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Research Article

Left Jordan derivations on certain semirings

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Abstract

We determine conditions under which a left Jordan derivation defined on an MA-semiring S is a left derivation on this semiring and prove when a left Jordan derivation on S implies the commutativity of S.

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1. Introduction

In the last thirty years, many papers have been written about the role of additive mappings in rings, and especially in the commutativity of rings. The first important results in this context were obtained by Posner (cf. [14]). After that, much attention was paid to various types of derivations (see for example [2–4,6,7,17]). In the last decade, some people tried to generalize these results to various types of semirings that play an important role in theoretical computer science (cf. [9] and [10]). Semirings can be characterized on the one hand by certain ideals (cf. [8] or [18]), on the other by various additive mappings (cf. [13, 15, 16]). The results obtained for semirings are similar to those for rings, but the methods of proofs are quite different.

In this note, which is a continuation of our previous article [1], we generalize the result of Brešar and Vukman (cf. [7]) that the existence of a non-zero left Jordan derivation forces a prime ring to be commutative. Namely, we prove that the existence of such a derivation on a prime, 2- and 3-torsion free MA-semiring forces the commutativity of this semiring (Theorem 4.3). Next we prove that each left Jordan derivation on a closed Lie ideal of a prime 2-torsion free MA-semiring is a left derivation on this Lie ideal (Theorem 4.5).

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2. Preliminaries

For completeness, we recall some definitions that will be used in this paper.

By a *semiring* $(S, +, \cdot)$ we mean a nonempty set S equipped with two binary operations $+$ and $\cdot$ (called addition and multiplication) such that the multiplication is left and right distributive with respect to the addition, $(S, +)$ is a semigroup with neutral element 0, and $(S, \cdot)$ is a semigroup with zero 0, i.e. $0a = a0 = 0$ for all $a \in S$. If the semigroup $(S, \cdot)$ is commutative, then we say that the semiring S is commutative.

A semiring S is *additively inverse*, if for every $a \in S$ there exists a uniquely determined element $a' \in S$ such that

$$a + a' + a = a \quad \text{and} \quad a' + a + a' = a'. \tag{2.1}$$

One can prove (cf. [12]) that in an additively inverse semiring S for all $a, b \in S$ we have

$$(ab)' = a'b = ab', \quad (a+b)' = b' + a', \quad a'b' = ab, \quad (a')' = a, \quad 0' = 0. \tag{2.2}$$

Also the following implication is valid

$$a + b = 0 \quad \textit{implies} \quad b = a' \ \textit{ and } \ a + a' = 0. \tag{2.3}$$

Note that in general $a + a' \neq 0$. $a + a' = 0$ if and only if there exists some $b \in S$ with $a + b = 0$.

By the *center* of S we mean the set $Z(S) = \{a \in S : ax = xa \text{ for all } x \in S\}$. An additively inverse semiring S with commutative addition such that

$$a + a' \in Z(S), \tag{2.4}$$

where $Z(S)$ is the center of S, is called an *MA-semiring* (cf. [11, 13]). Then also $[a, a] \in Z(S)$ for every $a \in S$, see the following definition.

The *commutator* of elements $x, y \in S$ is defined as $[x, y] = xy + y'x$. By (2.3) and (2.2), $[x, y] = 0$ implies $xy = yx$. It is not difficult to see that in MA-semirings:

$$[x, y]' = [x, y'] = [x', y] = [y, x], \quad [x', y'] = [x, y], \quad [x, yx] = [x, y]x. \tag{2.5}$$

Also the following Jacobi identities are satisfied:

$$[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z. \tag{2.6}$$

For the proof see [11].

Recall that an additive subsemigroup U of S is an *ideal* of S if $US \subseteq U$ and $SU \subseteq U$. An additive subsemigroup U of S such that $[U, S] \subseteq U$ is called a *Lie ideal* of S. A Lie ideal U with the property $UU \subseteq U$ is called *closed*. Note that in general Lie ideals are not ideals. A semiring S is *n-torsion free*, where $n > 1$ is an integer, if $nx = 0$ holds only for $x = 0$. S is *prime* if $xSy = \{0\}$ implies $x = 0$ or $y = 0$.

