ALKÜ Fen Bilimleri Dergisi 2019, Sayı 1(1): 40-47

On Some Properties of Δ^m-Statistical Convergence in a Paranormed Space

Çiğdem Asma BEKTAŞ[1*], Emine ÖZÇELİK[2]
[1] *Fırat Üniversitesi, Fen Fakültesi, Matematik Bölümü, Elazığ, Türkiye.*
[2] *Fırat Üniversitesi, Fen Fakültesi, Matematik Bölümü, Elazığ, Türkiye.*
**cbektas@firat.edu.tr*

Abstract

In this study, we introduce the concepts of strongly (Δ^m, p)-Cesàro summability, Δ^m-statistical Cauchy sequence and Δ^m-statistical convergence in a paronormed space. We give some certain properties of these concepts and some inclusion relations between them.

Keywords: Statistical convergence, statistical Cauchy, paranormed space, difference sequence.

1. BACKGROUND AND PRELIMINARIES

Fast [1] and Steinhaus [2] introduced the concept of statistical convergence for sequences of real numbers. Several authors studied this concept with related topics [3-5].

The asymptotic density of $K \subset N$ is defined as,

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : k \in K\}\right|$$

where K be a subset of the set of natural numbers N and denoted by $\delta(K)$. $|.|$ indicates the cardinality of the enclosed set.

A sequence (x_k) is called statistically covergent to L provided that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |x_k - L| > \varepsilon\}\right| = 0$$

for each $\varepsilon > 0$. It is denoted by $st - \lim_{k\to\infty} x_k = L$.

A sequence (x_k) is called statistically Cauchy sequence provided that there exist a number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\left|x_k-x_N\right|\ge\varepsilon\right\}\right|=0$$

for every $\varepsilon>0$.

Definition 1.1 $f:[0,\infty)\to[0,\infty)$ is called the modulus function which satisfies the following conditions. For $\forall\, x,y\in[0,\infty)$

i) $f(x)=0\Leftrightarrow x=0$,

ii) $f(x+y)\le f(x)+f(y)$,

iii) f is increasing function,

iv) f is continuous from the right at 0.

Definiton 1.2 [1] (x_k) is called convergent (or g-convergent) to L in a paranormed space (X,g) provided that $k_0\in\mathbb{Z}^+$ such that $g(x_k-L)<\varepsilon$ for $k\ge k_0$ for every $\varepsilon>0$.

It is written by $g-\lim_{k\to\infty}x_k=L$ and L is called the g-limit of the sequence (x_k).

Definiton 1.3 [1] (x_k) is called statistically covergent to L in a paranormed space (X,g) if for each $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:g(x_k-L)>\varepsilon\right\}\right|=0.$$

It is written by $g(st)-\lim_{k\to\infty}x_k=L$. The set of these sequences is indicated by S_g.

Definiton 1.4 [1] A sequence (x_k) is called statistically Cauchy sequence in a paranormed space (X,g) provided that there is a number $N=N(\varepsilon)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{j\le n:g(x_j-x_N)\ge\varepsilon\right\}\right|=0$$

for every $\varepsilon>0$. In brief we called such as these sequences $g(st)$-Cauchy.

2. MAIN RESULTS

Definiton 2.1 (x_k) is called Δ^m-convergent (or $g(\Delta^m)$-convergent) to L in a paranormed space (X,g) provided that $k_0\in\mathbb{Z}^+$ such that $g(\Delta^m x_k-L)<\varepsilon$ for $k\ge k_0$ and for every $\varepsilon>0$.

In this case it is written by $g(\Delta^m)-\lim_{k\to\infty}x_k=L$ and L is called the $g(\Delta^m)$-limit of the sequence (x_k).

Definiton 2.2 [6] (x_k) is called Δ^m-statistically convergent to L in a paranormed space (X,g) if for each $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n: g(\Delta^m x_k - L) > \varepsilon\right\}\right| = 0.$$

In this case, we write $g\left(st,\Delta^m\right)-\lim_{k\to\infty} x_k = L$. The set of these sequences is indicated by $S_g\left(\Delta^m\right)$.

