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Araştırma Makalesi (Research Article)

Approximation by a New Parametric Construction of Baskakov Operators

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Abstract

In this paper, we introduce a parametric family of Baskakov type operators depending on a real-valued function ξ and on two auxiliary sequences of functions u_n and v_n. We call these operators the α-Baskakov operators. Under a natural nonnegativity assumption on the coefficients, we investigate their weighted approximation properties on $[0,\infty)$. We establish a weighted Korovkin type convergence theorem, obtain quantitative estimates for the rate of convergence by means of the weighted modulus of continTamma uity associated with ξ and prove a Voronovskaya type asymptotic formula. Finally, we briefly discuss a numerical example illustrating the approximation behavior of the proposed operators on a compact interval.

Keywords: *Baskakov operators, weighted approximation, rate of convergence, Voronovskaya type theorem.*

Baskakov Operatörlerinin Yeni Bir Parametrik Yapısı ile Yaklaşım

Özet

Bu çalışmada, reel değerli bir ξ fonksiyonuna ve u_n ile v_n yardımcı fonksiyon dizilerine bağlı parametrik bir Baskakov tipi operatör ailesi tanıtılmaktadır. Bu operatörler α -Baskakov operatörleri olarak adlandırılmaktadır. Katsayılar üzerindeki doğal bir negatif olmama varsayımı altında, bu operatörlerin $[0,\infty)$ üzerindeki ağırlıklı yaklaşım özellikleri incelenmektedir. Ağırlıklı Korovkin tipinde bir yakınsaklık teoremi elde edilmekte, ξ ile ilişkili ağırlıklı süreklilik modülü yardımıyla yakınsama hızına ilişkin nicel tahminler verilmekte ve Voronovskaya tipinde bir asimptotik formül ispatlanmaktadır. Son olarak, önerilen operatörlerin kompakt bir aralık üzerindeki yaklaşım davranışını gösteren sayısal bir örnek kısaca ele alınmaktadır.

Anahtar Kelimeler: *Baskakov operatörleri, ağırlıklı yaklaşım, yakınsama hızı, Voronovskaya tipi teorem.*

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1. Introduction

In 1957, Baskakov [1] introduced the classical positive linear operators on $[0,\infty)$ defined by

$$V_n(f;x)=\sum_{k=0}^{\infty}p_{n,k}f\left(\frac{k}{n}\right),\qquad x\in[0,\infty),n\in\mathbb{N} \tag{1}$$

where

$$p_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}. \tag{2}$$

In 2011, Cárdenas-Morales et al. [2] defined generalized Bernstein operators by $B_n(f\circ\xi^{-1})$ and also obtained a better degree of approximation depending on ξ. This type of approximation operator generalized the Korovkin set from $\{1,x,x^2\}$ to $\{1,\xi,\xi^2\}$. Accordingly, Patel et al. [3] introduced generalized Baskakov operators and obtained the following operator:

$$V_n^{\xi}f(x)\coloneqq V_n^{\xi}(f;x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\left(\xi(x)\right)^k}{\left(1+\xi(x)\right)^{n+k}}(f\circ\xi^{-1})\left(\frac{k}{n}\right), \tag{3}$$

for $x\in[0,\infty),n\in\mathbb{N}$ and for f defined on $[0,\infty)$ such that $\xi(x)$ is a function satisfying the following conditions:

i. $\xi(0)=0$ and $\inf\limits_{x\in[0,\infty)}\xi'(x)\geq 1$,

ii. ξ is a continuously differentiable function on $[0,\infty)$.

Chen et al. [4] introduced a parametric extension of the Bernstein polynomials by means of a parameter α, where $\alpha\in[0,1]$ and investigated several of its approximation properties. Later, the operator family proposed in [4] attracted considerable attention and was further developed in numerous studies [5-12]. In 2021, Usta [13] defined a new construction of the Baskakov operator given in [3] which depends on $u_n(x)$ and $v_n(x)$, where $u_n(x)$ and $v_n(x)$ are sequences of functions defined on $\Omega\subset[0,\infty)$, for $x\in\Omega,n\in\mathbb{N}$:

$$R_n^{\xi}(f;x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\left(1+u_n(x)\right)^{n+1}\left(\xi(x)+v_n(x)\right)^k}{\left(1+\xi(x)+v_n(x)+u_n(x)\right)^{n+k-1}}(f\circ\xi^{-1})\left(\frac{k}{n}\right). \tag{4}$$

Here, ξ is a function satisfying conditions (i) and (ii). On the other hand, $\lim\limits_{n\to\infty}u_n(x)=0$ and $|u_n(x)|\leq u_n$, $\lim\limits_{n\to\infty}v_n(x)=0$ and $|v_n(x)|\leq v_n$.

In this paper, we introduce a nonnegative real parametric extension of the Baskakov operators, referred to as the α-Baskakov operators. Their definition is based on a real-valued function ξ and two sequences of functions $u_n(x)$ and $v_n(x)$ defined on $[0,\infty)$.