An important role in investigations of such semirings, in particular in investigations of their commutativity, play derivations defined as additive mappings $f\colon S \to S$ (i.e. endomorphisms of the additive semigroup of S) satisfying additional conditions similar to derivatives of real functions (cf. for example [11, 16]). Similarly as in rings, an additive mapping d is called a *derivation* if we have $d(xy) = d(x)y + xd(y)$ for all $x, y \in S$, and is called a *Jordan derivation* in case when $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in S$. Obviously, every derivation is a Jordan derivation. The converse need not be true in general. A special role plays a *left derivation* defined as an additive mapping $f\colon S \to S$ such that $f(xy) = xf(y) + yf(x)$ for all $x, y \in S$ (cf. [3, 5, 7]). An additive mapping $f\colon S \to S$ satisfying the identity

$$f(x^2) + 2x'f(x) = 0 \tag{2.7}$$

is called a *left Jordan derivation* or a *Jordan left derivation*. By (2.3), for such a derivation we have $f(x^2) = 2xf(x)$. The converse statement is not true. This means that a left Jordan derivation of S is a generalization of a left Jordan derivation defined for rings [17].

3. Lie ideals and left Jordan derivations

We start with some identities that will be used later.

Lemma 3.1. *If $f : S \to S$ is an additive mapping on an additively inverse semiring S, then $f(x') = f(x)'$ for all $x \in S$.*

Proof. Since for every $x \in S$ the element $x' \in S$ is uniquely defined, from $f(x) = f(x + x' + x) = f(x) + f(x') + f(x)$ and $f(x') = f(x' + x + x') = f(x') + f(x) + f(x')$ we deduce $f(x') = f(x)'$. □

Proposition 3.2. *Let f be a left Jordan derivation defined on a 2-torsion free MA-semiring S. Then for all $x, y \in S$ we have:*

(i) $[x, x]f(x) = 0$ *and* $x[y, y]f(x) = 0$,
(ii) $f(x + x') = 0$ *if S is prime.*

Proof. (i). By using (2.7) and (2.3), we get $2(x + x')f(x) = 0$. Because S is 2-torsion free, the last implies $(x+x')f(x) = 0$. Consequently, $x(x+x')f(x) = 0$, i.e. $[x, x]f(x) = 0$. From $(x + x')f(x) = 0$ we also get $y^2(x + x')f(x) = 0$. This, by (2.2) and (2.4), gives $x[y, y]f(x) = 0$.

(ii). As in (i) we obtain $(x + x')f(x) = 0$ whence applying (2.2) and (2.4), we get $xS(f(x) + f(x)') = 0$. So, $xSf(x + x') = 0$. This proves (ii). □

Proposition 3.3. *Let f be a left Jordan derivation defined on a closed Lie ideal U of a 2-torsion free MA-semiring S. Then for all $u, v, w \in U$ we have*

(i) $f(uv + vu) = 2uf(v) + 2vf(u)$,
(ii) $f(uvu) = u^2f(v) + 3uvf(u) + vuf(u)'$,
(iii) $[u, u]f(v) = 0$,
(iv) $f(uvw + wvu) = (uw + wu)f(v) + 3uvf(w) + 3wvf(u) + vuf(w)' + vwf(u)'$,
(v) $[u, v]uf(u) + u'[u, v]f(u) = 0$,
(vi) $[u, v](f(uv) + uf(v)' + vf(u)') = 0$.

Proof. (i). Since f is a left Jordan derivation, (2.3) together with (2.7) imply $f(x^2) = 2xf(x)$. Thus for $x = u + v$ we get

$$\begin{aligned}2(u + v)f(u + v) &= f((u + v)^2) = f(u^2 + v^2 + uv + vu)\\ &= 2uf(u) + 2vf(v) + f(uv + vu),\end{aligned}$$

whence, adding $2u'f(u) + 2v'f(v)$ and using (2.3) and (2.7) we obtain (i).

