathbf{T} \times \mathbf{j}}{|\mathbf{T} \times \mathbf{j}|} = \frac{iT_k - kT_i}{\sqrt{1 + T_j^2}}. \quad (4)$$

$$\mathbf{j}' = \mathbf{k}' \times \mathbf{T} = \frac{1}{\sqrt{1 + T_j^2}} (iT_k - kT_i) \times (iT_i + jT_j + kT_k),$$

or

$$\mathbf{j}' = \frac{1}{\sqrt{1 + T_j^2}} (-iT_iT_j - j(1 + T_j^2) - kT_jT_k). \quad (5)$$

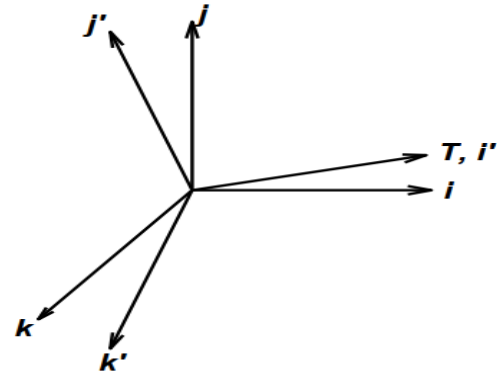


Figure 2: The angular orientation of the local coordinate system $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ according to the global coordinate system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are spacelike, timelike and spacelike vectors, respectively.

Substituting (4) and (5) into (3) gives

$$\mathbf{N} = -\left(\frac{-iT_iT_j - j(1 + T_j^2) - kT_jT_k}{\sqrt{1 + T_j^2}}\right) \cosh\theta + \left(\frac{iT_k - kT_i}{\sqrt{1 + T_j^2}}\right) \sinh\theta,$$

or

$$\mathbf{N} = \frac{1}{\sqrt{1 + T_j^2}} (\cosh\theta (iT_iT_j + j(1 + T_j^2) + kT_jT_k) + \sinh\theta (iT_k - kT_i)).$$

Now collecting the components of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ separately,

$$N_i = \frac{1}{\kappa} \frac{dT_i}{ds} = \frac{T_iT_j \cosh\theta + T_k \sinh\theta}{\sqrt{1 + T_j^2}}, \quad (6)$$

$$N_j = \frac{1}{\kappa} \frac{dT_j}{ds} = \cosh\theta \sqrt{1 + T_j^2}, \quad (7)$$

$$N_k = \frac{1}{\kappa} \frac{dT_k}{ds} = \frac{T_jT_k \cosh\theta - T_i \sinh\theta}{\sqrt{1 + T_j^2}}. \quad (8)$$

By the integration of the equation (7) with the constant initial values T_{j0}, s_0 , we get

$$\int_{T_{j0}}^{T_j} \frac{dT_j}{\sqrt{1 + T_j^2}} = \int_{s_0}^s \kappa \cosh\theta d\sigma, \quad (9)$$

turns to

$$\sinh^{-1}T_j = \sinh^{-1}T_{j0} + \int_{s_0}^s \kappa \cosh\theta d\sigma;$$

$$T_j = \sinh \left[\sinh^{-1} T_{j0} + \int_{s_0}^s \kappa \cosh \theta d\sigma \right] = \sinh \delta ;$$

here $\delta = \sinh^{-1} T_{j0} + \int_{s_0}^s \kappa \cosh \theta d\sigma$ and using the hyperbolic sine addition formula, then

$$T_j = T_{j0} \cosh \int_{s_0}^s \kappa \cosh \theta d\sigma + \sqrt{1 + T_{j0}^2} \sinh \int_{s_0}^s \kappa \cosh \theta d\sigma. \quad (10)$$