The structure of the paper is as follows. Section 2 provides the necessary preliminary material. In Section 3, the new α-Baskakov operators are defined by means of $u_n(x)$, $v_n(x)$ and $\xi(x)$. In addition, we present the construction of the proposed operators and compute their moments as well as central moments. Section 4 is devoted to the investigation of their convergence properties in a weighted space. In Section 5, the rate of convergence of these newly defined parametric Baskakov operators is established by using the weighted modulus of continuity. A Voronovskaya type theorem is proved in Section 6. Finally, Section 7 contains graphical illustrations and numerical examples.

2. Preliminaries

Let f be a continuous function on $[0,\infty)$ and let $\rho(x)$ be a weight function defined by $\rho(x)=1+\xi^2(x)$. Then $B_\rho[0,\infty)$ denotes the normed space

$$B_\rho[0,\infty)=\left\{f:[0,\infty)\to\mathbb{R}:|f(x)|\leq M_f.\rho(x),x\in[0,\infty)\right\} \tag{5}$$

where M_f is a positive constant depending on f, endowed with the norm

$$\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}. \tag{6}$$

In addition, we consider the following subspaces:

$$C_\rho[0,\infty) = \{ f \in B_\rho[0,\infty) : f \text{ is continuous on } [0,\infty)\}, \tag{7}$$

$$C_\rho^k[0,\infty) = \left\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} = k_f,\ \ k_f \text{ is constant depending on } f\right\}, \tag{8}$$

$$C_\rho^*[0,\infty) = \left\{ f \in C_\rho[0,\infty) : \frac{|f(x)|}{\rho(x)} \text{ is uniformly continuous on } [0,\infty)\right\}. \tag{9}$$

Another important tool in weighted approximation theory is the weighted modulus of continuity introduced by Holhoş [14]. For each $f \in C_\rho[0,\infty)$ and every $\delta > 0$, it is defined by

$$\omega_\xi(f,\delta) = \sup_{\substack{x,y\geq 0\\ |\xi(x)-\xi(y)|\leq\delta}} \frac{|f(x)-f(y)|}{\rho(x)+\rho(y)}. \tag{10}$$

Lemma 2.1. [14] If $f \in C_\rho^*[0,\infty)$, then

$$\lim_{\delta\to 0}\omega_\xi(f,\delta) = 0.$$

3. Construction of the Operators

For every $x \in [0,\infty)$ and $n \in \mathbb{N}$, define

$$\varphi_{n,k}^{\xi}(x) = \binom{n+k-1}{k} \frac{\left(1+v_n(x)\right)^n \left(\xi(x)+u_n(x)\right)^k}{(1+\xi(x)+u_n(x)+v_n(x))^{n+k}},\ \ k\geq 0. \tag{11}$$

where $u_n(x) \geq 0$ and $v_n(x) \geq 0$.

Then, for $f \in C_\rho[0,\infty)$ and $0 \leq \alpha \leq 1$, we define the α-Baskakov operator by

$$W_n^{\xi,\alpha}(f;x) = \alpha \sum_{k=0}^{\infty} \varphi_{n,k}^{\xi}(x)(f\circ\xi^{-1})\left(\frac{k}{n}\right) + (1-\alpha)\sum_{k=0}^{\infty} \varphi_{n,k}^{\xi}(x)(f\circ\xi^{-1})\left(\frac{k+1}{n}\right). \tag{12}$$

Equivalently,

$$W_n^{\xi,\alpha}(f;x) = \sum_{k=0}^{\infty} P_{n,k}^{\xi,\alpha}(x)(f\circ\xi^{-1})\left(\frac{k}{n}\right), \tag{13}$$

where

$$P_{n,0}^{\xi,\alpha}(x) = \alpha\varphi_{n,0}^{\xi}(x), \tag{14}$$

and for $k \geq 1$,

$$P_{n,k}^{\xi,\alpha}(x) = \alpha\varphi_{n,k}^{\xi}(x) + (1-\alpha)\varphi_{n,k-1}^{\xi}(x). \tag{15}$$

Here, ξ is a function satisfying conditions (i) and (ii). On the other hand, $\lim_{n\to\infty} u_n(x) = 0$ and $|u_n(x)| \leq u_n$, $\lim_{n\to\infty} v_n(x) = 0$ and $|v_n(x)| \leq v_n$.

It is clear that the operators in (12) are linear and positive. If $\alpha = 1$ and $v_n(x) = u_n(x) = 0$, then the operators in (12) reduce to those given by Patel et al. [3]. Furthermore, if in addition $\xi(x) = x$, then the classical Baskakov operators given in (1) are obtained.