(ii). Since U is closed, $uv + vu \in U$ for all $u, v \in U$. So, (i) for $v = uv + vu$ gives

$$\begin{aligned}f(u(uv + vu) + (uv + vu)u) &\overset{(i)}{=} 2uf(uv + vu) + 2(uv + vu)f(u)\\ &\overset{(i)}{=} 2u(2uf(v) + 2vf(u)) + 2(uv + vu)f(u)\\ &= 4u^2f(v) + 6uvf(u) + 2vuf(u).\end{aligned}$$

Consequently,

$$f(u(uv + vu) + (uv + vu)u) = 4u^2f(v) + 6uvf(u) + 2vuf(u). \tag{3.1}$$

On the other hand, by (i) and (2.7), we also have

$$f(u(uv + vu) + (uv + vu)u) = 2u^2f(v) + 4vuf(u) + 2f(uvu). \tag{3.2}$$

Replacing x by $x+y$ in (2.7) and applying (2.3), we can see that $f(uv+vu)+f(uv+vu)' = 0$ for all $u, v \in U$. In particular

$$f(u(uv + vu) + (uv + vu)u) + f(u(uv + vu) + (uv + vu)u)' = 0,$$

whence, by (3.1) and (3.2), we have

$$4u^2f(v)+6uvf(u)+2vuf(u)+(2u^2f(v)+4vuf(u)+2f(uvu))'=0$$

or

$$2(2u^2f(v)+u^2f(v)'+vuf(u)+2vuf(u)'+3uvf(u)+f(uvu)')=0.$$

Since S is a 2-torsion free, from the above, using (2.1) for $a=u^2f(v)$ and $a=vuf(u)'$, we obtain

$$u^2f(v)+vuf(u)'+3uvf(u)+f(uvu)'=0. \tag{3.3}$$

This, by (2.3), implies (ii).

(iii). Since $[u,u]f(v)=u^2f(v)+u^2f(v)'$ and $u^2f(v)+u^2f(v)'=0$, by (3.3) and (2.3), we have $[u,u]f(v)=0$.

(iv). Putting $u+w$ instead of u in (3.3), we obtain

$$\left.\begin{aligned} f((u+w)v(u+w))'+u^2f(v)+(uw+wu)f(v)+w^2f(v)\\ +vuf(u)'+vuf(w)'+vwf(u)'+vwf(w)'\\ +3uvf(u)+3wvf(u)+3uvf(w)+3wvf(w)=0 \end{aligned}\right\} \tag{3.4}$$

On the other hand

$$f((u+w)v(u+w))=f(uvu)+f(wvw)+f(uvw+wvu),$$

whence, by (ii), we get

$$\left.\begin{aligned} f((u+w)v(u+w))=u^2f(v)+3uvf(u)+v'uf(u)+w^2f(v)\\ +3wvf(w)+v'wf(w)+f(uvw+wvu). \end{aligned}\right\} \tag{3.5}$$

Applying (2.3) to (3.4) we can see that

$$u^2f(v)+u^2f(v)'=0,\quad w^2f(v)+w^2f(v)'=0,\quad 3uvf(u)+3uvf(u)'=0,$$
$$3wvf(w)+3wvf(w)'=0,\quad vuf(u)+vuf(u)'=0,\quad vwf(w)+vwf(w)'=0.$$

Now, comparing (3.4) with (3.5) and using the above identities we obtain

$$f(uvw+wvu)'+(uw+wu)f(v)+3uvf(w)+3wvf(u)+vuf(w)'+vwf(u)'=0, \tag{3.6}$$

which in view of (2.3) completes te proof of (iv).