The equation (6) can be solved by the relation $T_k = \sqrt{1 + T_j^2 - T_i^2} = \sqrt{\cosh^2 \delta - T_i^2}$ and introducing the new variable β , then we arrive

$$T_i = \cosh \delta \cos \beta, \quad (11)$$

$$T_j = \sinh \delta, \quad (12)$$

$$T_k = \cosh \delta \sin \beta. \quad (13)$$

To confirm that this parametrization (11)-(13) satisfies the condition for Lorentzian vector to be unit, we compute

$$\begin{aligned} T_i^2 - T_j^2 + T_k^2 &= \cosh^2 \delta \cos^2 \beta - \sinh^2 \delta + \cosh^2 \delta \sin^2 \beta, \\ &= \cosh^2 (\cos^2 \beta + \sin^2 \beta) - \sinh^2 \delta, \\ &= \cosh^2 - \sinh^2 \delta, \\ &= 1. \end{aligned}$$

Now, we differentiate T_i in (11) according to the arc length parameter s ;

$$\begin{aligned} \frac{dT_i}{ds} &= \frac{d}{ds} (\cosh \delta \cos \beta), \\ \frac{dT_i}{ds} &= \sinh \delta \frac{d\delta}{ds} \cos \beta - \cosh \delta \sin \beta \frac{d\beta}{ds} \end{aligned} \quad (14)$$

is obtained.

From the relation

$$\delta = \sinh^{-1} T_{j0} + \int_{s_0}^s \kappa \cosh \theta d\sigma,$$

it follows that

$$\frac{d\delta}{ds} = \kappa \cosh \theta.$$

Substituting this into (14), we get

$$\frac{dT_i}{ds} = \kappa \cosh \theta \sinh \delta \cos \beta - \cosh \delta \sin \beta \frac{d\beta}{ds}. \quad (15)$$

On the other hand, substituting (11)-(13) into equation (6) yields, then

$$\frac{dT_i}{ds} = \kappa \cosh \theta \sinh \delta \cos \beta + \kappa \sinh \theta \sin \beta. \quad (16)$$

Comparing (15) and (16) gives the common term,

$$\begin{aligned} -\cosh \delta \sin \beta \frac{d\beta}{ds} &= \kappa \sinh \theta \sin \beta \\ \frac{d\beta}{ds} &= -\frac{\kappa \sinh \theta}{\cosh \delta}; \end{aligned} \quad (17)$$

or

$$\beta = \beta_0 - \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma = \cos^{-1} \left(\frac{T_i}{\cosh \delta} \right). \quad (18)$$

From the equations (11) and (18),

$$T_i = \cosh \delta \left(\cos \beta_0 \cos \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma + \sin \beta_0 \sin \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma \right).$$

Since $\beta_0 = \cos^{-1} \left(\frac{T_{i0}}{\cosh \delta_0} \right)$, we obtain

$$T_i = T_{i0} \frac{\cosh \delta}{\cosh \delta_0} \cos \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma - T_{k0} \frac{\cosh \delta}{\cosh \delta_0} \sin \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma, \quad (19)$$

where $\cosh \delta = \sqrt{1 + T_j^2}$. Using the equation (10),

$$\cosh^2 \delta = 1 + \left[T_{j0} \cosh \int_{s_0}^s \kappa \cosh \theta d\sigma + \sqrt{1 + T_{j0}^2} \sinh \int_{s_0}^s \kappa \cosh \theta d\sigma \right]^2$$

or

$$\cosh \delta = \sqrt{1 + T_{j0}^2} \cosh \int_{s_0}^s \kappa \cosh \theta d\sigma + T_{j0} \sinh \int_{s_0}^s \kappa \cosh \theta d\sigma \quad (20)$$

is found.