Lemma 3.1. For the operators $W_n^{\xi,\alpha}$ and for all $x \in [0,\infty)$, we have

i. $W_n^{\xi,\alpha}(e_0;x) = 1$
ii. $W_n^{\xi,\alpha}(e_1;x) = \psi_n(x) + \frac{1-\alpha}{n}$,
iii. $W_n^{\xi,\alpha}(e_2;x) = \frac{n+1}{n}\psi_n^2(x) + \frac{3-2\alpha}{n}\psi_n(x) + \frac{1-\alpha}{n^2}$,
iv. $W_n^{\xi,\alpha}(e_3;x) = \frac{(n+1)(n+2)}{n^2}\psi_n^3(x) + \frac{3(n+1)(2-\alpha)}{n^2}\psi_n^2(x) + \frac{7-6\alpha}{n^2}\psi_n(x) + \frac{1-\alpha}{n^3}$,

v. $W_n^{\xi,\alpha}(e_4;x)=\frac{(n+1)(n+2)(n+3)}{n^3}\psi_n^4(x)+\frac{(n+1)(n+2)(10-4\alpha)}{n^3}\psi_n^3(x)+\frac{(n+1)(25-18\alpha)}{n^3}\psi_n^2(x)+\frac{15-4\alpha}{n^3}\psi_n(x)+\frac{1-\alpha}{n^4}$

where $e_j(t)=(\xi(t))^j$, $j=0,1,2,3,4$ and $\psi_n(x)=\frac{\xi(x)+u_n(x)}{1+v_n(x)}$.

Proof.

$$\sum_{k=0}^{\infty}\varphi_{n,k}^{\xi}(x)=1\,, \tag{16}$$

$$\sum_{k=0}^{\infty}k\varphi_{n,k}^{\xi}(x)=n\psi_n(x), \tag{17}$$

$$\sum_{k=0}^{\infty}k^2\varphi_{n,k}^{\xi}(x)=n(n+1)\psi_n^2(x)+n\psi_n(x), \tag{18}$$

$$\sum_{k=0}^{\infty}k^3\varphi_{n,k}^{\xi}(x)=n(n+1)(n+2)\psi_n^3(x)+3n(n+1)\psi_n^2(x)+n\psi_n(x), \tag{19}$$

and

$$\sum_{k=0}^{\infty}k^4\varphi_{n,k}^{\xi}(x)=n(n+1)(n+2)(n+3)\psi_n^4(x)+6n(n+1)(n+2)\psi_n^3(x)+7n(n+1)\psi_n^2(x)+n\psi_n(x). \tag{20}$$

Now, for $e_j(t)=(\xi(t))^j$,

$$(e_j\circ\xi^{-1})\left(\frac{k}{n}\right)=\left(\frac{k}{n}\right)^j. \tag{21}$$

Thus,

$$W_n^{\xi,\alpha}(e_j;x)=\frac{\alpha}{n^j}\sum_{k=0}^{\infty}k^j\varphi_{n,k}^{\xi}(x)+\frac{(1-\alpha)}{n^j}\sum_{k=0}^{\infty}(k+1)^j\varphi_{n,k}^{\xi}(x). \tag{22}$$

From (22), the first three moments can be obtained explicitly as follows. For $e_0(t)=1$, we have

$$W_n^{\xi,\alpha}(e_0;x)=\alpha\sum_{k=0}^{\infty}\varphi_{n,k}^{\xi}(x)+(1-\alpha)\sum_{k=0}^{\infty}\varphi_{n,k}^{\xi}(x)=1. \tag{23}$$

For $e_1(t)=\xi(t)$, using (17), we obtain

$$W_n^{\xi,\alpha}(e_1;x)=\frac{\alpha}{n}\sum_{k=0}^{\infty}k\varphi_{n,k}^{\xi}(x)+\frac{1-\alpha}{n}\sum_{k=0}^{\infty}(k+1)\varphi_{n,k}^{\xi}(x)=\alpha\psi_n(x)+(1-\alpha)\left(\psi_n(x)+\frac{1}{n}\right)=\psi_n(x)+\frac{1-\alpha}{n}. \tag{24}$$

Similarly, for $e_2(t)=(\xi(t))^2$, using (18), we have

$$\begin{aligned}W_n^{\xi,\alpha}(e_2;x)&=\frac{\alpha}{n^2}\sum_{k=0}^{\infty}k^2\varphi_{n,k}^{\xi}(x)+\frac{(1-\alpha)}{n^2}\sum_{k=0}^{\infty}(k+1)^2\varphi_{n,k}^{\xi}(x)\\&=\alpha\left(\frac{n+1}{n}\psi_n^2(x)+\frac{1}{n}\psi_n(x)\right)+(1-\alpha)\left(\frac{n+1}{n}\psi_n^2(x)+\frac{3}{n}\psi_n(x)+\frac{1}{n^2}\right)\\&=\frac{n+1}{n}\psi_n^2(x)+\frac{3-2\alpha}{n}\psi_n(x)+\frac{1-\alpha}{n^2}.\end{aligned} \tag{25}$$

The remaining identities for e_3 and e_4 are obtained in the same way by using (19), (20) and the binomial expansion of $(k+1)^j$. This completes the proof.