Definiton 2.3 $\left(x_k\right)$ is called Δ^m-statistically Cauchy (or $g\left(st,\Delta^m\right)$-Cauchy) sequence in a paranormed space $\left(X,g\right)$ provided that there exists a number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n: g(\Delta^m x_k - \Delta^m x_N) \ge \varepsilon\right\}\right| = 0$$

for every $\varepsilon > 0$.

Theorem 2.4 If $\left(x_k\right)$ is Δ^m-statistically convergent in a paranormed space $\left(X,g\right)$, then its $g\left(st,\Delta^m\right)$ limit value is unique.

Theorem 2.5 If $g\left(\Delta^m\right)-\lim_{k\to\infty} x_k = L$ then $g\left(st,\Delta^m\right)-\lim_{k\to\infty} x_k = L$ But converse case is not true.

Proof. Assume that $g\left(\Delta^m\right)-\lim_{k\to\infty} x_k = L$. Then for every $\varepsilon > 0$, there is $N\in\mathbb{Z}^+$ such that $g\left(\Delta^m x_n - L\right) < \varepsilon$ for all $n \ge N$. We have

$$A(\varepsilon) = \left\{k\in N : g(\Delta^m x_k - L\ ) \ge \varepsilon\right\} \subset \{1,2,3,\ldots\}$$

and $\delta(A(\varepsilon)) = 0$. Hence, $g\left(st,\Delta^m\right)-\lim_{k\to\infty} x_k = L$.

Let us show the converse case is not true with an example.

Example 2.1

Let choose $p_k = \dfrac{1}{k}$ for all $k\in\mathbb{N}$. Then, we have

$$\ell\left(\Delta^m, p\right) = \left\{x = (x_k) : \sum_{k=0}^{\infty}\left|\Delta^m x_k\right|^{\frac{1}{k}} < \infty\right\}.$$

The paranorm on this space is

$$g(x) = \left(\sum_{k=0}^{\infty}\left|\Delta^m x_k\right|^{\frac{1}{k}}\right).$$

If $\left(x_k\right)$ defined by

$$\Delta^m x_k = \begin{cases} k, & if\ k = n^2, n\in\mathbb{N} \\ 0, & otherwise \end{cases}$$

we have

$$K(\varepsilon) = \left|\left\{k\le n : g(\Delta^m x_k) \ge \varepsilon\right\}\right| \quad , 0<\varepsilon<1.$$

So we see that

$$g\left(\Delta^m x_k\right)=\begin{cases} k^{\frac{1}{k}}, & k=n^2, n\in\mathbb{N} \\ 0, & otherwise \end{cases}$$

and

$$\lim_{k\to\infty} g\left(\Delta^m x_k\right)=\begin{cases} 1, & if\ k=n^2, n\in\mathbb{N} \\ 0, & otherwise \end{cases}.$$

Therefore $\left(x_k\right)$ is not Δ^m-convergent to a number in $\left(X,g\right)$. Since $\delta(K(\varepsilon))=0$, $\left(x_k\right)$ is Δ^m-statistically convergent to 0 in $\left(X,g\right)$.

Theorem 2.6 Let $g\left(st,\Delta^m\right)-\lim_{k\to\infty} x_k = L_1$ and $g\left(st,\Delta^m\right)-\lim_{k\to\infty} x_k = L_2$. Then

i) $g\left(st,\Delta^m\right)-\lim_{k\to\infty}\left(x_k \mp y_k\right)=L_1 \mp L_2$,

ii) $g\left(\Delta^m,st\right)-\lim_{k\to\infty}\alpha x_k = \alpha L_1,\ \alpha\in\mathbb{R}$.

Theorem 2.7 $\left(x_k\right)$ in a paranormed space $\left(X,g\right)$ is Δ^m-statistically convergent to L if and only if provided that

$$K=\left\{k_1<k_2<k_3<....<k_n<...\right\}\subseteq\mathbb{N} \text{ with } \delta(K)=1$$

and $g(\Delta^m x_{k_n}-L)\to 0$ as $n\to\infty$.