In order to get the solution for T_k , by using the equations (13) and (18), we obtain

$$T_k = \cosh \delta \sin \left(\beta_0 - \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma \right);$$

in other words by thanks to $\beta_0 = \cos^{-1} \left(\frac{T_{i0}}{\cosh \delta_0} \right)$, the component T_k occurs as follows;

$$T_k = -T_{i0} \frac{\cosh \delta}{\cosh \delta_0} \sin \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma + T_{k0} \frac{\cosh \delta}{\cosh \delta_0} \cos \int_{s_0}^s \frac{\kappa \sinh \theta}{\cosh \delta} d\sigma. \quad (21)$$

Furthermore, from (10), (19) and (21) and using the Serret-Frenet relation $\mathbf{T}' = \kappa \mathbf{N}$, we clarify the definition of the curvature in the Lorentzian setting as

$$\kappa = \left\| \frac{dT}{ds} \right\|. \quad (22)$$

Computing the principal normal vector $\mathbf{N}$ by using (11)-(13) into (6)-(8), respectively;

$$N_i = \cos \beta \sinh \delta \cosh \theta + \sin \beta \sinh \theta, \quad (23)$$

$$N_j = \cosh \theta \cosh \delta, \quad (24)$$

$$N_k = \sin \beta \sinh \delta \cosh \theta - \cos \beta \sinh \theta. \quad (25)$$

From $\mathbf{B} = \mathbf{T} \times \mathbf{N}$:

$$\begin{aligned} B_i &= T_j N_k - N_j T_k \\ &= -\sin \beta \cosh \theta - \cos \beta \sinh \delta \sinh \theta, \end{aligned} \quad (26)$$

$$\begin{aligned} B_j &= T_i N_k - T_k N_i \\ &= -\cosh \delta \sinh \theta, \end{aligned} \quad (27)$$

$$\begin{aligned} B_k &= T_i N_j - T_j N_i \\ &= \cos \beta \cosh \theta - \sin \beta \sinh \delta \sinh \theta. \end{aligned} \quad (28)$$

The new expression for the curvature κ is as a function of θ by (7) and (24):

$$\kappa = \frac{1}{\cosh \theta} \frac{d\delta}{ds}. \quad (29)$$

The curvature κ in (29) is written with respect to the components of $\mathbf{T}$ and $\mathbf{N}$:

$$\kappa = \frac{\sqrt{1 + T_j^2}}{N_j}. \quad (30)$$

The torsion τ depending on θ becomes by (24), (27), (29) and $N_j = \frac{1}{\tau} \frac{dB_j}{ds}$ from (2) as follows:

$$\begin{aligned} \cosh \theta \cosh \delta &= \frac{1}{\tau} \frac{d(-\sinh \theta \cosh \delta)}{ds}, \\ \cosh \theta \cosh \delta &= \frac{1}{\tau} \left(-\kappa \sinh \delta \cosh \theta \sinh \theta - \cosh \delta \cosh \theta \frac{d\theta}{ds} \right), \\ \tau &= -\kappa \tanh \delta \sinh \theta - \frac{d\theta}{ds} \end{aligned} \quad (31)$$

or in terms of the componets of $\mathbf{T}$ and $\mathbf{B}$, then

$$\tau = -\frac{d\theta}{ds} + \frac{\kappa T_j B_j}{1 + T_j^2}. \quad (32)$$

In particular, if θ in (31) is constant, then $\tau = -\kappa \tanh \delta \sinh \theta$. In this case, the torsion is directly proportional to the curvature. Consequently, the ratio $\frac{\tau}{\kappa}$ becomes

$$\frac{\tau}{\kappa} = -\tanh \delta \sinh \theta,$$

which is constant whenever δ is also constant. Therefore, when both θ and δ are constant, the curve exhibits a helical behaviour in Minkowski space.

DISCUSSION

The expression of the angle θ is obtained by the integration of (32) as the following:

$$\theta = \theta_0 + \int_{s_0}^s \left(\frac{\kappa T_j B_j}{1 + T_j^2} - \tau \right) d\sigma. \quad (33)$$

The equation (33) shows a unique value of the angle θ for each specific value of the torsion τ .