Lemma 3.2. For the operators $W_n^{\xi,\alpha}$,

i. $W_n^{\xi,\alpha}(e_1-\xi(x);x)=\psi_n(x)-\xi(x)+\frac{1-\alpha}{n}$,

ii. $W_n^{\xi,\alpha}\left((e_1-\xi(x))^2;x\right)=(\psi_n(x)-\xi(x))^2+\frac{\psi_n^2(x)+(3-2\alpha)\psi_n(x)-2(1-\alpha)\xi(x)}{n}+\frac{1-\alpha}{n^2}$,

iii. $$W_n^{\xi,\alpha}\left(\left(e_1-\xi(x)\right)^4;x\right)=\left(\psi_n(x)-\xi(x)\right)^4+\frac{2\left(\psi_n(x)-\xi(x)\right)^2\left(3\psi_n^2(x)-(5-2\alpha)\psi_n(x)-2(1-\alpha)\xi(x)\right)}{n}+\frac{11\psi_n^4(x)+\left(6(5-2\alpha)-8\xi(x)\right)\psi_n^3(x)+\left(25-18\alpha-12(2-\alpha)\xi(x)\right)\psi_n^2(x)-4(7-6\alpha)\xi(x)\psi_n(x)+6(1-\alpha)\xi^2(x)}{n^2}+\frac{6\psi_n^4(x)+(20-8\alpha)\psi_n^3(x)+(25-18\alpha)\psi_n^2(x)+(15-14\alpha)\psi_n(x)-4(1-\alpha)\xi(x)}{n^3}+\frac{1-\alpha}{n^4}.$$

Proof. The result follows immediately from the linearity of $W_n^{\xi,\alpha}$ and the identities obtained in Lemma 3.1.

4. Direct Theorems in Weighted Spaces

In this section, we examine the convergence behavior of the newly introduced parametric Baskakov operators in weighted spaces. To this end, we first recall the following weighted Korovkin type theorem established by Gadjiev [15].

Lemma 4.1. [15] For each $n \geq 1$, let $L_n: C_\rho[0,\infty) \longrightarrow B_\rho[0,\infty)$ be a positive linear operator such that

$$|L_n(\rho;x)| \leq M_n\rho(x), \qquad x \geq 0 \tag{26}$$

where $M_n > 0$ is a constant depending on n.

Theorem 4.2. [15] Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from $C_\rho[0,\infty)$ into $B_\rho[0,\infty)$. If

$$\lim_{n\to\infty}\|L_n(t^i)-t^i\|_\rho=0, \ \ i=0,1,2 \tag{27}$$

then

$$\lim_{n\to\infty}\|L_n(f;\cdot)-f\|_\rho=0, \tag{28}$$

for every $f \in C_\rho^k[0,\infty)$.

Accordingly, our main result is as follows.

Theorem 4.3. Let $\rho(x)=1+\xi^2(x)$ and let $W_n^{\xi,\alpha}: C_\rho[0,\infty) \longrightarrow B_\rho[0,\infty)$ be the operators defined by (12). Then, for each $f \in C_\rho^k[0,\infty)$,

$$\lim_{n\to\infty}\left\|W_n^{\xi,\alpha}(f;\cdot)-f\right\|_\rho=0. \tag{29}$$

Proof. (i) Let $f \in C_\rho^k[0,\infty)$. By Lemma 4.1, we have

$$W_n^{\xi,\alpha}(f;x)=W_n^{\xi,\alpha}\left(\frac{f}{\rho}\rho;x\right) \leq \|f\|_\rho W_n^{\xi,\alpha}(\rho;x) \leq \|f\|_\rho M_n \cdot \rho(x) \leq M_f \cdot \rho(x) \tag{30}$$

where $M_f > 0$ is a constant depending on f. It follows from (10) that $W_n^{\xi,\alpha}(f;\cdot) \in B_\rho[0,\infty)$.

(ii) From Lemma 3.1, it follows that

$$\lim_{n\to\infty}\left\|W_n^{\xi,\alpha}(1;\cdot)-1\right\|_\rho=0.$$

$$\left\|W_n^{\xi,\alpha}(\xi;\cdot)-\xi\right\|_\rho=\sup_{x\in[0,\infty)}\frac{\left|W_n^{\xi,\alpha}(\xi;\cdot)-\xi\right|}{1+\xi^2(x)} \leq \sup_{x\in[0,\infty)}\frac{\left|\frac{u_n(x)-\xi(x)v_n(x)}{1+v_n(x)}-\frac{1-\alpha}{n}\right|}{1+\xi^2(x)} \leq \frac{u_n}{1-v_n}+\frac{v_n}{2(1-v_n)}+\frac{1-\alpha}{n}. \tag{31}$$