(v). Since the Lie ideal U is closed, we have $uv\in U$ for all $u,v\in U$. Thus, in (3.6) we can replace w by uv, whence we obtain

$$\left.\begin{aligned} f((uv)^2+uv^2u)'+u^2vf(v)+uvuf(v)+3uvf(uv)\\ +3uv^2f(u)+vuf(uv)'+vuvf(u)'=0. \end{aligned}\right\} \tag{3.7}$$

On the other hand, by (2.7) and (ii), we get

$$f((uv)^2+uv^2u)=2uvf(uv)+2u^2vf(v)+3uv^2f(u)+v^2uf(u)'.$$

This together with (3.7), after reduction, gives

$$u[u,v]f(v)'+[u,v]f(uv)+v[u,v]f(u)'=0 \tag{3.8}$$

by (3.7) and (2.1).

Now, replacing v by $u+v$, using (iii) and the fact that $f(x^2)=2xf(x)$, we get

$$2[u,v]uf(u)+[u,v]f(uv)+2u[u,v]f(u)'+u[u,v]f(v)'+v[u,v]f(u)'=0.$$

This, by (3.8) and 2-torsion freeness of S, completes the proof of (v).

(vi). If in (v) we replace u by $u+v$, then after reordering components we obtain the expression

$$W(u,v) = \Big([u,v]uf(u) + u'[u,v]f(u)\Big) + \Big([v,v]uf(u) + [v,v]uf(v) + [v,v]vf(u)$$
$$+[v,v]vf(v) + u'[v,v]f(u) + u'[v,v]f(v) + v'[v,v]f(u) + v'[v,v]f(v)\Big) + \Big([u,v]uf(v)$$
$$+[u,v]vf(u) + [u,v]vf(v) + u'[u,v]f(v) + v'[u,v]f(u) + v'[u,v]f(v)\Big) = 0.$$

By (v) the first parenthesis is zero. Since $[x,x] \in Z(S)$ for all $x \in S$, the second parenthesis can be rewritten in the form

$$u[v,v]f(u) + u[v,v]f(v) + v[v,v]f(u+u') + v[v,v]f(v) + (u[v,v]f(u))'$$
$$+(u[v,v]f(v))' + v'[v,v]f(v),$$

which, by Proposition 3.2, means that also the second parenthesis is zero. Thus $W(u,v)$ reduces to

$$[u,v]uf(v) + [u,v]vf(u) + [u,v]vf(v) + u'[u,v]f(v) + v'[u,v]f(u) + v'[u,v]f(v) = 0.$$

This implies (vi) because by (3.8) and (2.3) we have

$$u'[u,v]f(v) + v'[u,v]f(u) = [u,v]f(uv)'$$

and

$$[u,v]vf(v) + v'[u,v]f(v) = 0$$

by (v), (2.2) and (2.5). □

Proposition 3.4. *Let f be a left Jordan derivation defined on a closed Lie ideal U of a prime 2-torsion free MA-semiring S. Then for all $u,v,w \in U$ we have*

(i) $[u,v]f([u,v]) = 0$,
(ii) $(vu^2 + 2uv'u + u^2v)f(v) = 0$.

Proof. (i). Using Proposition 3.3 (vi) we get

$$[u,v](f(v'u) + uf(v) + vf(u)) = 0$$

and

$$[u,v](f(uv) + uf(v') + vf(u')) = 0.$$

By adding these two equations and using Proposition 3.2 (ii) we get (i).

(ii). Since the Lie ideal U is closed, $4vuv \in U$ for all $u,v \in U$. By (2.7) and (i) we also have

$$f([u,v]^2) = 2[u,v]f([u,v]) = 0.$$

Thus

$$\begin{aligned} 0 = f([u,v]^2) &= f((uv+v'u)^2) = f(uvuv + vuvu + u'v^2u + v'u^2v) \\ &= f(u(vuv) + (vuv)u) + f(u'v^2u + v'u^2v). \end{aligned}$$

From this, after application of Proposition 3.3 (i), we get

$$2uf(vuv) + 2vuvf(u) + f(u'v^2u + v'u^2v) = 0.$$

This, by Proposition 3.3 (ii), gives

$$2uv^2f(u) + 6uvuf(v) + 2u^2vf(v)' + 2vuvf(u) + u^2f(v^2)' + 3uv^2f(u)'$$
$$+v^2uf(u) + v^2f(u^2)' + 3vu^2f(v)' + u^2vf(v) = 0,$$

which due to the fact that $f(x^2) = 2xf(x)$, can be written as

$$(2uv^2 + 3u'v^2)f(u) + 6uvuf(v) + (4u^2v' + u^2v)f(v) + 2vuvf(u)$$
$$+ (v^2u + 2v^2u')f(u) + 3vu^2f(v)' = 0.$$