The angle θ can be shown according to the componets of the vector fields $\mathbf{T}, \mathbf{N}, \mathbf{B}$ by (12), (24) and (27):

$$\theta = -\sinh^{-1} \frac{B_j}{\sqrt{1 + T_j^2}} = \cosh^{-1} \frac{N_j}{\sqrt{1 + T_j^2}} = -\tanh^{-1} \frac{B_j}{N_j}. \quad (34)$$

Constant θ

For the constant value of angle θ , we get an explicit solution. Without the loss of generality, we choose the initial conditions of angle $\beta_0 = 0$, $\delta_0 = 0$, $s_0 = 0$.

Since expression of T_i in (11), we obtain $T_{i0} = \cosh 0 \cos 0 = 1$. Hence, these initial conditions represent only a normalization of the reference frame in Minkowski space.

By using the equations (17) and (29), we get δ and β for the constant θ , respectively;

$$\delta = \int_{s_0}^s \kappa(\sigma) \cosh \theta_0 d\sigma, \quad (35)$$

$$\beta = \tanh^{-1} \theta_0 \tan^{-1}(\sinh \delta) \quad (36)$$

which leads to the following expressions of T_i, T_j, T_k and τ :

$$T_i = \cos[\tanh^{-1} \theta_0 \tan^{-1}(\sinh \delta)] \cosh \int_{s_0}^s \kappa(\sigma) \cosh \theta_0 d\sigma, \quad (37)$$

$$T_j = \sinh \int_{s_0}^s \kappa(\sigma) \cosh \theta_0 d\sigma, \quad (38)$$

$$T_k = \sin[\tanh^{-1} \theta_0 \tan^{-1}(\sinh \delta)] \cosh \int_{s_0}^s \kappa(\sigma) \cosh \theta_0 d\sigma, \quad (39)$$

$$\tau = -\kappa(s) \sinh \theta_0 \tanh \int_{s_0}^s \kappa(\sigma) \cosh \theta_0 d\sigma. \quad (40)$$

Constant κ

For constant $\kappa = \kappa_0$, but $\theta \neq \theta_0$, the functions δ and β are determined through the integration of (17) and (29).

As an illustrative example (chosen for computational convenience rather than for a specific interpretation), we take the curvature $\kappa = \kappa_0$ and the angle $\theta = \kappa_0 s$. Thus δ and β become respectively:

$$\delta = \sinh \kappa_0 s, \quad (41)$$

$$\beta = - \int_{s_0}^s \frac{\kappa_0 \sinh \kappa_0 \sigma}{\cosh(\sinh \kappa_0 \sigma)} d\sigma \quad (42)$$

which turns to new expressions for T_i, T_j, T_k and τ :

$$T_i = \cosh(\sinh \kappa_0 s) \cos \int_{s_0}^s \frac{\kappa_0 \sinh \kappa_0 \sigma}{\cosh(\sinh \kappa_0 \sigma)} d\sigma, \quad (43)$$

$$T_j = \sinh(\sinh \kappa_0 s), \quad (44)$$

$$T_k = - \cosh(\sinh \kappa_0 s) \sin \int_{s_0}^s \frac{\kappa_0 \sinh \kappa_0 \sigma}{\cosh(\sinh \kappa_0 \sigma)} d\sigma, \quad (45)$$

$$\tau = -\kappa_0 - \kappa_0 \tanh(\sinh \kappa_0 s) \sinh \kappa_0 s. \quad (46)$$

Constant κ and θ

If both the curvature and the angle are constant, i.e. $\kappa = \kappa_0$, $\theta = \theta_0$, then we get the following relations for T_i, T_j, T_k and τ by (37)-(40):

$$T_i = \cosh(\kappa_0 s \cosh \theta_0) \cos[\tanh^{-1} \theta_0 \tan^{-1}(\sinh \delta)], \quad (47)$$

$$T_j = \sinh(\kappa_0 s \cosh \theta_0), \quad (48)$$

$$T_k = \cosh(\kappa_0 s \cosh \theta_0) \sin[\tanh^{-1} \theta_0 \tan^{-1}(\sinh \delta)], \quad (49)$$

$$\tau = -\kappa_0 \sinh \theta_0 \tanh(\kappa_0 s \cosh \theta_0). \quad (50)$$

In particular, when $\theta_0 = 0$, we obtain $\tau = 0$. Hence, in the case where both the curvature and the rotation angle are constant and $\theta_0 = 0$, the curve is necessarily planar.