Consequently,

$$\lim_{n\to\infty}\left\|W_n^{\xi,\alpha}(\xi;\cdot)-\xi\right\|_\rho=0.$$

Next, by the moment formula for $W_n^{\xi,\alpha}(\xi^2;\cdot)$, we obtain

$$\begin{aligned}\left\|W_n^{\xi,\alpha}(\xi^2;\cdot)-\xi^2\right\|_\rho &= \sup_{x\in[0,\infty)}\frac{\left|W_n^{\xi,\alpha}(\xi^2;\cdot)-\xi^2\right|}{1+\xi^2(x)}\\ &\le \sup_{x\in[0,\infty)}\frac{\left|\left(\psi_n^2(x)-\xi^2(x)\right)+\frac{\psi_n^2(x)}{n}-\frac{3-2\alpha}{n}\psi_n(x)+\frac{1-\alpha}{n^2}\right|}{1+\xi^2(x)}\\ &\le \sup_{x\in[0,\infty)}\frac{|\psi_n^2(x)-\xi^2(x)|}{1+\xi^2(x)}+\frac{1}{n}\sup_{x\in[0,\infty)}\frac{|\psi_n^2(x)|}{1+\xi^2(x)}+\frac{3-2\alpha}{n}\sup_{x\in[0,\infty)}\frac{|\psi_n(x)|}{1+\xi^2(x)}\\ &\quad+\frac{1-\alpha}{n^2}\\ &\le \frac{u_n+2v_n+u_n^2+v_n^2}{(1-v_n)^2}+\frac{1+u_n+u_n^2}{n(1-v_n)^2}+\frac{(3-2\alpha)}{n}\frac{\frac{1}{2}+u_n}{1-v_n}+\frac{1-\alpha}{n^2}\end{aligned} \tag{32}$$

Hence,

$$\lim_{n\to\infty}\left\|W_n^{\xi,\alpha}(\xi^2;\cdot)-\xi^2\right\|_\rho=0. \tag{33}$$

Thus, all the conditions of Theorem 4.2 are fulfilled. Consequently, an application of Theorem 4.2 yields $\lim_{n\to\infty}\left\|W_n^{\xi,\alpha}(f;\cdot)-f\right\|_\rho=0$ for $f\in C_\rho^k[0,\infty)$. This completes the proof.

5. Rate of Convergence of $W_n^{\xi,\alpha}$

This section is devoted to obtaining the rate of convergence of $W_n^{\xi,\alpha}$ in terms of the weighted modulus of continuity $\omega_\xi(f;\delta)$.

Theorem 5.1. [14] Let $L_n: C_\rho[0,\infty)\longrightarrow B_\rho[0,\infty)$ be a sequence of positive linear operators with

i. $\|\mathrm{L}_n(\xi^0;\cdot)-\xi^0\|_{\rho^0}=a_n$,
ii. $\|\mathrm{L}_n(\xi^1;\cdot)-\xi^1\|_{\rho^{\frac{1}{2}}}=b_n$,
iii. $\|\mathrm{L}_n(\xi^2;\cdot)-\xi^2\|_{\rho}=c_n$,
iv. $\|\mathrm{L}_n(\xi^3;\cdot)-\xi^3\|_{\rho^{\frac{3}{2}}}=d_n$

where a_n, b_n, c_n and d_n tends to zero as $n\to\infty$. Then

$$\|\mathrm{L}_n(f)-f\|_{\rho^{\frac{3}{2}}}\le(7+4a_n+2c_n)\omega_\xi(f;\delta_n)+a_n\|f\|_\rho\,, \tag{34}$$

for every $f\in C_\rho[0,\infty)$, where

$$\delta_n=2\sqrt{(a_n+2b_n+c_n)(1+a_n)}+a_n+3b_n+3c_n+d_n. \tag{35}$$

Theorem 5.2. Let $\{W_n^{\xi,\alpha}\}_{n\ge1}$ be the sequence of positive linear operators defined by (12). Then, we get

$$\left\|W_n^{\xi,\alpha}(f)-f\right\|_{\rho^{\frac{3}{2}}}\le(7+2c_n)\omega_\xi(f;\delta_n) \tag{36}$$

where

$$\delta_n=2\sqrt{2b_n+c_n}+3b_n+3c_n+d_n \tag{37}$$

and

$$b_n=\frac{u_n+v_n}{1-v_n}+\frac{1-\alpha}{n} \tag{38}$$

$$c_n=\frac{u_n+2v_n+u_n^2+v_n^2}{(1-v_n)^2}+\frac{1+u_n+u_n^2}{n(1-v_n)^2}+\frac{(3-2\alpha)}{n}\frac{\frac{1}{2}+u_n}{1-v_n}+\frac{1-\alpha}{n^2} \tag{39}$$

$$d_n=\frac{u_n^3+3u_n^2+3u_n+v_n^3+3v_n^2+3v_n}{(1-v_n)^3}+\frac{(3n+2)(1-u_n)^3}{n^2(1-v_n)^3}$$

$$+\frac{(3n+3)(2-\alpha)(1+u_n+u_n^2)}{n^2(1-v_n)^2}+\frac{(7-6\alpha)\left(\frac{1}{2}+u_n\right)}{n^2(1-v_n)}+\frac{1-\alpha}{n^3}. \tag{40}$$