Proof. Assume that $\left(x_k\right)$ is Δ^m-statistically convergent to L, that is, $g\left(st,\Delta^m\right)-\lim_{k\to\infty} x_k = L$. Now, write

$$K_r=\left\{n\in\mathbb{N}: g\left(\Delta^m x_{k_n}-L\right)\geq\frac{1}{r}\right\},\quad M_r=\left\{n\in\mathbb{N}: g\left(\Delta^m x_{k_n}-L\right)<\frac{1}{r}\right\} \text{ for } r=1,2,... .$$ Then $\delta\left(K_r\right)=0$. Hence,

$$M_1\supset M_2\supset M_3\supset ...\supset M_i\supset M_{i+1}\supset ... \quad (1)$$

and

$$\delta\left(M_r\right)=1,\ r=1,2,... . \quad (2)$$

We need to prove $\left(x_{k_n}\right)$ is $g\left(\Delta^m\right)$-convergent to L for $n\in M_r$. On contrary suppose that $\left(x_{k_k}\right)$ is not $g\left(\Delta^m\right)$-convergent to L. Therefore, there is $\varepsilon>0$ such that $g\left(\Delta^m x_{k_n}-L\right)\geq\varepsilon$ for infinitely many terms. Let $M_\varepsilon=\left\{n\in\mathbb{N}: g\left(\Delta^m x_{k_n}-L\right)<\varepsilon\right\}$ and $\varepsilon>1/r$, $r\in\mathbb{N}$. Then $\delta\left(M_\varepsilon\right)=0$ and by (1), $M_r\subset M_\varepsilon$. Hence, $\delta\left(M_r\right)=0$. This contradicts (2). Consequently $\left(x_{k_n}\right)$ is $g\left(\Delta^m\right)$-convergent to L.

Now we assume that there is a set $K=\{k_1<k_2<k_3<....<k_n<...\}$ with $\delta(K)=1$, such that $g(\Delta^m)-\lim_{n\to\infty} x_{k_n}=L$. Then, $N\in\mathbb{Z}^+$ such that $g(\Delta^m x_{k_n}-L)<\varepsilon$ for $n>N$. We choose $K_\varepsilon=\{n\in\mathbb{N}: g(\Delta^m x_n-L)\geq\varepsilon\}$ and $K^c=\{k_{N+1},k_{N+2},....\}$. $\delta(K^c)=1$ and $K_\varepsilon\subseteq\mathbb{N}-K^c$. This implies that $\delta(K_\varepsilon)=0$. Hence, $g(st,\Delta^m)-\lim_{k\to\infty} x_k=L$.

Theorem 2.8 (x_k) in a paranormed space (X,g) is Δ^m-statistically convergent if and only if it is Δ^m-statistically Cauchy.

2. STRONGLY SUMMABILITY IN A PARANORMED SPACE WITH A MODULUS FUNCTION

Definition 3.1 (x_k) is called strongly (Δ^m, p)-Cesàro summable to L in a paranormed space (X,g) $(0<p<\infty)$ if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(g\left(\Delta^m x_j-L\right)\right)^p=0.$$

This definition is a special case of Definition 3.5 which was given by Altundağ in [6]. In this case, we write

$$x_k\to L\left[C_1,g,\Delta^m\right]_p.$$

L is called the $\left[C_1,g,\Delta^m\right]_p$-limit of (x_k).

Theorem 3.2 If $x_k\to L\left[C_1,g,\Delta^m\right]_p$ $(0<p<\infty)$, then (x_k) is Δ^m-statistically convergent to L in a paranormed space (X,g).