On Timelike Curves

When the curve is timelike, the normal vector $\mathbf{N}$ and the binormal vector $\mathbf{B}$ become spacelike. In this case, the principal normal vector lies in the spacelike $(\mathbf{i}', \mathbf{k}')$ -plane, and its angular characterization can be written as

$$\mathbf{N} = \cos \theta \mathbf{i}' + \sin \theta \mathbf{k}'.$$

With this definition of the causal character of the Serret-Frenet frame, the same curvature-angle based methodology applied for spacelike curves can be adapted to the timelike case through the corresponding Lorentzian Serret-Frenet equations.

Examples

Let us now discuss the geometric meaning of the examples corresponding to special curvature function.

Case 1: $\kappa = \kappa_0 s^2$

If we take the curvature $\kappa = \kappa_0 s^2$ with the constant angle θ , then the components T_i, T_j, T_k and the torsion τ via (37)-(40) become as the followings:

$$T_i = \cosh\left[\frac{\kappa_0}{3} s^3 \cosh \theta_0\right] \cos\left[\tanh \theta_0 \tan^{-1}\left(\sinh\left[\frac{\kappa_0}{3} s^3 \cosh \theta_0\right]\right)\right],$$

$$T_j = \sinh\left[\frac{\kappa_0}{3} s^3 \cosh \theta_0\right],$$

$$T_k = \cosh\left[\frac{\kappa_0}{3} s^3 \cosh \theta_0\right] \sin\left[\tanh \theta_0 \tan^{-1}\left(\sinh\left[\frac{\kappa_0}{3} s^3 \cosh \theta_0\right]\right)\right],$$

and

$$\tau = -\kappa_0 s^2 \sinh \theta_0 \tanh\left[\frac{\kappa_0}{3} s^3 \cosh \theta_0\right].$$

For the polynomial curvature $\kappa(s) = \kappa_0 s^2$, the curvature vanishes at $s = 0$ and increases quadratically as s grows. Unlike the classical Euler (linear) clothoid [8], characterized by $\kappa(s) = as$, the present curvature law is quadratic rather than linear; therefore, the curve does not belong to the standard clothoid family but may instead be interpreted as a higher order polynomial spiral-type curve.

Figure 3 visualizes the spacelike curve generated by the polynomial curvature $\kappa(s) = \kappa_0 s^2$ together with its curvature, torsion, and tangent vector components.

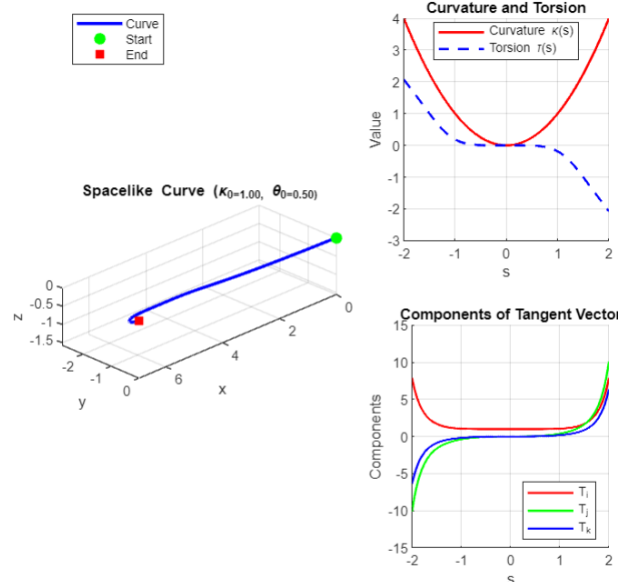


Figure 3: Spacelike curve with the curvature $\kappa_0 s^2$ and its associated curvature, torsion, and tangent vector components.