Since $2a + 3a' = (a' + a + a') + a + a' = a'$ and $4a' + a = 3a'$, the last expression reduces to

$$u'v^2f(u) + 6uvuf(v) + 3u^2v'f(v) + 2vuvf(u) + v^2u'f(u) + 3vu^2f(v)' = 0,$$

i.e. to

$$3(u^2v' + 2uvu + v'u^2)f(v) + (u'v^2 + 2vuv + v^2u')f(u) = 0,$$

which, by (2.2), implies

$$3(vu^2 + 2uv'u + u^2v)f(v) + (uv^2 + 2vu'v + v^2u)f(u) = 0. \tag{3.9}$$

By Proposition 3.3(v), we have $(u^2v + 2uv'u + vu^2)f(u) = 0$ and u replaced by $u + v$ in this expression gives

$$\begin{aligned}((u^2v + vvv + vuv + uv^2) &+ (2uv'u + 2v^2u' + 2u'v^2 + 2vv'v)\\ &+ (vu^2 + vvv + vuv + v^2u))f(u + v) = 0,\end{aligned}$$

which, by (2.3), implies

$$((u^2v + 2uv'u + vu^2) + (uv^2 + v^2u + 2v^2u' + 2u'v^2 + 2vuv))f(u + v) = 0.$$

Since $a + a' + a = a$, the last equation reduces to

$$((u^2v + 2uv'u + vu^2) + (v^2u' + u'v^2 + 2vuv))f(u + v) = 0. \tag{3.10}$$

Observe that the condition (v) from Proposition 3.3 can be rewritten in the form

$$(v'u^2 + u^2v' + 2uvu)f(u) = 0.$$

So, $(vu^2 + u^2v + 2uv'u)'f(u) = 0$.

Applying the last expression to (3.10) we obtain

$$(u^2v + 2uv'u + vu^2)f(v) + (v^2u + uv^2 + 2vu'v)'f(u) = 0.$$

Adding this equation to (3.9), applying Proposition 3.2 (ii) and using 2-torsion freeness of S, we get (ii). □

Lemma 3.5. *Let U be a closed Lie ideal of a prime MA-semiring S such that $[U, U] \neq 0$. Then*

(i) *the ideal generated by $S[U, U]S$ is included in U,*
(ii) *there exists an ideal $M \subseteq U$ such that $[[M, S], S] \neq 0$,*
(iii) *$aUb = 0$ implies $a = 0$ or $b = 0$.*

Proof. (i). Let $u, v \in U$. By the assumption $[u, v], [u, s], [u, vs], v[u, s] \in U$ for all $s \in S$. Since $v'[u, s] = v[u, s'] \in U$, also $[u, vs] + v'[u, s] \in U$. But,

$$\begin{aligned}[u, vs] + v'[u, s] &\overset{(2.6)}{=} v[u, s] + [u, v]s + v'[u, s]\\ &= vus + vs'u + uvs + v'us + v'us + v's'u\\ &= (v' + v + v')us + (vs + vs')u + uvs\\ &= v'us + (vs + vs')u + uvs\\ &\overset{(2.4)}{=} v'us + u(vs + vs') + uvs\\ &= v'us + uv(s + s' + s) = v'us + uvs = [u, v]s.\end{aligned}$$

So, $[u, v]s \in U$. In particular, $[u, v]st'$ and $[[u, v]s, t]$ are in U for all $s, t \in S$. Thus,

$$[[u, v]s, t] + [u, v]st' \in U.$$

From this, using (2.1) and (2.4), we obtain

$$\begin{aligned}[[u, v]s, t] + [u, v]st' &= [u, v]st + t'[u, v]s + [u, v]st'\\ &= [u, v]s(t + t') + t'[u, v]s \overset{(2.4)}{=} (t + t')[u, v]s + t'[u, v]s\\ &= (t' + t + t')[u, v]s = t'[u, v]s \overset{(2.2)}{=} t[u, v]'s \overset{(2.5)}{=} t[v, u]s\end{aligned}$$

for all $s, t \in S$. Therefore $S[U, U]S \subseteq U$.