Proof. Since $W_n^{\xi,\alpha}(e_0;x)=1$, we immediately have

$$a_n=\left\|W_n^{\xi,\alpha}(\xi^0;\cdot)-\xi^0\right\|_{\rho^0}=0. \tag{41}$$

From Lemma 3.1 (ii)

$$\frac{\left|W_n^{\xi,\alpha}(\xi;\ x)-\xi(x)\right|}{\sqrt{1+\xi^2(x)}}\le\frac{|u_n|}{(1-v_n)\sqrt{1+\xi^2(x)}}+\frac{|v_n|\xi(x)}{(1-v_n)\sqrt{1+\xi^2(x)}}+\frac{1-\alpha}{n\sqrt{1+\xi^2(x)}}. \tag{42}$$

Using $|u_n(x)|\le u_n$, $|v_n(x)|\le v_n$ and

$$\sup_{x\in[0,\infty)}\frac{1}{\sqrt{1+\xi^2(x)}}\le 1,\quad \sup_{x\in[0,\infty)}\frac{\xi(x)}{\sqrt{1+\xi^2(x)}}\le 1\,. \tag{43}$$

Therefore,

$$\left\|W_n^{\xi,\alpha}(\xi^1;\cdot)-\xi^1\right\|_{\rho^{\frac{1}{2}}}=\sup_{x\in[0,\infty)}\frac{\left|W_n^{\xi,\alpha}(\xi^1;\cdot)-\xi^1\right|}{\sqrt{1+\xi^2(x)}}\le\frac{u_n+v_n}{1-v_n}+\frac{1-\alpha}{n}=b_n. \tag{44}$$

Similarly, c_n and d_n can be derived from Lemma 3.1 (iii) and (iv).

6. Voronovskaya Type Theorem of $W_n^{\xi,\alpha}$

In this section, we prove a Voronovskaya type theorem for the newly introduced parametric Baskakov operator.

Theorem 6.1. Let $f\in C_\rho[0,\infty)$ and $x\in[0,\infty)$. Assume that $(f\circ\xi^{-1})'$ and $(f\circ\xi^{-1})''$ exist at $\xi(x)$. Also, if $(f\circ\xi^{-1})''$ is bounded on $[0,\infty)$ and

$$\lim_{n\to\infty}n\big(\psi_n(x)-\xi(x)\big)=\eta(x) \tag{45}$$

then we have

$$\lim_{n\to\infty}n\left(W_n^{\xi,\alpha}(f;x)-f(x)\right)=(\eta(x)+1-\alpha)\,(f\circ\xi^{-1})'\big(\xi(x)\big)+\frac{\xi(x)\big(1+\xi(x)\big)}{2}(f\circ\xi^{-1})''\big(\xi(x)\big). \tag{46}$$

Proof. By Taylor's formula, we may write

$$\begin{aligned}f(t)-f(x)&=(f\circ\xi^{-1})\big(\xi(t)\big)-(f\circ\xi^{-1})\big(\xi(x)\big)\\&=\big(\xi(t)-\xi(x)\big)(f\circ\xi^{-1})'\big(\xi(x)\big)+\frac{1}{2}\big(\xi(t)-\xi(x)\big)^2(f\circ\xi^{-1})''\big(\xi(x)\big)\\&\quad+\Upsilon_x(t)\big(\xi(t)-\xi(x)\big)^2\end{aligned} \tag{47}$$

where

$$\Upsilon_x(t)=\frac{1}{2}\left[(f\circ\xi^{-1})''\big(\xi(\zeta)\big)-(f\circ\xi^{-1})''\big(\xi(x)\big)\right]. \tag{48}$$

Here $|\Upsilon_x(t)|\le M$, $\forall t$ and $\Upsilon_x(t)\longrightarrow 0$ as $t\longrightarrow x$.

Applying the operator $W_n^{\xi,\alpha}$ to both sides and using the identities obtained above, we get

$$\begin{aligned}n\left(W_n^{\xi,\alpha}(f;x)-f(x)\right)&\\&=(f\circ\xi^{-1})'\big(\xi(x)\big)nW_n^{\xi,\alpha}\big(\xi(t)-\xi(x);x\big)\\&\quad+\frac{1}{2}(f\circ\xi^{-1})''\big(\xi(x)\big)nW_n^{\xi,\alpha}\left(\big(\xi(t)-\xi(x)\big)^2;x\right)\\&\quad+nW_n^{\xi,\alpha}\left(\Upsilon_x(t)\big(\xi(t)-\xi(x)\big)^2;x\right).\end{aligned} \tag{49}$$