Case 2: $\kappa = \kappa_0 e^{-s^2}$

For another curvature value of the curvature as $\kappa = \kappa_0 e^{-s^2}$; T_i, T_j, T_k and τ in (37)-(40) turn to:

$$T_i = \cosh \left[\frac{\kappa_0 \sqrt{\pi}}{2} \operatorname{erf}(s) \cosh \theta_0 \right] \cos \left[\tanh \theta_0 \tan^{-1} \left(\sinh \left[\frac{\kappa_0 \sqrt{\pi}}{2} \operatorname{erf}(s) \cosh \theta_0 \right] \right) \right],$$

$$T_j = \sinh \left[\frac{\kappa_0 \sqrt{\pi}}{2} \operatorname{erf}(s) \cosh \theta_0 \right],$$

$$T_k = \cosh \left[\frac{\kappa_0 \sqrt{\pi}}{2} \operatorname{erf}(s) \cosh \theta_0 \right] \sin \left[\tanh \theta_0 \tan^{-1} \left(\sinh \left[\frac{\kappa_0 \sqrt{\pi}}{2} \operatorname{erf}(s) \cosh \theta_0 \right] \right) \right],$$

and

$$\tau = -\kappa_0 e^{-s^2} \sinh \theta_0 \tanh \left[\frac{\kappa_0 \sqrt{\pi}}{2} \operatorname{erf}(s) \cosh \theta_0 \right].$$

For $\kappa = \kappa_0 e^{-s^2}$, the curvature is maximal at $s = 0$ and decays rapidly to zero as $s \rightarrow \infty$ [14]. Therefore, the curve bends most strongly near the origin and becomes asymptotically straight at infinity. Thus, the tangent vector approaches a limiting direction at infinity. Consequently, the torsion also decays exponentially. Hence, both bending and twisting are concentrated near the origin.

The results for this curvature case are visualized in Figure 4, where the curve together with its curvature, torsion, and tangent vector components are displayed.

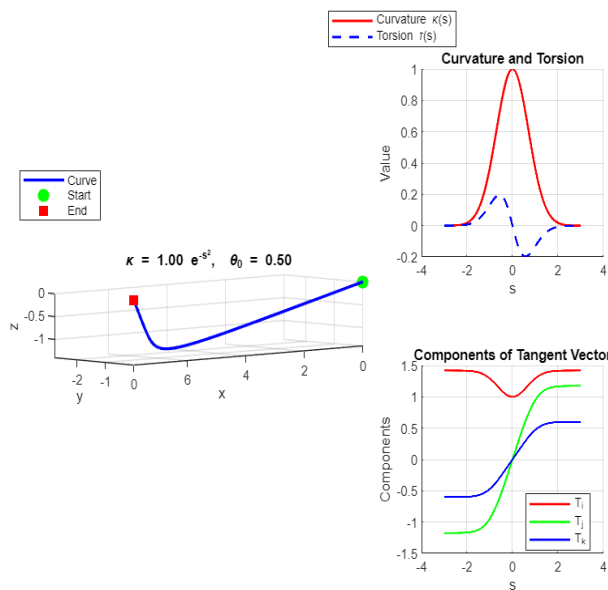


Figure 4: Spacelike curve with the curvature $\kappa_0 e^{-s^2}$ and its associated curvature, torsion, and tangent vector components.

CONCLUSION

In this paper, we developed a curvature-angle based formulation for spacelike curves with timelike principal normal vector fields in Minkowski 3-space. By rewriting the Serret-Frenet system in terms of the curvature κ and a rotation angle θ , we derived explicit expressions for the tangent vector field and obtained the corresponding torsion in selected cases, including

constant and prescribed curvature profiles. This formulation provides a new framework for reconstructing spacelike curves.