Let $M = \langle S[U,U]S \rangle$ be the ideal generated by $S[U,U]S$. Since $[U,U] \neq 0$, also $M \neq 0$. Moreover, each element of M is a finite sum of elements $s_i[u_i, v_i]t_i \in S[U,U]S \subseteq U$. So, $M \subseteq U$.

(ii). Let $M = \langle S[U,U]S \rangle$. Assume $[[M,S],S] = 0$. Then $[M,S] \subseteq Z(S)$. We have $[m,s]m = [m,sm]$ for all $m \in M$ and $s \in S$. Since $[m,sm] \in Z(S)$, we also have $s'[m,s]m = [m,sm]s'$. Hence,

$$0 = [[m,s],s]m = [m,s]sm + s'[m,s]m = [m,s]sm + [m,sm]s' \overset{(2.5)}{=} [m,s]sm + [m,s]ms' = [m,s][s,m].$$

Consequently, $[m,s]S[s,m] = 0$, which implies $[m,s] = 0$, therefore $ms = sm$. Also $[M,S] = 0$. Thus, for all $m \in M$, $s \in S$ and $u \in U$, we have

$$0 = [m,us] = mus + us'm = (mu)s + (mu)s' = (mu)s + s'(mu),$$

since M is an ideal and $mu \in M$. Hence

$$0 = mus + s'mu = usm + s'um = [u,s]m.$$

Therefore, $[U,S]M = 0$. In particular, $[U,U]M = 0$. Consequently, $[U,U]SM = 0$. But S is prime and $M \neq 0$, so $[U,U] = 0$ which is a contradiction. Hence, $[[M,S],S] \neq 0$.

(iii). Suppose that $Ub = 0$ for some $0 \neq b \in S$. Then $[U,S]b \subseteq Ub = 0$. Thus for all $u \in U$ and $s,t \in S$ we have $0 = [u,st]b = [u,s]tb + s[u,t]b = [u,s]tb$. Therefore, $[U,S]Sb = 0$, which implies $[U,S] = 0$. Consequently, $[U,U] = 0$, a contradiction. So, $Ub \neq 0$ for each $b \neq 0$.

Now let M as in the proof of (ii). Suppose $aUb = 0$ for some $a,b \in U \setminus \{0\}$. Since $[mau, s] \in [M,S] \subseteq U$ for all $m \in M$ and $u \in U$, from $aUb = 0$ we conclude $a[mau,s]b = 0$. From this, using (2.6) and $aUb = 0$, we deduce

$$0 = ama[u,s]b + am[a,s]ub + a[m,s]aub = am[a,s]ub = amasub.$$

Hence, $aMaSUb = 0$. Since S is prime and $Ub \neq 0$ for $b \neq 0$, we must have $aMa = 0$. But $aMSa \subseteq aMa = 0$ means that $aM = 0$ or $a = 0$. If $aM = 0$, then also $aSM = 0$. Hence $a = 0$ or $M = 0$. The case $M = 0$ is impossible because $M = S[U,U]S = 0$ implies $(S[U,U]S)S[U,U] = 0$, which gives $S[U,U] = 0$. Thus, $[U,U]S[U,U] = 0$ and consequently $[U,U] = 0$, This contradicts the assumption on $[U,U]$. So in any case we have $a = 0$. □

4. Left Jordan derivations on prime MA-semirings

In this section we characterize left Jordan derivations defined on closed Lie ideals and determine conditions under which an MA-semiring with left Jordan derivation is commutative.