Then,

$$nW_n^{\xi,\alpha}\left(\big(\xi(t)-\xi(x)\big)^2;x\right)=n\big(\psi_n(x)-\xi(x)\big)^2+\psi_n^2(x)+(3-2\alpha)\psi_n(x)-2(1-\alpha)\xi(x)+\frac{1-\alpha}{n} \tag{50}$$

and now, as $n \to \infty$, under the usual assumptions, $\psi_n(x) \to \xi(x)$ and $n\big(\psi_n(x) - \xi(x)\big)^2 \to 0$ from $\psi_n(x) = \frac{\xi(x)+u_n(x)}{1+v_n(x)}$. Therefore, we obtain

$$\lim_{n\to\infty} nW_n^{\xi,\alpha}\left(\big(\xi(t)-\xi(x)\big)^2; x\right) = \xi^2(x) + (3-2\alpha)\xi(x) - 2(1-\alpha)\xi(x) = \xi^2(x) + \xi(x). \quad (51)$$

Now, we prove that $nW_n^{\xi,\alpha}\left(\Upsilon_x(t)\big(\xi(t)-\xi(x)\big)^2; x\right) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary. Then one can choose $\delta > 0$ such that $|\Upsilon_x(t)| < \varepsilon$ whenever $|\mathrm{t}-\mathrm{x}| < \delta$. Moreover by condition (ii), we have

$$|\xi(t) - \xi(x)| < \xi(\sigma)|\mathrm{t}-\mathrm{x}| \geq |\mathrm{t}-\mathrm{x}|. \quad (52)$$

Hence, if $|\xi(t) - \xi(x)| < \delta$, then

$$\left|\Upsilon_x(t)\big(\xi(t)-\xi(x)\big)^2\right| < \varepsilon\big(\xi(t)-\xi(x)\big)^2. \quad (53)$$

On the other hand, if $|\xi(t) - \xi(x)| \geq \delta$ then using the bound $|\Upsilon_x(t)| \leq M$, we obtain

$$\left|\Upsilon_x(t)\big(\xi(t)-\xi(x)\big)^2\right| \leq \frac{M}{\delta^2}\big(\xi(t)-\xi(x)\big)^4. \quad (54)$$

Simple computations yield

$$W_n^{\xi,\alpha}\left(\big(\xi(t)-\xi(x)\big)^4; x\right) = O\left(\frac{1}{n^2}\right). \quad (55)$$

It follows that

$$\lim_{n\to\infty} nW_n^{\xi,\alpha}\left(\Upsilon_x(t)\big(\xi(t)-\xi(x)\big)^2; x\right) = 0 \quad (56)$$

which finishes the proof of Theorem 6.1.

7. Graphical Results

In this section, we apply the new parametric Baskakov operators introduced in this paper to approximate an illustrative example. Let $f(x) = e^{-\frac{x}{2}}\sin(2x) + \frac{x}{x+1} + 2$. The maximum absolute approximation error on the interval [0,4] is computed by

$$E_{n,\alpha} = \max_{x\in[0,4]}\left|W_n^{\xi,\alpha}(f;x) - f(x)\right|.$$

a) In the first example, we study the convergence behavior of the operators $\xi(x) = 2x$, $u_n(x) = \frac{x}{n+1}$ and $v_n(x) = \frac{x^2}{n+4}$.

Let $\alpha = 0.3$. The approximation of $f(x)$ by the operators $W_n^{\xi,\alpha}$ for $n = 10, 20, 50, 100$ on the interval [0,4] is presented in Figure 1.

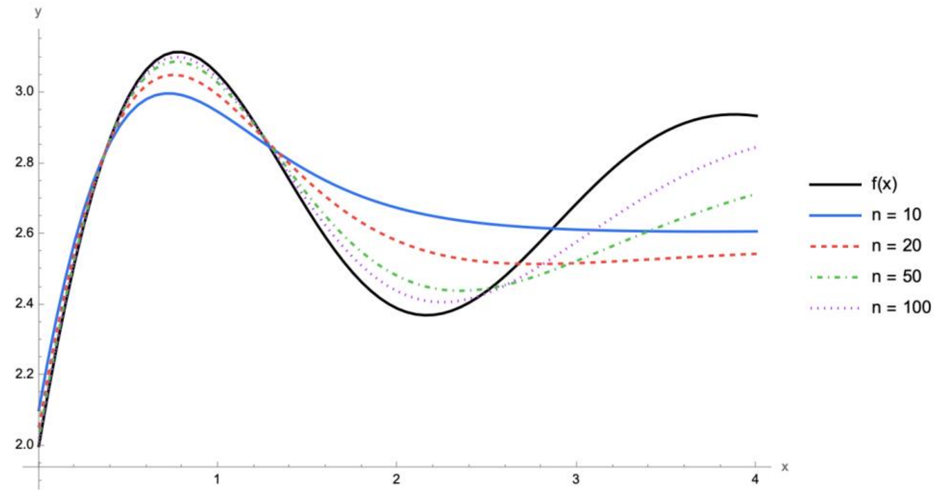

Figure 1. Convergence of the operators $1\boldsymbol{W_n^{\xi,\alpha}}$ for $\boldsymbol{\alpha = 0.3}$ and $\boldsymbol{n = 10, 20, 50}$ and $\mathbf{100}$