Proof. Let $x_k\to L\left[C_1,g,\Delta^m\right]_p$. We have the inequaltiy

$$\frac{1}{n}\sum_{k=1}^{n}\left(g\left(\Delta^m x_k-L\right)\right)^p\geq\frac{1}{n}\sum_{\substack{k=1\\ \left(g\left(\Delta^m x_k-L\right)\right)^p\geq\varepsilon}}^{n}\left(g\left(\Delta^m x_k-L\right)\right)^p$$

$$\geq\frac{\varepsilon^p}{n}\left|K_\varepsilon\right|$$

as $n\to\infty$. Since $\frac{1}{n}\sum_{k=1}^{n}\left(g\left(\Delta^m x_k-L\right)\right)^p\to 0$, we have $\lim_{n\to\infty}\frac{1}{n}\left|K_\varepsilon\right|=0$ and so $\delta(K_\varepsilon)=0$, where $K_\varepsilon=\left\{k\leq n: g\left(\left(\Delta^m x_k-L\right)\right)^p\geq\varepsilon\right\}$. So (x_k) is Δ^m-statistically convergent to L in a paranormed space (X,g).

Theorem 3.3 If Δ^m-statistically convergent to L in a paranormed space (X,g) and $(x_k)\in\ell_\infty$, then $x_k \to L\left[C_1,g,\Delta^m\right]_p$.

Proof. Suppose that $(x_k)\in\ell_\infty$ and Δ^m-statistically convergent to L in a paranormed space (X,g). Then, we have $\delta(K_\varepsilon)=0$ for $\varepsilon>0$. Since $(x_k)\in\ell_\infty$, there is a $M>0$ such as $g(\Delta^m x_k - L)\leq M$ $(k=1,2,3,...)$. We obtain the equality

$$\frac{1}{n}\sum_{k=1}^{n}\left(g\left(\Delta^m x_k - L\right)\right)^p = \frac{1}{n}\sum_{\substack{k=1\\ k\notin K_\varepsilon}}^{n}\left(g\left(\Delta^m x_k - L\right)\right)^p + \frac{1}{n}\sum_{\substack{k=1\\ k\notin K_\varepsilon}}^{n}\left(g\left(\Delta^m x_k - L\right)\right)^p$$
$$= S_1(n)+S_2(n)$$

where

$$S_1(n)=\frac{1}{n}\sum_{\substack{k=1\\ k\notin K_\varepsilon}}^{n}\left(g\left(\Delta^m x_k - L\right)\right)^p$$

and

$$S_2(n)=\frac{1}{n}\sum_{\substack{k=1\\ k\in K_\varepsilon}}^{n}\left(g\left(\Delta^m x_k - L\right)\right)^p.$$

If $k\notin K_\varepsilon$, then $S_1(n)<\varepsilon^p$. Moreover, we have

$$S_2(n)\leq\left(\sup\left(g\left(\Delta^m x_k - L\right)\right)\right).\left(\frac{|K_\varepsilon|}{n}\right)\leq M.\frac{|K_\varepsilon|}{n}\to 0$$

as $n\to\infty$ and for $k\in K_\varepsilon$. Since $\delta(K_\varepsilon)=0$, $x_k \to L\left[C_1,g,\Delta^m\right]_p$.

Definition 3.4 (x_k) is called strongly (Δ^m,p)-Cesàro summable to L with respect to a modulus function f in a paranormed space (X,g) provided that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\left(g\left(\Delta^m x_k - L\right)\right)^p\right)=0 \quad (0<p<\infty)$$

and we write $x_k \to L\left(w\left(f,g,\Delta^m,p\right)\right)$. We note that if we choose $p_k = p$ in Definition 3.5 which was given in [6] we obtain the definition of strong (Δ^m,p)-Cesàro summability.

Corollary 3.5

i) Let $x_k \to L\left(w\left(f,g,\Delta^m,p\right)\right)$ and f be any modulus function. Then, (x_k) is Δ^m-statistically convergent to L in a paranormed space (X,g).

ii) A modulus function f is bounded if and only if $S_g\left(\Delta^m\right) = w\left(f, g, \Delta^m, p\right)$.

4. CONCLUSION

In this paper, the concepts of strongly $\left(\Delta^m, p\right)$-Cesàro summability, Δ^m-statistical Cauchy sequence and Δ^m-statistical convergence in a paronormed space are examined. Some new properties of these concepts in paranormed spaces are obtained.

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Geliş/Received: 31 Ağu 2018/31 Aug 2018
Kabul Ediliş/Accepted: 3 Ara 2018/3 Dec 2018