Lemma 4.1. *If on a prime MA-semiring S there 2-torsion free is a non-zero left Jordan derivation f, then each element of the form $[a,[a,x]]$, where $a,x \in S$ and $f(a) \neq 0$, is nilpotent of index 2, i.e. $a^2 = 0$.*

Proof. Let $f(a) \neq 0$ for some $a \in S$. Then, by Proposition 3.3 (v), for all $w \in S$ we have $[[a,w],a]f(a) = 0$, which by (2.2) gives $[[a,w],a]'f(a) = 0' = 0$. Consequently,

$$[a,[a,w]]f(a) = 0. \tag{4.1}$$

From this, putting $w = xy$, where $x,y \in S$, and using the right Jacobi identity (2.6), we obtain

$$(x[a,[a,y]] + [a,[a,x]]y + 2[a,x][a,y])f(a) = 0,$$

which, by (4.1), gives

$$([a,[a,x]]y + 2[a,x][a,y])f(a) = 0.$$

Replacing in this expression y by $[a,yz]$ and applying (4.1), we get

$$[a,[a,x]][a,yz]f(a) = 0$$

for all $x, y, z \in S$, whence, by (2.6), we deduce

$$([a,[a,x]]y[a,z] + [a,[a,x]][a,y]z)f(a) = 0. \tag{4.2}$$

This, by (4.1), for $z = [a, z]$ gives

$$[a,[a,x]][a,y][a,z]f(a) = 0.$$

Now, replacing y by $[a, y]$ in (4.2) and using the last identity, we obtain

$$[a,[a,x]][a,[a,y]]zf(a) = 0.$$

This expression is valid for all $x, y, z \in S$, so it implies $[a,[a,x]]^2 Sf(a) = 0$. But S is prime and $f(a) \neq 0$, thus $[a,[a,x]]^2 = 0$ for all $x \in S$. This completes the proof. □

Lemma 4.2. *Let a prime MA-semiring S be 2- and 3-torsion free. Then $f(a) = 0$ for any left Jordan derivation f defined on S and any nilpotent element a of index 2.*

Proof. Let $a \in S$ be a nilpotent element of index 2. Since S is 2-torsion free, $0 = f(a^2) = 2af(a)$ implies $af(a) = 0$. Thus, Proposition 3.3(ii), gives

$$f(a(xay + yax)a) = 3axayf(a) + 3ayaxf(a).$$

On the other hand, $f(aya) = 3ayf(a)$, by Proposition 3.3(ii), and

$$f(a(xay + yax)a) = f(ax(aya) + (aya)xa) = 9axayf(a) + 3ayaxf(a),$$

by Proposition 3.3 (iv).

These two identities together with Proposition 3.2 (ii) and (2.3) imply

$$3axayf(a) + 3ayaxf(a) + 9axayf(a)' + 3ayaxf(a)' = 0.$$

Hence,

$$3[ax, ay]f(a) + 3[ay, ax]f(a) + 6axayf(a)' = 0$$

or

$$3[ax, ay](f(a) + f'(a)) + 6axayf(a)' = 0.$$

This, by Proposition 3.2(ii), gives $6axayf(a) = 0$. But S is 2- and 3-torsion free, therefore $axayf(a) = 0$ for all $x, y \in S$, so $aSaSf(a) = 0$. Since S is prime and $a \neq 0$, the last implies $aSf(a) = 0$. Consequently $f(a) = 0$. □

The following theorem establishes the commutativity condition on S.

Theorem 4.3. *If on a prime, 2- and 3-torsion free MA-semiring S there exists a non-zero left Jordan derivation, then S is commutative.*

Proof. Let f be a left Jordan derivation on S and $0 \neq a \in S$ be such that $f(a) \neq 0$. Then, by Lemmas 4.1 and 4.2, we have $f([a,[a,x]]) = 0$, that is

$$f(a^2x + xa^2) + 2f(axa)' = 0$$

for all $x \in S$. Hence, applying Proposition 3.3 (i) and (ii), we get

$$6xaf(a) + 6axf(a)' + 2a^2f(x) + 2a^2f(x)' = 0,$$

which, by (2.3), implies $6(xa + a'x)f(a) = 0$. But S is 2- and 3-torsion free, so $[x, a]f(a) = 0$. From this, replacing x by yx and using the first Jacobi identity, we obtain $[y, a]xf(a) = 0$. Since $f(a) \neq 0$ and S is prime, we have $[y, a] = 0$ for all $y \in S$, so $a \in Z(S)$.