Table 1. Maximum absolute approximation errors of the $W_n^{\xi,\alpha}$ operator on [0, 4].

n	α = 0.1	α = 0.3	α = 0.6	α = 0.9
10	0.3335	0.3313	0.3280	0.3246
20	0.3973	0.3978	0.3984	0.3991
50	0.2659	0.2667	0.2678	0.2689
100	0.1524	0.1528	0.1534	0.1541

Figure 1 illustrates the convergence behavior of the operators for different values of n. The numerical results in Table 1 show that the approximation improves for sufficiently large values of n, especially for n =50 and $n = 100$. Although the decrease in the error is not strictly monotone for all values of n, the results are consistent with the asymptotic convergence behavior of the operators.

b) In the second example, we study the convergence behavior of the operators $\xi(x) = 2x$, $u_n(x) = \frac{x}{n^2+1}$ and $v_n(x) = \frac{x}{n+4}$.

Let $\alpha = 0.3$. We approximate $f(x)$ by the operators $W_n^{\xi,\alpha}$ for $n = 10, 20, 50, 100$ on the interval [0,4]; the results are presented in Figure 2.

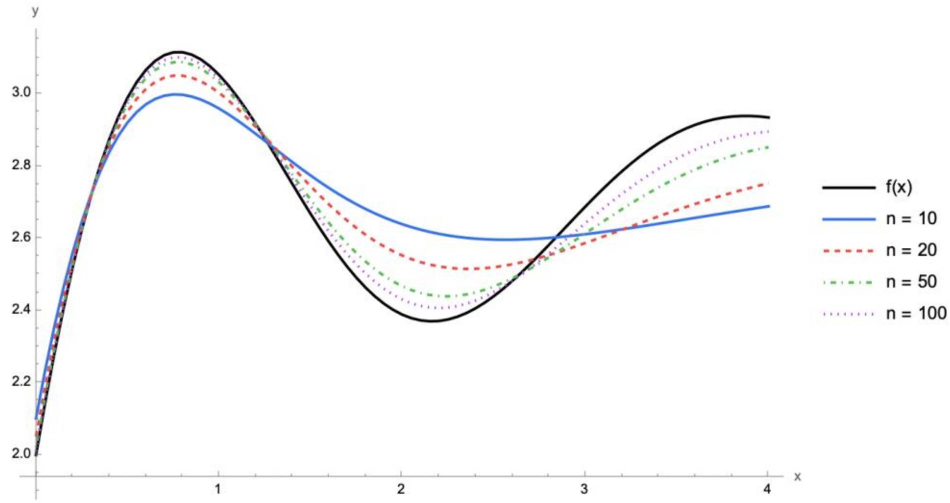

Figure 2. Convergence of the operators $2\boldsymbol{W_n^{\xi,\alpha}}$ for $\boldsymbol{\alpha = 0.3}$ and $\boldsymbol{n = 10, 20, 50}$ and $\mathbf{100}$

Table 2. Maximum absolute approximation errors of $W_n^{\xi,\alpha}$ operator on [0, 4].

n	α = 0.1	α = 0.3	α = 0.6	α = 0.9
10	0.2640	0.2649	0.2664	0.2678
20	0.2180	0.2192	0.2209	0.2225
50	0.1244	0.1250	0.1259	0.1268
100	0.0708	0.0711	0.0716	0.0721

Figure 2 shows that the operators approximate the function more accurately as n increases. This observation is also supported by the maximum absolute errors in Table 2, where the error values decrease regularly for larger n. Compared with the first example, Figure 2 and Table 2 indicate a better approximation behavior. This improvement can be attributed to the different choices of the auxiliary functions u_n and v_n. In particular, the maximum absolute errors in Table 2 are smaller and decrease more regularly as n increases. Therefore, for the selected functions u_n and v_n, the operators exhibit a more accurate approximation performance than in the previous example.

8. Conclusions

In this paper, we introduced a new parametric family of Baskakov type operators generated by the function ξ and the auxiliary sequences u_n and v_n. We established the corresponding moment estimates and showed that these operators possess effective weighted approximation properties on $[0, \infty)$. More precisely, a weighted Korovkin type convergence theorem was proved, quantitative estimates for the rate of convergence were obtained by means of the weighted modulus of continuity and a Voronovskaya type asymptotic result was derived. Since the proposed construction includes certain known operators as special cases, it provides a unified and flexible framework for further studies in weighted approximation theory. The numerical examples are also consistent with the theoretical findings and illustrate the influence of the parameters on the approximation behavior.

Conflict of Interest

The authors declare no conflict of interest.

Author Contribution

All authors contributed equally to this work.

Ethics Consent

Ethics committee approval is not required for this work.

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