The use of a rotation based representation in the Lorentzian setting proves particularly effective for analyzing the evolution of the Serret-Frenet frame and clarifying how curvature and angular parameters govern the geometric behavior of the curve. In this sense, the approach offers a structured method for studying invariant based characterizations of curves and for generating curve families determined by prescribed curvature functions.

Although the present analysis is restricted to spacelike curves, the structure of the formulation naturally extends to timelike curves through the appropriate Serret-Frenet equations. In Lorentz geometry, many fundamental curves—such as magnetic trajectories, elasticae, and uniformly accelerated relativistic motions—are naturally described by hyperbolic functions arising from the structure of Minkowski spacetime, and the flexibility of this approach allows them to be treated within a geometric framework.

ACKNOWLEDGEMENT

The authors appreciate the anonymous reviewers for their precious comments, which helped to make the quality of this study better.

KAYNAKÇA

- [1] Altunkaya B., Kula L., 2018. On timelike rectifying slant helices in Minkowski 3-space International Electronic Journal of Geometry 11 (1), 17-25.
- [2] Arreaga, G., Capovilla, R. and Guven, J. Frenet-Serret dynamics, Class. Quantum Grav. 18, 5065–5083, 2001. DOI:10.1088/0264-9381/18/23/304.
- [3] Beem, J.K., Ehrlich, P., Easley, K. Global Lorentzian Geometry, NewYork: Marcel Dekker Inc, 1906.
- [4] Couto, I.T., Lymberopoulos, A. Introduction to Lorentz Geometry, Curves and Surfaces (First Edition). NewYork: CRC Press, 2021.
- [5] Calin, O., Chang, C. Geometric Mechanics on Riemannian and Lorentzian Manifolds. Birkhauser, Boston, 2005.
- [6] Dede M., Yaylı Y. 2014. Curve reconstruction in Lorentz Minkowski 3 space. Glob. J. Adv. Res. Class. Mod. Geom., 3(1).
- [7] Divjak, B. The general solution of the Frenet system of differential equations for curves in the pseudo-Galilean space G_1^3 , Mathematical Communications 2, 143-147, 1997.
- [8] Frego, M. 2022. Closed form parametrisation of 3D cloithoids by arclength with both linear varying curvature and torsion, Applied Mathematics and Computation, Volume 421. <https://doi.org/10.1016/j.amc.2021.126907>
- [9] Formiga, J. B., Romero, C. On the differential geometry of curves in Minkowski space, American Journal of Physics 74(11), 1012-1016, 2006. DOI:10.1119/1.2232644.
- [10] İlarslan, K. 2005. Spacelike normal curves in Minkowski space E_1^3 . Turkish Journal of Mathematics, 29(1).

- [11] İlarıslan, K., Uçum, A., Mladenov I. M. 2015. Sturmian Spirals in Lorentz-Minkowski Plane. J. Geom. Symmetry Phys. 37: 25-42 (2015). DOI: 10.7546/jgsp-37-2015-25-42
- [12] Lopez, R. Differential geometry of curves and surfaces in Lorentz-Minkowski space, Int. Electron. J. Geometry, 7(1), 44-107, 2014.
- [13] O'Neill, B. Semi-Riemannian Geometry: With Applications to Relativity (First Edition). New York: Academic Press, 1983.
- [14] Olver, F.W.J. et al. 2010. NIST Handbook of Mathematical Functions, Section 7.1: Error Function, Cambridge Univ. Press.
- [15] Rindler, W. 2006. Relativity, Special, General and Cosmology. Second Edition, Oxford University Press, Oxford
- [16] Ruffa, A. A. A novel solution to the Frenet-Serret Equations, Bull. Pure Appl. Math. 2(1), 70-78, 2008.
- [17] Wald, R. M. 1984. General Relativity, University of Chicago Press, 1984. DOI: 10.7208/chicago/9780226870373.001.0001.