For the proof that S is commutative, we will divide S into two disjoint subsets

$$A = \{a \in S \mid f(a) \neq 0\} \subseteq Z(S) \quad \text{and} \quad B = \{a \in S \mid f(a) = 0\}.$$

Consider the sum $a + b$, where $a \in A$ and $b \in B$. If $a + b \in B$, then $0 = f(a + b) = f(a) + f(b) = f(a)$, a contradiction. So, $a + b \in A$. In this case for each $y \in S$ we have $0 = [y, a + b] = [y, b]$. So, $B \subseteq Z(S)$. Hence S is commutative. □

Corollary 4.4. *A non-commutative, prime,* 2- *and* 3-*torsion free* MA-*semiring does not allow non-zero left Jordan derivations.*

This theorem generalizes the result of Ashraf and Rehman from [3].

Theorem 4.5. *A left Jordan derivation defined on a closed Lie ideal of a prime* 2-*torsion free* MA-*semiring* S *is a left derivation on this Lie ideal.*

Proof. Let U be a closed Lie ideal of S. If $[U,U]=0$, then $uv=vu$ for all $u,v\in U$. So, $f(x^2)+2x'f(x)=0$ for $x=u+v$ gives $2f(uv)+2u'f(v)+2v'f(u)=0$. Since S is 2-torsion free, the last implies $f(uv)+u'f(v)+v'f(u)=0$. This, by (2.3), means that $f(uv)=uf(v)+vf(u)$.

Consider now the case $[U,U]\neq 0$. By Proposition 3.3 (v), for all $u,v\in U$, we have $(u^2v+2uv'u+vu^2)f(u)=0$. In particular, for $u=[w,z]$ we have

$$([w,z]^2v+2[w,z]v'[w,z]+v[w,z]^2)f([w,z])=0.$$

Since by Proposition 3.4 (i), $[u,v]f([u,v])=0$, the above expression reduces to

$$[w,z]^2vf([w,z])=0.$$

It holds for all $w,v,z\in U$, so $[w,z]^2Uf([w,z])=0$. This, by Lemma 3.5, implies $[w,z]^2=0$ or $f([w,z])=0$. If $f([w,z])=0$, then $f(wz)=f(zw)$ and, as in the previous case, $f(wz)=wf(z)+zf(w)$.

If $f([w,z])\neq 0$, then $[w,z]^2=0$. In this case, from the equation mentioned in Proposition 3.4 (ii), for $v=[w,z]$, after reduction by Proposition 3.4(i), we obtain

$$(2u[w,z]'u+[w,z]u^2)f([w,z])=0.$$

Replacing in this equation u by $u+v$, we see that

$$(2u[w,z]'v+2v[w,z]'u+[w,z]uv+[w,z]vu)f([w,z])=0,$$

whence, putting $u=y[w,z]$, applying Proposition 3.4(i) and the fact that $[w,z]^2=0$, we get $[w,z]y[w,z]vf([w,z])=0$, and in the consequence, $([w,z]y[w,z])Uf([w,z])=0$. So, $f([w,z])=0$ or $[w,z]U[w,z]=0$. The case $f([w,z])=0$ was considered earlier, so we will consider only the case when $[w,z]U[w,z]=0$. In this case, by Lemma 3.5 (iii), we have $[w,z]=0$. Thus $wz=zw$ for all $w,z\in U$, but then, as was proved, f is a left derivation on U. $\square$

As a consequence we obtain

Theorem 4.6. *Any left Jordan derivation on a prime* 2-*torsion free* MA-*semiring is a left derivation on this* MA-*semiring.*

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